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Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Tony Pantev

Talk Title: Shifted symplectic structures and applications

Date: 2 / 7 / 19 **Time:** 11 : 45 [✓ am] / pm (circle one)

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Shifted symplectic structures and applications

Tony Pantev

University of Pennsylvania

Introductory workshop
'Derived algebraic geometry and
Birational geometry of moduli spaces'
MSRI, January 2019

Outline

- based on joint works with D.Calaque, L.Katzarkov, B.Toën, G.Vezzosi, M.Vaquié
- shifted symplectic geometry
- derived Darboux theorems
- applications

Symplectic structures

Recall: For X a smooth scheme$/\mathbb{C}$ is a **symplectic structure** is an $\omega \in H^0(X, \Omega_X^{2,cl})$ such that its adjoint $\omega^\flat : T_X \to \Omega_X^1$ is a sheaf isomorphism.

Note: Does not work for X singular (or stacky or derived):

- T_X and Ω_X^1 are too crude as invariants and get promoted to complexes $\mathbb{T}_X$ and $\mathbb{L}_X$. **Details**
- A form being closed is not just a condition but rather an extra structure. **Details**

Definition: X derived Artin stack locally of finite presentation (so that $\mathbb{L}_X$ is perfect).

- A n-shifted 2-form $\omega : \mathcal{O}_X \to \mathbb{L}_X \wedge \mathbb{L}_X[n]$ - i.e. $\omega \in \pi_0(\mathcal{A}^2(X; n))$ - is nondegenerate if its adjoint $\omega^\flat : \mathbb{T}_X \to \mathbb{L}_X[n]$ is an isomorphism (in $D_{qcoh}(X)$).
- The space of n-shifted symplectic forms $Sympl(X; n)$ on $X/\mathbb{C}$ is the subspace of $\mathcal{A}^{2,cl}(X; n)$ of closed 2-forms whose underlying 2-forms are nondegenerate i.e. we have a homotopy cartesian diagram of spaces

$$\begin{tikzcd}
Sympl(X, n) \arrow[r] \arrow[d] & \mathcal{A}^{2,cl}(X, n) \arrow[d] \\
\mathcal{A}^2(X, n)^{nd} \arrow[r] & \mathcal{A}^2(X, n)
\end{tikzcd}$$

Shifted symplectic structures: examples (i)

- Nondegeneracy: a duality between the stacky (positive degrees) and the derived (negative degrees) parts of $\mathbb{L}_X$.
- $G = GL_n \rightsquigarrow BG$ has a canonical 2-shifted symplectic form whose underlying 2-shifted 2-form is

  $$k \to (\mathbb{L}_{BG} \wedge \mathbb{L}_{BG})[2] \simeq (\mathfrak{g}^\vee[-1] \wedge \mathfrak{g}^\vee[-1])[2] = Sym^2 \mathfrak{g}^\vee$$

  given by the dual of the trace map $(A, B) \mapsto tr(AB)$.
- Same as above (with a choice of G-invariant symm bil form on $\mathfrak{g}$) for G reductive over k.
- The n-shifted cotangent bundle $T^\vee X[n] := \mathrm{Spec}_X(\mathrm{Sym}(\mathbb{T}_X[-n]))$ has a canonical n-shifted symplectic form.

Shifted symplectic structures: examples (ii)

Theorem: [PTVV] Let F be a derived Artin stack and let $\omega \in Symp(F, n)$. Suppose X is $\mathcal{O}$-compact and equipped with an $\mathcal{O}$-orientation $[X] : \mathbb{H}(X, \mathcal{O}_X) \longrightarrow \mathbb{C}[-d]$ of dimension d. If the derived mapping stack $MAP(X, F)$ is a derived Artin stack locally of finite presentation over $\mathbb{C}$, then, $MAP(X, F)$ carries a canonical $(n - d)$-shifted symplectic structure.

Remark:

0) Analog to Alexandrov-Kontsevich-Schwarz-Zaboronsky result.
1) A d-dimensional $\mathcal{O}$-orientation on X is a variant of a Calabi-Yau structure of dimension d;
2) A compact oriented topological d-manifold has an $\mathcal{O}$-orientation of dimension d (Poincaré duality).

Lagrangian structures

Let (Y, ω) be a n-shifted symplectic derived stack. A lagrangian structure on a map $f : X \to Y$ is a

- path γ in $\mathcal{A}^{2,\mathrm{cl}}(X; n)$ from $f^*\omega$ to 0 (isotropic structure),
- which is non-degenerate, i.e. the induced map $\theta_\gamma : \mathbb{T}_f \to \mathbb{L}_X[n-1]$ is an equivalence.

