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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

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Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Emanuele Macri

Talk Title: Derived categories of cubic fourfolds and non-commutative K3 surfaces

Date: 2 / 8 / 19 **Time:** 11 : 00 [✓ am] / pm (circle one)

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MODULI OF OBJECTS IN KUZNETSOV COMPONENTS

EMANUELE MACRI

1. Kuznetsov components

This is joint work with Bayer, Lahoz, Nuer, Perry, Stellari.

Let X be smooth projective over $\mathbf{C}$. We look at $D^b(X) := D^b(\mathrm{Coh}(X))$. A *Kuznetsov component* is a certain admissible (triangulated) subcategory $\mathcal{D} \subset D^b(X)$. We ask for an adjoint p to the inclusion $\mathcal{D} \hookrightarrow D^b(X)$. The goal is to understand these categories by studying their moduli space of objects.

1.1. Motivation.

- The first motivation is speculative: we believe that these is related to rationality. More precisely, the $\mathcal{D}$ might provide an interesting birational invariant.
- Many geometric constructions on X can be realized as moduli spaces on $\mathcal{D}$.

Example 1.1. Let X be a surface. Then we have the following conjecture.

Conjecture 1.2 (Orlov)**.** *X is rational if and only if $D^b(X)$ has a "full exceptional collection", meaning it can be decomposed into pieces which are derived categories of points.*

Moduli spaces of objects in these components are not interesting in this case (being zero-dimensional), but the existence is fundamental in studying properties for all moduli spaces of sheaves on such surfaces.

Example 1.3. Let X be a Fano 3-fold, $\rho(X) = 1$, $i(X) = 2$.

Let $\mathcal{D}_X := \langle \mathcal{O}_X, \mathcal{O}_X(1) \rangle^{\perp}$, meaning the subcategory of objects C such that $\mathrm{Ext}^*(\mathcal{O}_X, C) = \mathrm{Ext}^*(\mathcal{O}_X(1), C) = 0$.

Then X is rational if and only if $\mathcal{D}_X$ is "geometric", i.e. $\mathcal{D}_X \cong D^b(\text{curve})$ or it has a full exceptional collection. (Unfortunately, at the moment the proof is indirect. You find two lists of objects satisfying the respective properties, and verify that they coincide.)

Given $\ell \subset X$, with ideal sheaf $\mathscr{I}_\ell$, associate $p(\mathscr{I}_\ell) \in \mathcal{D}_X$ where p is the projection left adjoint to $\mathcal{D}_X \hookrightarrow \mathcal{D}^b(X)$. This shows that the moduli space of lines is already "a moduli space in $\mathcal{D}_X$", which means $\mathcal{D}_X$ is uninteresting for this purpose.

Let X be a cubic 3-fold. Then from a moduli perspective $p(\kappa(x))$ (the projection of the skyscraper sheaf) looks better once you project to $\mathcal{D}_X$. It is the resolution of the blowup of the theta divisor.

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Example 1.4. Let X be a cubic 4-fold, and define

$$\mathcal{D}_X = \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle^{\perp}.$$

The expectation is that rationality of X should have something to do with $\mathcal{D}_X$ be geometric, and more precisely that $\mathcal{D}_X \cong D^b(\mathrm{K3})$.

The moduli spaces are interesting. For a line $\ell \subset X$, $p(\mathscr{I}_\ell)$ has the same deformation space as the moduli space (i.e. Hilbert scheme) of lines.

On the other hand, the moduli space of $\kappa(x)$ has more deformations – the associated moduli space contains X.

2. Moduli spaces in $\mathcal{D}_X$

2.1. Cohomology.

Example 2.1. Let X be a cubic 4-fold. We form the Kuznetsov component $\mathcal{D}_X$ as before. Properties:

1. Serre duality: for all $E, F \in \mathcal{D}_X$, we have $\mathrm{Ext}^*(E,F) = \mathrm{Ext}^{2-*}(F,E)^\vee$. (This suggests that $\mathcal{D}_X$ is "smooth".)
2. $H(\mathcal{D}_X; \mathbf{Z}) = K_{\mathrm{top}}(\mathcal{D}_X)$. This is a lattice $U^4 \oplus E_8(-1)^2$, which looks like $H^*(\mathrm{K3}; \mathbf{Z})$. It has a weight 2 Hodge structure (e.g. from Hochschild cohomology).
3. There is a Mukai vector $v\colon K(\mathcal{D}_X) \to H_{\mathrm{Hdg}}(\mathcal{D}_X; \mathbf{Z})$ given by "$v = \mathrm{ch}\cdot\mathrm{Td}$". We have $\chi(E,F) = -(v(E), v(F))$ where χ is the Euler characteristic.

In general, the structure we fix is $\mathcal{D}_X \subset D^b(X)$ with left adjoint p, and $v\colon K_0(\mathcal{D}_X) \to \Lambda$ where Λ is a finite rank abelian group.

2.2. Stability.

We need an abstract notion of stability in derived categories in order to define moduli spaces.

Example 2.2. Let $X = Q_1 \cap Q_2 \subset \mathbf{P}^5$. Then we get $\mathcal{D}_X \cong D^b(C)$ where C is a genus 2 curve. In this case we want "stability" to mean Mumford stability for coherent sheaves on C.

