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Talk Title: DAG III: examples

Date: 2 / 4 / 19 **Time:** 11 : 30 [✓ am] / pm (circle one)

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DAG III: EXAMPLES

BEN ANTIEAU

1. Picard stacks

Let $\underline{\mathrm{Pic}} = B\,\mathrm{GL}_1$ and $\underline{\mathrm{Br}} = B^2\,\mathrm{GL}_1$. (In other language, $\underline{\mathrm{Pic}}$ classifies GL_1-torsors and $\underline{\mathrm{Br}}$ classifies GL_1-*gerbes.*)

Remark 1.1. For a discrete ring R, $\underline{\mathrm{Br}}(R)$ is valued in 2-truncated spaces $\mathcal{S}_{\leq 2}$. This shows that it cannot be represented by an Artin stack, as Artin stacks are valued in groupoids.

Example 1.2. Let $\mathcal{X} = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}(-4)[1])$. Let's try to compute $\underline{\mathrm{Br}}(\mathcal{X})$. There is a hypercohomology spectral sequence that can be used to understand this.

Definition 1.3. Let $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(\mathrm{dAff}_k)$. We define the kth *étale homotopy sheaf* $\pi_k^{\text{ét}}(\mathcal{F})$ of X to be the étale sheafification of $U \hookrightarrow \pi_k(\mathcal{F}(U))$.

In these terms, we have

$$\pi_i^{\text{ét}}(\underline{\mathrm{Br}}) \cong \begin{cases} 0 & i > 0, 1 \\ \mathbf{G}_m & i = 2 \\ \pi_{i-2}(\mathcal{O}) & i \geq 3 \end{cases}$$

Here by $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$ we mean $\mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}(-4)[1]$, viewed as a ring by the square-zero extension structure. By definition, $\pi_0\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{\mathbf{P}^3}$ and $\pi_1\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{\mathbf{P}^3}(-4)$.

There is a spectral sequence

$$E_2^{st} = H^s_{\text{ét}}(\mathcal{X}, \pi_t^{\text{ét}}\underline{\mathrm{Br}}) \implies \pi_{t-s}\underline{\mathrm{Br}}(\mathcal{X})$$

The differentials have bidegree $|d_r| = (r, r-1)$, hence the spectral sequence collapses on E_2 in our case.

For $t = 3$, we get

$$\begin{aligned} H^0(\mathbf{P}^3, \mathcal{O}(-4)) &= 0, \\ H^1(\mathbf{P}^3, \mathcal{O}(-4)) &= 0 \\ H^1(\mathbf{P}^3, \mathcal{O}(-4)) &= 0, \\ H^3(\mathbf{P}^3, \mathcal{O}(-4)) &\approx k. \end{aligned}$$

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For $t = 2$, we get

$$\begin{aligned} H^0(\mathbf{P}^3, \mathbf{G}_m) &\cong k^\times \\ H^1(\mathbf{P}^3, \mathbf{G}_m) &\cong \mathbf{Z}, \\ H^2(\mathbf{P}^3, \mathbf{G}_m) &\cong \mathrm{Br}(k) \end{aligned}$$

Hence we get an exact sequence

$$0 \to \underbrace{k}_{H^3(\mathbf{P}^3, \mathcal{O}(-4))} \to \pi_0 \underline{\mathrm{Br}}(\mathcal{X}) \to \mathrm{Br}(k) \to 0$$

and we similarly see $\pi_1 \underline{\mathrm{Br}}(\mathcal{X}) \cong \mathrm{Pic}(\mathbf{P}^3) \cong \mathbf{Z}$, $\pi_2(\underline{\mathrm{Br}}(\mathcal{X})) \cong k^\times$.

Theorem 1.4 (Toën). *Every class in $\pi_0 \operatorname{Br}(X)$ is represented by a derived Azumaya algebra.*

2. Moduli of objects

Let k be a commutative ring and A be a dg algebra over k.

Definition 2.1. We say A is *hfp* ("homotopically finitely presented") if $A \in \mathrm{Alg}_k^\omega$ (the $^\omega$ notation means categorically compact objects).

Example 2.2. If P is a perfect complex over k, then its tensor algebra $T(P)$ is hfp.

Example 2.3. Let X/k be smooth and projective. Then $D_{qc}(X) \cong D(A)$ where A is hfp.

Definition 2.4. For $R \in \mathrm{sCAlg}_k$, let $\mathcal{M}_A(R)$ be the category of representations of A in $\mathrm{Perf}(R)$, i.e. $\mathrm{Fun}^{\mathrm{exact}}(\mathrm{Perf}(A), \mathrm{Perf}(R))$. (Here all categories are regarded as stable ∞-categories.)

Remark 2.5. Since $\mathrm{Perf}(A)$ is generated by A, a functor in $\mathrm{Fun}^{\mathrm{exact}}(\mathrm{Perf}(A), \mathrm{Perf}(R))$ can be identified with $P \in \mathrm{Perf}(R)$ which has an A-action.

Example 2.6. Let X/k be smooth and proper. Then $\mathcal{M}_X \cong \mathcal{M}_A$ where A is such that $D_{qc}(X) \cong \mathcal{D}(A)$. We have $\mathcal{M}_X(R) = \mathrm{Perf}(X_R)$.

Definition 2.7. For any category $\mathcal{C}$, let $\iota\mathcal{C} \subset \mathcal{C}$ be the maximal subgroupoid. Let $M_A = \iota\mathcal{M}_A$.

Now, $\pi_0 M_R$ is equivalent to the category of isomorphic classes of objects in the triangulated category $\mathrm{Ho}(M_A(R))$.