Examples:

- usual smooth lagrangians $L \hookrightarrow (Y, \omega)$ where (Y, ω) is a smooth (0)-symplectic scheme.
- there is a bijection between lagrangian structures on the canonical map $X \to (\mathrm{Spec}\,\mathbb{C}, \omega_{n+1})$ and n-shifted symplectic structures on X (thus lagrangian structures generalize shifted symplectic structures)

Shifted symplectic structures: examples (iii)

Theorem: [PTVV] Let (F, ω) be n-shifted symplectic derived Artin stack, and $L_i \to F$ a map of derived stacks equipped with a Lagrangian structure, $i = 1, 2$. Then the homotopy fiber product $L_1 \times_F L_2$ is canonically a $(n-1)$-shifted derived Artin stack.

In particular, if $F = Y$ is a smooth symplectic Deligne-Mumford stack (e.g. a smooth symplectic variety), and $L_i \hookrightarrow Y$ is a smooth closed lagrangian substack, $i = 1, 2$, then the derived intersection $L_1 \times_F L_2$ is canonically (-1)-shifted symplectic.

Remark: An important special case is the derived critical locus $\mathbb{R}Crit(f)$ for f a global function on a smooth symplectic Deligne-Mumford stack Y. Here

$$\begin{tikzcd}
\mathbb{R}Crit(f) \arrow[r] \arrow[d] & Y \arrow[d, "df"] \\
Y \arrow[r, "0"'] & T^{\vee}Y
\end{tikzcd}$$

Local models (i)

Recall: In classical symplectic geometry the local structure of a symplectic manifold is described by the **Darboux theorem:**

Local models (i)

Recall: In classical symplectic geometry the local structure of a symplectic manifold is described by the **Darboux theorem:** *a symplectic structure is locally (in the C^∞ or analytic setting) or formally (in the algebraic setting) isomorphic to the standard symplectic structure on a cotangent bundle.*

Local models (i)

In the derived and stacky setting there are two natural incarnations of an n-shifted symplectic cotangent bundle:

(a) The shifted cotangent bundle $T_M^\vee[n] = \mathbb{R}\mathbf{Spec}_{/M}\left(\mathrm{Sym}^\bullet_{\mathcal{O}_M}\left(T_M[-n]\right)\right)$, equipped with n-th shift of the standard symplectic form;

(b) The derived critical locus $\mathbf{Rcrit}(\mathbf{w})$ of an $n+1$ shifted function $\mathbf{w} : M \to \mathbb{A}^1[n+1]$, equipped with the inherited n-shifted symplectic form $\omega_{\mathbf{Rcrit}(\mathbf{w})}$.

Note: **(a)** is a special case of **(b)** corresponding to the zero shifted function.

Local models (ii)

Remark: • Shifted cotangent bundles are too restrictive to serve as local models of shifted symplectic structures.

• Derived critical loci of shifted functions have enough flexibility to provide local models. This leads to a remarkable shifted version of the Darboux theorem:

Local models (ii)

Theorem: [BBBJ'2013] Let X be a derived Deligne-Mumford stack, and let ω be an n-shifted symplectic structure on X, with $n < 0$.

Then, étale locally (X, ω) is isomorphic to $\left(\mathbf{Rcrit}(\mathbf{w}), \omega_{\mathbf{Rcrit}(\mathbf{w})}\right)$ for some shifted function $\mathbf{w} : M \to \mathbb{A}^1[n+1]$ on a derived scheme M.

O.Ben-Bassat, C.Brav, V.Bussi, D.Joyce

Local models (ii)

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Then, étale locally (X, ω) is isomorphic to $\big(\mathbf{Rcrit}(\mathbf{w}), \omega_{\mathbf{Rcrit}(\mathbf{w})}\big)$ for some shifted function $\mathbf{w} : M \to \mathbb{A}^1[n+1]$ on a derived scheme M.

Question: Find additional geometric structures that will ensure a global existence of a potential?

Local models (iii)

Answer: Potentials always exist in the presence of isotropic foliations.

Local models (iii)

Theorem: Let X be a derived stack, locally of f.p. and let ω be an n-shifted symplectic structure on X. Assume:

- ω is exact, i.e. $[\omega] = 0 \in H^{\bullet}_{DR}(X)$;
- (X, ω) is equipped with an isotropic foliation

$$(\mathscr{L}, h) = (L, \boldsymbol{\alpha}, \epsilon; h).$$

Then there exists

- a shifted function $f : [X/\mathscr{L}] \to \mathbb{A}^1[n+1]$, and
- a symplectic map $s : X \to \mathbf{Rcrit}(f)$ of n-shifted symplectic

stacks, i.e. $s^*\omega_{\mathbf{Rcrit}(f)} = \omega$.

