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Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Janos Kollar

Talk Title: Moduli of canonical models

Date: 1 / 31 / 19 **Time:** 11 : 00 (am) / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Introduction to Moduli spaces

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Moduli spaces of algebraic varieties I

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Princeton University

January, 2019

Main question, old style

– Can we parametrize all varieties in a natural way?

Main questions, new style

– What is a "good family" of algebraic varieties?
– Can we describe all "good families" in an "optimal" manner?

Moduli of curves, analytic theory I

Riemann (1857), *Theorie der Abel'schen Funktionen*

Riemann surfaces of genus g depend on $3g - 3$ parameters

Fricke–Klein (1897-1912), *Vorlesungen über die Theorie der automorphen Funktionen*, (1300 pp.)

T_g exists and is contractible:

$T_g =$ discrete, cocompact representations $\pi_1(C) \to \mathrm{PGL}_2(\mathbb{R}) = \mathrm{Aut}(\text{unit disc})$, modulo conjugation

complex structure not natural, not considered much

Siegel (1935), construction of A_g as analytic space

very precise, modern feel, mostly arithmetic

Moduli of curves, analytic theory II

Teichmüller (1940–44), complete theory of T_g

complex structure + functorial aspects

Weil (1958), Bourbaki seminar: "As for M_g there is virtually no doubt that it can be provided with the structure of an algebraic variety"

Grothendieck (1960), Cartan Seminar,
T_g represents a functor (based on Teichmüller?)
projective families over analytic bases

worth reading: A'Campo-Ji-Papadopoulos:
On the early history of moduli and Teichmüller spaces

Moduli of curves, algebraic theory I

Cayley (1862), *A new analytic representation of curves in space*, Constructs moduli of space curves.

$C \mapsto$ (all lines meeting C)

General theory: van der Waerden, Chow, Hodge-Pedoe

Hilbert (1890), *Über die Theorie der algebraischen Formen*, finite generation of rings of invariants

("Theologie" according to Gordan)

BUT: nobody seems to have taken its Proj

Hurwitz (1891), *Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, M_g is irreducible

Moduli of curves, algebraic theory II

Severi (1915), *Sulla classificazione delle curve algebriche e sul teorema d'esistenza di Riemann*,

M_g unirational for $g \leq 10$

existence? not clear what he thinks

uses the word "Mannigfaltigkeiten" (after Riemann) not "varietà"

Claim: there is a family over a rational variety that gives almost all curves of a fixed genus.

Weil, Matsusaka (1946–56) field of definition/field of moduli

M_g, A_g should be defined over $\mathbb{Z}$, so

$k_C :=$ residue field of $[C] \in M_g$.

Aim: finding k_C from C (without knowing M_g).

Moduli of curves, algebraic theory II

Satake (1956-60), Baily-Borel (1966), Compactifying A_g,
not yet as a moduli space: points at infinity are
lower dimensional Abelian varieties

may have been a serious stumbling block

solved by V. Alexeev

Example – Hypersurfaces

– $X_d \subset \mathbb{P}^n$ of degree d.
– Equation: $\sum_I a_I x^I = 0$ where
$I = (i_0, \dots, i_n)$ and $i_0 + \cdots + i_n = d$.

Classical claim. All degree d hypersurfaces in $\mathbb{P}^n$ "naturally" form a projective space $\mathbb{P}^N$ where $N = \binom{n+d}{n} - 1$:

$$X_d = \left(\sum\nolimits_I a_I x^I = 0\right) \leftrightarrow \{a_I\}.$$

– works over any field
– counts multiplicities
– similar: hypersurface sections of any $Y^n \subset \mathbb{P}^M$.