Definition 2.3 (Bridgeland, Kontsevich-Soibelman)**.** A *stability condition* on $\mathcal{D}_X$ is a pair $\sigma = (Z, \mathcal{A})$ where

- $Z\colon \Lambda \to \mathbf{C}$,
- $\mathcal{A}$ is the heart of a t-structure

such that

1. $Z(v(\mathcal{A} - 0)) \subset \mathbf{H} \cup \mathbf{R}_{<0}$. We define $\mu_\sigma = -\frac{\mathrm{Re}\ z}{\mathrm{Im}\ z}$ to be the "slope". We say $E \in \mathcal{A}$ is *σ-semistable* if for all $F \subset E$, $\mu(F) \leq \mu(E)$.
2. Harder-Narasimhan filtrations exist.
3. We want a kind of discreteness of semistable objects. To codify this, we ask for a quadratic form Q on $\Lambda_{\mathbf{R}}$ such that $Q|_{\ker Z} < 0$.
4. Semistable objects of fixed Mukai vector v_0 satisfy universal openness and boundedness.

Example 2.4. In the previous example, $\mathcal{A} = \mathrm{Coh}(C)$ and $Z = -\deg + i\,\mathrm{rank}$.

2.3. **Existence of moduli spaces.**

Theorem 2.5.

1. *(Bridgeland) The set of stability conditions* $\mathrm{Stab}(\mathcal{D}_X)$ *has the structure of a complex manifold with dimension being* $\operatorname{rank}\Lambda$.
2. *Fix* $\sigma \in \mathrm{Stab}(\mathcal{D}_X)$ *and* $v_0 \in \Lambda$. *Then* $M_\sigma(v_0)$ *exists as an Artin stack of finite type over* $\mathbf{C}$ *[Lieblich-Toda]. It has a good moduli space [Alper, Halpern-Leistner, Heinloth] which is a proper and separated algebraic space [Abramovich-Polishchuk].*

   There is a canonical line bundle $\ell_\sigma(v_0)$ *on this good moduli space, which is a strictly nef real divisor class [BM].*
3. *(BLMS) If* X *is a Fano 3-fold or cubic 4-fold, then there is a Kuznetsov component* $\mathcal{D}_X$ *with* $\mathrm{Stab}(\mathcal{D}_X) \neq \emptyset$.

3. Relative stability

We now want to extend this notion to families. Issues: we want to be able to do semistable reduction, we want to deform, and we want the moduli space to be proper.

Consider a smooth projective family $\mathcal{X} \to S$. We will want a subcategory $\mathcal{D}_{\mathcal{X}/S} \subset D^b(\mathcal{X})$ which is S-linear, meaning it's preserved by $D_{\mathrm{perf}}(S)$.

We want to have a notion of stability varying in fibers. This is subtle, but we finally found a definition that seems to work. To begin, we want a family of functions $v_s\colon K_{num}(\mathcal{D}_S) \to \Lambda$ for $s \in S$, which are constant over S.

Definition 3.1 (BLMNPS)**.** A collection $\underline{\sigma} = \{\sigma_s = (Z_s, \mathcal{A}_s)\}_{s\in S}$ of stability conditions is a *stability condition on* $\mathcal{D}_{\mathcal{X}/S}$ *over* S if

1. Z_S is locally constant, hence we get $Z\colon \Lambda \to \mathbf{C}$.
2. Universal open-ness of stability.
3. For all smooth curves $C \to S$, the collection $\{\sigma_c\}$ "integrates" over C. This means we want a global heart, and for the stability conditions to be induced by global HN filtrations and stability on this heart.
4. A uniform Q.
5. Boundedness.

Theorem 3.2.

1. $\mathrm{Stab}\,\mathcal{D}_{\mathcal{X}/S}$ *is a manifold.*
2. $M_{\underline{\sigma}}(\underline{v_0}) \to S$ *exists and has a good moduli space which is proper and separated over* S.
3. *If* $\mathcal{X} \to S$ *is a smooth family of Fano 3-folds or cubic 3-folds, then the Kuznetsov component* $\mathcal{D}_{\mathcal{X}/S}$ *has* $\mathrm{Stab}(\mathcal{D}_{\mathcal{X}/S}) \neq \emptyset$.

Remark 3.3. Λ is "not constant". Assume $\underline{v}$ factors through $\mathcal{N}(\mathcal{D}_{\mathcal{X}/S})$. Then $\ell_{\underline{\sigma}}(\underline{v})$ exists and is strictly S-nef.

4. Applications

Let X be a cubic 4-fold. Then we have defined $\mathcal{D}_X$.

Corollary 4.1. *Fix a primitive vector* $v_0 \in H_{\mathrm{Hdg}}(\mathcal{D}_X; \mathbf{Z})$*. Let* $\sigma \in \mathrm{Stab}(\mathcal{D}_X)$ *is generic with respect to* v_0*. Then* $M_\sigma(v_0) \neq \emptyset$ *if and only if* $v_0^2 \geq -2$*. In such a case* $M_\sigma(v_0)$ *is smooth projective deformation equivalent to the Hilbert scheme of a K3 surface of dimension* $v_0^2 + 2$ *with* $\ell_\sigma(v_0)$ *ample.*

We consider a relative situation over a curve. We need to prove the relative moduli space is smooth. If it's not smooth, one needs to "increase" Λ. This lets one deform to the case of the K3 surface, which we know.