Moreover, if $(\mathscr{L}, h)$ is Lagrangian, then s is étale.

Local models (iv)

Note: This connects directly to the **[BBBJ'2013]** Darboux theorem because of the following result:

Theorem: Let X be a derived stack, locally of f.p. and let ω be any n-shifted closed p-form on X with $n < 0$. Then ω is exact, i.e. $[\omega] = 0 \in H^{\bullet}_{DR}(X) = \mathbb{H}^{\bullet}\left(\mathcal{A}^{0,\mathrm{cl}}(X)\right)$.

Note: $[\omega] \in H^{p+n}_{DR}(X)$ and in general $H^{p+n}_{DR}(X) \neq 0$. So the statement is not a triviality.

Examples (i)

(1) Derived critical loci. Let Z be a smooth scheme, $\mathbf{w} : Z \to \mathbb{A}^1$ a regular function. Consider $X = \mathbf{Rcrit}(\mathbf{w})$ with its inherited (-1)-shifted symplectic structure $\omega_{\mathbf{Rcrit}(\mathbf{w})}$. Let $\imath : X \to Z$ be the natural map, and let $\mathscr{L}_\imath = \left(\mathbb{L}_{X/Z}, \mathbf{res}, d_{DR}\right)$ be the associated tangential foliation. Then:

Claim: • The foliation $\mathscr{L}_\imath$ has a natural Lagrangian structure h.

• The quotient $[X/\mathscr{L}_\imath] = \widehat{Z}_{\mathbf{crit}(\mathbf{w})}$ is the formal completion of Z along $\mathbf{crit}(\mathbf{w}) = t_0(X)$.

• The potential $f : \widehat{Z}_{\mathbf{crit}(\mathbf{w})} \to \mathbb{A}^1$ associated with h is given by $f = \mathbf{w}_{|\widehat{Z}_{\mathbf{crit}(\mathbf{w})}}$.

Examples (ii)

Variant: If $Z \in \mathbf{dSt}_{\mathbb{C}}$ is a derived stack locally of finite type, $\mathbf{w} : Z \to \mathbb{A}^1[n]$ is an n-shifted function, and $X = \mathbf{Rcrit}(\mathbf{w}) \xrightarrow{\imath} Z$, then

Claim:

- The foliation $\mathscr{L}_\imath$ has a natural Lagrangian structure h.
- The quotient $[X/\mathscr{L}_\imath] = \widehat{X}_\imath$ is the relative completion of X along $\imath$.
- The potential $f : \widehat{Z}_{\mathbf{crit}(\mathbf{w})} \to \mathbb{A}^1[n]$ associated with h is given by $f = \mathbf{w}_{|\widehat{X}_\imath}$.

Examples (iii)

(2) Cotangent bundles. If M is a smooth manifold, and

- $X = T^{\vee}M$,
- $\omega =$ (the standard symplectic structure).

Then: The natural projection $\pi : X \to M$ gives rise to a tangential foliation $\mathscr{L}_{\pi} = (L_{\pi}, \mathbf{res}, d_{DR})$ which is Lagrangian.

In this case:

- $[X/\mathscr{L}_{\pi}] = (X/M)_{DR}$,
- $f = 0$ viewed as a 1-shifted function,

and we get an identification $\mathbf{Rcrit}(f) = T^{\vee}_{M}[1-1] = X$ together with the natural 0-shifted symplectic forms.

Examples (iv)

(3) Twisted cotangent bundles. Suppose M is a smooth manifold over $\mathbb{C}$ and

$$\eta \in \mathbb{H}^1\left(M, \Omega_M^{\geq 1}[1]\right) = \mathbb{H}^2\left(M, \Omega_M^1 \xrightarrow{d} \Omega_M^{2,\mathrm{cl}}\right).$$

Such η gives rise to an algebraic symplectic manifold - the $\boxed{\textbf{twisted cotangent bundle}}$ $(\pi_\eta : X_\eta \to M, \omega_\eta)$.

Note:

- The tangential foliation $\mathscr{L}_{\pi_\eta}$ is Lagrangian.
- If ω_η is exact, then (X_η, ω_η) will be symplectically isomorphic to $\mathbf{Rcrit}(f)$ for a 1-shifted function f on $\left[X_\eta/\mathscr{L}_{\pi_\eta}\right] = (X_\eta/M)_{DR}$.

