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Speaker's Name: Karl Schwede

Talk Title: Birational algebraic geometry in positive characteristic

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BIRATIONAL ALGEBRAIC GEOMETRY IN POSITIVE CHARACTERISTIC

KARL SCHWEDE

1. What goes wrong?

At least 3 things go "wrong" in characteristic p algebraic geometry:

1. Resolution of singularities. In characteristic p, we don't know resolution of singularities for dimension ≥ 4.
2. Bertini theorems (general elements of a basepoint free linear system of a nonsingular variety may be singular).
3. Kodaira-type vanishing. (More specifically, Kawamata-Viehweg.)

I'm not going to say anything about getting around the first two issues, but I'll explain methods for getting around the third one.

2. Kodaira vanishing

Let X be a smooth projective variety over $\mathbf{C}$. Let $\mathscr{L}$ be an ample line bundle on X, and $\omega_X = \wedge^{\dim X}\Omega^1_{X/\mathbf{C}}$.

Theorem 2.1 (Kodaira vanishing). $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ *for* $i > 0$.

This fails in characteristic $p > 0$. To motivate the replacement, we want to show an argument that uses resolution of singularities to weaken the smoothness hypotheses.

Definition 2.2. Let X be a variety over $\mathbf{C}$. We say that X has *rational singularities* if

1. For any resolution of singularities $\pi\colon Y \to X$, we have $\pi_*\omega_Y \xrightarrow{\sim} \omega_X$, where ω_X is the canonical bundle in the classical sense.
2. X is Cohen-Macaulay.

Remark 2.3. This isn't the usual definition of rational singularities – we have used Kempf's criterion to reformulate it.

Theorem 2.4. *If X has rational singularities, and $\mathscr{L}$ is a big and nef line bundle (which would be implied by ampleness), and X is projective, then* $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ *for all* $i > 0$.

We will give the proof because it will provide useful motivation.

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Proof. Take $\pi \colon Y \to X$ a resolution of singularities. By definition,

$$H^i(X, \omega_X \otimes \mathscr{L}) = H^i(X, \pi_* \omega_Y \otimes \mathscr{L}).$$

By Grauer-Remmert vanishing, the higher direct images of $\pi_* \omega_Y$ vanish. So this is the same as $H^i(X, (R\pi_* \omega_Y) \otimes \mathscr{L}) = H^i(Y, \omega_Y \otimes \pi^* \mathscr{L})$. Although ampleness is not preserved by pulling back through blowups, the pullback of a big and nef line bundle under blowup is still big and nef. This is actually good enough to get $H^i(Y, \omega_Y \otimes \pi^* \mathscr{L}) = 0$. ☐

Remark 2.5. The sheaf $\pi_* Y$ turns out to be independent of the choice of resolution $\pi \colon Y \to X$.

3. CHARACTERISTIC p

In characteristic $p > 0$, Kodaira vanishing can fail even for smooth surfaces.

Let X be projective over k with characteristic $p > 0$, and $\mathscr{L}$ be ample. Can we find a proper surjective map $\pi \colon Y \to X$ (not necessarily birational) such that

$$H^i(Y, \omega_Y \otimes \pi^* \mathscr{L}) = 0 \text{ for } i > 0?$$

Now we don't have resolution of singularities, and even if we did, they needn't have this property, since Kodaira vanishing can fail.

But we have a map that does not exist in characteristic 0, namely the Frobenius! We have

$$F \colon \mathcal{O}_X \to F_* \mathcal{O}_X$$

sending $z \mapsto z^p$. Iterating this, we get $\mathcal{O}_X \to F^e_* \mathcal{O}_X$. Noting that $Y = X$, we want

$$H^i(X, \omega_X \otimes F^{e*} \mathscr{L})$$

to vanish. But note that $F^{e*} \mathscr{L} = \mathscr{L}^{p^e}$. For large enough e, this will vanish by Serre's theorem (in fact this is a characterization of $\mathscr{L}$ being ample!).

Remark 3.1. Deligne observed that this trick could be used to give an algebraic proof of Kodaira vanishing *in characteristic* 0.

Definition 3.2. Let X be projective and smooth. Let L be ample, $z \in X$ and $\pi \colon Y \to X$ be the blowup of X at z. Let E be the exceptional divisor. We define

$$\epsilon(L, z) := \sup\{t > 0 \mid \pi^* L - tE \text{ ample}, t \in \mathbf{Q}\}.$$

If L is (very) ample, then $\pi^* L - tE$ is ample for small t. This is called the *Seshadri constant* at z.

You can think of $\epsilon(L, z)$ as giving some measure of "how positive L is at z".