Hypersurfaces with coordinate changes

Claim. Let $X_i \subset \mathbb{P}^n$ be hypersurfaces and $\phi : X_1 \cong X_2$ an isomorphism. Then ϕ extends to a linear coordinate change $\Phi : \mathbb{P}^n \cong \mathbb{P}^n$, except possibly in the following cases

- $n = 1$
- $n = 2$ and $\deg X_i \leq 3$ (Castelnuovo, Serrano)
- $n = 3$ and $\deg X_i = 4$ (needs Lefschetz)

Determinantal examples

$W \subset \mathbf{P}^n_x \times \mathbf{P}^n_y$: intersection of $n+1$ bidegree $(1,1)$:
$\sum_{ijk} a^k_{ij} x_i y_j = 0$.

Projections:
$W_x = \left(\det(\sum_i a^k_{ij} x_i) = 0\right) \subset \mathbf{P}^n_x$ and
$W_y = \left(\det(\sum_j a^k_{ij} y_j) = 0\right) \subset \mathbf{P}^n_y$.

Oguiso: For $n = 3$ we get smooth degree 4 surfaces,
that are not even Cremona equivalent.

One should study:

$$\mathrm{Hyp}_{d,n} := \{\text{Hypersurfaces of degree } d \text{ in } \mathbb{P}^n\}/\mathrm{PGL}_{n+1}.$$

$\mathrm{Hyp}_{d,n}$ is a horrible space

Closure of a subset $U \subset \mathrm{Hyp}_{d,n}$:

given $X_t := \big(F(x_0, \ldots, x_n; t) = 0\big)$,
if $[X_t] \in U$ for $t \neq 0$ then $[X_0] \in \bar{U}$.

Fix $X := F(x_0, \ldots, x_n)$ and let
$F(x, t) := F(x_0, \ldots, x_r, tx_{r+1}, \ldots, tx_n)$.

- $X_t \cong X$ for $t \neq 0$ and
- $X_0 = F(x_0, \ldots, x_r, 0, \ldots, 0)$.

Corollary. $[(x_0^d = 0)]$ is the only closed point of $\mathrm{Hyp}_{d,n}$.

Trying to fix it

- $\mathrm{Hyp}_{d,n}^{\mathrm{reduced}}$
  only closed points are $[F(x_0, x_1, 0, \ldots, 0) = 0]$.
- $\mathrm{Hyp}_{d,n}^{\mathrm{normal}}$
  only closed points are $[F(x_0, x_1, x_2, 0, \ldots, 0) = 0]$.

these are: cones with large singular set.

- $\mathrm{Hyp}_{d,n}^{\mathrm{isolated,non-cone}}$ example:

$X_t := (x_0^{d/2} + t^{d/2} x_1^{d/2}) x_1^{d/2} + x_2^d + \cdots x_n^d$

– $X_t \cong X_1$ if $t \neq 0$ (apply $(x_0, x_1) \mapsto (tx_0, t^{-1} x_1)$)
  1 isolated singularity

– X_0: 2 isolated singularities of multiplicity $d/2$.

Mori example

Consider $\deg G(x) = d, \deg F(x) = de$, $\deg z = d$ and

$$X_t := \left(z^e - F(x) = G(x) - tz = 0\right).$$

- for $t \neq 0$: X_t smooth hypersurface of degree de

$$X_t := \left(G^e(x) - t^e F(x) = 0\right).$$

- for $t = 0$: X_0 is not a hypersurface but a degree e cover of $(G(x) = 0)$ ramified along $(F(x) = 0)$.

Question. Any prime degree examples for $\dim \geq 3$?
Ottem-Schreieder: no for degrees 5 and 7.

Enter Mumford

"When [Zariski] spoke the words algebraic variety, there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too ... Especially, I became obsessed with a kind of passion flower in this garden, the moduli spaces of Riemann."

Nomen est omen

Providential notation: M_g short for Mumford_g

Structure: *DM*-stack David Mumford-stack*

∗ *apologies to Deligne*

Mumford's main ideas

Geometric invariant theory

Construction of M_g: Take 5-canonical embedding $C \to \mathbb{P}^{9g-10}$ and then quotient by PGL_{9g-9}.