For $P \in \pi_0(M_R)$, a perfect R-module with an action of A, we have

$$\pi_i(M_A(R), P) \cong \begin{cases} \mathrm{Hom}_{\mathrm{Ho}(M_A(R))}(P, P)^\times & i = 1 \\ \mathrm{Ext}^{1-i}_{\mathrm{Ho}(M_A(R))}(P, P) & i \geq 2 \end{cases}$$

Theorem 2.8 (Toen-Vaquié). *Let A be hfp over k.*

(1) M_A is locally geometric and lfp.

(2) For $\operatorname{Spec} R \xrightarrow{P} M_A$*, we have*

$$P^* \mathbf{L}_{M_A} \cong R\operatorname{Hom}(P, P)^\vee[-1].$$

Proof. We have a map $M_A \to M_k$ that forgets the A-action. This is representable, so for the purpose of the proof we can replace A by k.

Let $M_k^{[a,b]}(R) \subset M_k(R)$ be the subcategory of perfect R-modules with Tor-amplitude in the range $[a, b]$. It is true that

$$\varinjlim M_k^{[a,b]} \cong M_k$$

(ultimately coming from quasicompactness of Spec R).

(1) Next, we use that

$$M_k^{[0,0]} = \coprod_{n \geq 0} \mathrm{BGL}_n \,.$$

(2) Assume that $M_k^{[a,b]}$ is geometric for every $b - a \leq n - 1$, for $n \geq 1$. We want to understand $Q \in M_k^{[a,b]}$. We can make a triangle

$$C \to P \to Q \to C[1]$$

where $C \in [0, n-1]$, $P \in [0, 0]$ surjects onto Q in H_0. This lets us induct.

(3) Consider the diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \iota\mathrm{Fun}(\Delta^1, M_k) \arrow[d] \\
M_k^{[0,n-1]} \times M_k^{[0,0]} \arrow[r] & \iota\,\mathrm{Fun}(\partial\Delta^1, M_k) \cong M_k \times M_k
\end{tikzcd}$$

What is the fiber of U over a pair $(C, P) \in M_k^{[0,n-1]} \times M_k^{[0,0]}$? We need to give a morphism $C \to P$, which is a point of $C^\vee \otimes P$, so the fiber is Spec $\mathrm{Sym}(C^\vee \otimes P)$. So the morphism $U \to M_k^{[0,n-1]} \times M_k^{[0,0]}$ is geometric, hence (by induction) U is geometric. We claim that the map $U \to M_k^{[0,n]}$ (obtained by gluing C and P) is smooth and surjective.

Rather than give the details, we point out an example: if $P = k, C = k[n-1]$, then $U = B^{n-1}\mathbf{G}_a$.

□

3. Derived Picard stack

Let X/k be smooth and proper, geometrically connected, and d-dimensional.

Let $\underline{\mathrm{dPic}_X}(R) \subset M_X(R)$ be the subcategory of invertible objects in $\mathrm{Perf}(X_R)$. Exercise: show that this is open. Our formalism implies that it is geometric.

$\mathbf{L}_{\underline{\mathrm{dPic}_X}, \mathscr{L}[d]} \cong R\Gamma(X, \mathcal{O}_X)^\vee[-1]$ is concentrated in degrees $[-1, d-1]$, which tells us that it is represented by a derived Artin stack.

The adjunction $\mathrm{sCAlg}_k \leftrightarrow \mathrm{CAlg}_k$ induces

$$\begin{aligned}
\mathrm{Shv}_{\text{ét}}(\mathrm{dAff}_k) &\leftarrow \mathrm{Shv}_{\text{ét}}(\mathrm{Aff}_k) \colon cl_* \\
\mathrm{Shv}_{\text{ét}}(\mathrm{dAff}_k) &\leftarrow \mathrm{Shv}_{\text{ét}}(\mathrm{Aff}_k) \colon cl_! \\
cl^* \colon \mathrm{Shv}_{\text{ét}}(\mathrm{dAff}_k) &\to \mathrm{Shv}_{\text{ét}}(\mathrm{Aff}_k)
\end{aligned}$$

For example, for $R \in \mathrm{sCAlg}_k$ and $S \in \mathrm{sCAlg}_k$, we have

(i) $cl^* \operatorname{Spec} R \cong \operatorname{Spec} \pi_0 R$,
(ii) $cl_! \operatorname{Spec} S = \operatorname{Spec} S$, and
(iii) $cl_* \operatorname{Spec} S(R) = \operatorname{Hom}(S, \pi_0(R))$.

We have $cl^* \mathrm{dPic}_X \cong \coprod_{n \in \mathbf{Z}} \mathrm{Pic}_{X/k}$, where n tracks the homological degree. The map to $\pi_0^{\text{ét}}$ looks like the disjoint union of the classical Picard schemes. The fiber of $cl^* \mathrm{dPic}_X \to \pi_0^{\text{ét}} cl^* \mathrm{dPic}_X$ is $B\mathbf{G}_m$.

What about the derived picture? The map dPic_X to $\pi_0^{\text{ét}} \mathrm{dPic}_X$ has fiber $\mathrm{BGL}_{1, R\Gamma(X, \mathcal{O}_X)}$. This is the sheafification of the presheaf $R \mapsto \mathrm{BGL}_1(R\Gamma(X, \mathcal{O}_X) \otimes^L R)$.

What we want is more like $\mathrm{dPic}_X \to \mathrm{dPic}_X / \mathrm{BGL}_1$ whose fiber is BGL_1.