Proposition 3.3. *If* $\epsilon(L, z) > \dim X$*, then* $\omega_X \otimes L$ *is globally generated at* z*.*

Proof. If $n = \dim X$, then $\pi^* L - nE$ is ample (hence big and nef). Consider the maps.

$$H^0(\mathfrak{m}_z \otimes \omega_X \otimes \mathscr{L}) \longrightarrow H^0(\omega_X \otimes L) \longrightarrow H^0(\omega_X \otimes \mathscr{L}/\mathfrak{m}_x).$$

We want to show that the rightmost map is surjective. The sheaf intervening looks kinda like what it's in Kodaira vanishing, except for the $\mathfrak{m}_z$. So we try blowing up:

$$\begin{tikzcd}
H^0(\mathfrak{m}_z \otimes \omega_X \otimes L) \arrow[r] & H^0(\omega_X \otimes L) \arrow[r] & H^0(\omega_X \otimes L/\mathfrak{m}_x) \\
H^0(Y, \omega_Y(\pi^*L - nE)) \arrow[r] \arrow[u] & H^0(Y, \omega_Y(\pi^*L - (n-1)E)) \arrow[r] \arrow[u] & H^0(E, \omega_E(\pi^*L - nE|_E)) \arrow[u, "\sim"]
\end{tikzcd}$$

Use $\omega_Y = \pi^*\omega_X + (n-1)E$ to see why you get the numbers above. Continuing the sequence, we would see $H^1(Y, \omega_Y(\pi^*L - nE))$ which vanishing in characteristic 0 by Kodaira.

In characteristic p, we aren't done yet because we cannot invoke Kodaira vanishing. Let's try to use Frobenius again. Dualizing $\mathcal{O}_Y \to F^e_*\mathcal{O}_Y$ we get $F^e_*\omega_Y \to \omega_Y$, and also $F^e_*\omega_Y(E) \to \omega_Y(E)$. Hence we have a diagram

$$\begin{tikzcd}
H^0(X, \mathfrak{m}_z \otimes \omega_X \otimes L) \arrow[r] & H^0(X, \omega_X \otimes L) \arrow[r] & H^0(X, \omega_X \otimes L/\mathfrak{m}_x) \\
H^0(Y, \omega_Y(\pi^*L - nE)) \arrow[r] \arrow[u] & H^0(Y, \omega_Y(\pi^*L - (n-1)E)) \arrow[r] \arrow[u] & H^0(E, \omega_E(\pi^*L - nE|_E)) \arrow[u, "\sim"] \\
H^0(Y, F^e_*\omega_Y(p^e\pi^*L - p^enE)) \arrow[r] \arrow[u] & H^0(Y, F^e_*\omega_Y(E + p\pi^*L - p^enE)) \arrow[r] \arrow[u] & H^0(E, \omega_E(p^e\pi^*L - p^enE|_E)) \arrow[u]
\end{tikzcd}$$

and now the obstruction $H^1(Y, F^e_*\omega_Y(p^e\pi^*L - p^enE)) = 0$ for large enough e by ampleness.

We'll show shortly that since E is a projective space,

$$H^0(E, \omega_E(p^e\pi^*L - p^enE|_E)) \twoheadrightarrow H^0(E, \omega_E(\pi^*L - nE|_E)).$$

This will complete the proof. ☐

4. Frobenius splitting

Definition 4.1. A variety X over a perfect field k of characteristic p is *Frobenius-split* (or "F-split") if the map $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits. (This implies that $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ splits for all e.)

Proposition 4.2. *Toric varieties (including* $\mathbf{P}^n$*) are Frobenius split.*

Proof. We'll give the argument for $X = \operatorname{Spec} k[x_1^{\pm}, \dots, x^{\pm 1}]$. The splitting $\varphi\colon F^e_*\mathcal{O}_X \to \mathcal{O}_X$ will take

$$x_1^{\lambda_1} \dots x_n^{\lambda_n} \mapsto x_1^{\lambda_1/p^e} \dots x_n^{\lambda_n/p^e}$$

where our convention is that $x_i^{\lambda_i/p^e} = 0$ if $\lambda_i/p^e \notin \mathbf{Z}$. ☐

It is clear that if X is Frobenius split and $\mathscr{L}$ is any line bundle on X, then

1. $H^i(X, \mathscr{L}) \hookrightarrow H^i(X, F^e_*\mathscr{L}^{p^e})$ is injective, and
2. $H^i(X, F^e_*(\omega_X \otimes \mathscr{L}^{p^e})) \to H^i(X, \omega_X \otimes \mathscr{L})$ is surjective.

Remark 4.3. If X is Frobenius-split, then Kodaira vanishing holds for X since

$$H^i(X, F^e_* \omega_X \otimes \mathscr{L}^{p^e}) \twoheadrightarrow H^i(X, \omega_X \otimes \mathscr{L}).$$

Example 4.4. For abelian varieties, "Frobenius split" is equivalent to "ordinary". Varieties of general type are never Frobenius split. Indeed, note that

$$\begin{aligned} \mathscr{H}om(F^e_* \mathcal{O}_X, \mathcal{O}_X) &\cong \mathscr{H}om(F^e_* \mathcal{O}_X(p^e K_X), \mathcal{O}_X(K_X)) \\ &\cong F^e_* \mathscr{H}om(\mathcal{O}_X(p^e K_X), \mathcal{O}_X(K_X)) \\ &\cong F^e_* \mathcal{O}_X((1-p^e)K_X). \end{aligned}$$

But if X is of general type, $\mathcal{O}_X((1-p^e)K_X)$ cannot have global sections for large e.