Stable curves (with Deligne) Compactifying by stable curves gives a very nice $\bar{M}_g$.

Cohomology of M_g: This is a non-linear analog of Grassmannians. So $H^*(M_\infty, \mathbb{Z})$ governs many enumerative questions.

Geometric Invariant Theory of Hypersurfaces

There is a notion of stability.

- $\mathrm{Hyp}_{d,n}^{\mathrm{stable}}$ is as nice as possible:
  noncompact, nearly smooth algebraic variety, and
- $\mathrm{Hyp}_{d,n}^{\mathrm{semistable}}$ is less nice but
  compact algebraic variety.

Good property: smooth $\Rightarrow$ stable.

Bad properties:

– no idea what else is stable if $d \geq 4$
– semi-stable points correspond to
many different hypersurfaces.

Genus 2 curves or $\mathrm{Hyp}_{6,1}$

- C: smooth, projective curve of genus 2
- C: smooth, compact Riemann surface of genus 2

Structure theorem. There is a unique $\tau : C \to \mathbb{P}^1$ of degree 2 ramified at 6 points.

Affine equation: $z^2 = f_6(x)$ (no multiple roots)

Corollary. M_2, the set/space of all smooth, projective curves of genus 2 is

– $\{6 \text{ points in } \mathbb{P}^1\}/\mathrm{PGL}_2$, equivalently

– $(\mathrm{Sym}^6\mathbb{P}^1 \setminus \text{diagonals})/\mathrm{PGL}_2$.

Compactifying M_2

Typical example: 4-fold root for $t = 0$:
$f_6(x{:}y, t) = (x - ta_1 y) \cdots (x - ta_4 y)(x - a_5 y)(x - a_6 y)$
Coordinate change $x = tx', y = y'$ and dividing by t^4:
$(x' - a_1 y') \cdots (x' - a_4 y')(tx' - a_5 y')(tx' - a_6 y')$
which has ony 2-fold root.

Lemma. Same trick achieves: at most triple root at $t = 0$.

Triple root case: $x^3(x - a_4 y)(x - a_5 y)(x - a_6 y)$.
$x = tx', y = y'$ and dividing by $t^3 a_4 a_5 a_6$ we get
$(x')^3(\frac{t}{a_4}x' - y')(\frac{t}{a_5}x' - y')(\frac{t}{a_6}x' - y')$.
For $t = 0$ this becomes
$(x')^3(y')^3$: two triple roots.

GIT compactification $\bar{M}_2^{\mathrm{GIT}}$

Points correspond to:
•: two triple roots (unique point) and
•: at most double roots.

Corresponding curves:
•: $z^2 = x^3(x-1)^3$ rational with 2 cusps.
•: at most double roots $z^2 = f_6(x)$.
Irreducible with at most nodes, except:

- $z^2 = x^2(x-1)^2(x+1)^2$. Contract one of the components: rational with 1 triple point like the 3 coordinate axes.

End of old style story.

$\bar{M}_2^{\mathrm{GIT}}$ is an unpleasant compactification.

- Stacky problem at $z^2 = x^2(x-1)^2(x+1)^2$.
- Local universal families:

At 2 cusp point $z^2 = x^3(x-1)^3$, deformations are

$z^2 = \left(x^3 + ux + v\right)\left((x-1)^3 + s(x-1) + t\right)$.

Problem: $(u = v = 0)$ or $(s = t = 0)$ define disallowed curves.

Deligne–Mumford compactification $\bar{M}_2$

- at most double roots $z^2 = f_6(x)$: keep these
- $z^2 = x^2(x-1)^2(x+1)^2$: keep as is.
- $z^2 = x^3(x-1)^3$ change to:

double cover of pair of intersecting lines,
ramified at 3+3 pts plus the node:
= two elliptic curves meeting at a point.

Source of triple root problem: 3 choices

- contract one elliptic curve
- contract other elliptic curve
- blow up intersection point and contract both