Examples (v)

We are looking for a shifted function $f : (X_\eta/M)_{DR} \to \mathbb{A}^1[1]$, or equivalently for an element

$$f \in \mathbb{H}^1\left(M, \mathcal{H}^\bullet_{DR}(X/M)\right) = H^1\left(M, \mathcal{O}_M\right).$$

By construction $[\omega_\eta] = 0 \in H^2_{DR}(X_\eta)$ if and only if η is in the image of the map $d : H^1\left(M, \mathcal{O}_M\right) \to H^1\left(M, \Omega^{\geq 1}_M[1]\right)$.

Therefore ω_η is exact precisely when we can find $f \in H^1\left(M, \mathcal{O}_M\right)$ such that $\eta = df$. This f is the shifted function provided by the theorem, i.e.

$$(X_\eta, \omega_\eta) \cong \left(\mathbf{Rcrit}(f), \omega_{\mathbf{Rcrit}(f)}\right).$$

Note: Note that as in the classical case f is only unique up to a class in $H^1(M, \mathbb{C})$, i.e. up to a (locally) constant 1-shifted function on $(X_\eta/M)_{DR}$.

Examples (vi)

(4) Integrable systems. Let (X, ω) be an exact symplectic manifold, and let

$$h : X \to B$$

be a smooth completely integrable system structure on X. Again the tangential foliation $\mathscr{L}_h$ is Lagrangian and $[X/\mathscr{L}_h] = (X/B)_{DR}$ and by the theorem we can find

$$f : (X/B)_{DR} \to \mathbb{A}^1[1]$$

such that $(X, \omega) = \left(\mathbf{Rcrit}(f), \omega_{\mathbf{Rcrit}(f)}\right)$.

Examples (vii)

Now note that

$$\begin{tikzcd}
\mathrm{Map}_{\mathbf{dSt}_{\mathbb{C}}}\left((X/B)_{DR}, \mathbb{A}^1[1]\right) \arrow[r, equal] & H^1\left(B, \mathcal{H}^{\bullet}_{DR}(X/B)\right) \arrow[d, phantom, "\cup"] \\
& H^0\left(B, h_*\Omega^1_{X/B}\right)
\end{tikzcd}$$

If $\lambda \in H^0\left(X, \Omega^1_X\right)$ is such that $\omega = d\lambda$, then λ maps to a relative 1-form $\lambda^{rel} \in H^0\left(X, \Omega^1_{X/B}\right) = H^0\left(B, h_*\Omega^1_{X/B}\right)$. One now checks that $f = \lambda^{rel}$.

Note: The full form λ also plays a role in the picture. It defines the map $s : X \to \mathbf{Rcrit}(f)$.

Examples (viii)

Indeed, if A is a reduced $\mathbb{C}$-algebra, then $(X/B)_{DR}(A) = X(A)$, i.e. $((X/B)_{DR})_{\text{red}} = X$. In particular $f_{|((X/B)_{DR})_{\text{red}}} = 0$ as it is the image of $f = \lambda^{\text{rel}} \in H^0\left(B, h_*\Omega^1_{X/B}\right) \subset H^1\left(B, \mathcal{H}^{\bullet}_{DR}(X/B)\right)$ in $H^1(X, \mathcal{O})$.

Therefore $\mathbf{Rcrit}(f)(A) = \mathbf{Rcrit}(0)(A) = T^{\vee}X(A)$ i.e. $\mathbf{Rcrit}(f)_{\text{red}} = T^{\vee}X$.

Since X itself is reduced, the map $s : X \to \mathbf{Rcrit}(f)$ will factor as

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "s"'] & \mathbf{Rcrit}(f)_{\text{red}} \arrow[r, hook] & \mathbf{Rcrit}(f) \end{tikzcd}$$

and it can be checked that the map $X \to T^{\vee}X$ coincides with the section λ.

Higher Chern-Simons functionals (i)

Let M be a compact oriented C^∞ manifold of dimension $d = 2k + 1$. Choose a Morse-Smale function $\mu : M \to \mathbb{R}$.

a self-indexing Morse function, i.e. for every $x \in \mathbf{crit}(\mu)$ we have $\mu(x) = \mathrm{ind}_\mu(x)$.

Higher Chern-Simons functionals (i)

Let M be a compact oriented C^∞ manifold of dimension $d = 2k + 1$. Choose a Morse-Smale function $\mu : M \to \mathbb{R}$. Choose $c \in (k, k+1)$, and let $M^+ := \mu^{-1}((-\infty, c])$. Then

- M^+ is a manifold with boundary;
- the inclusion $M^+ \hookrightarrow M$ induces a homotopy equivalence between M^+ and the k-dimensional skeleton of M.

Fix a complex reductive group G, and let $\mathrm{Bun}_G(M) = \mathrm{Map}_{\mathbf{dSt}_{\mathbb{C}}}(M, BG)$ be the derived moduli stack of G-local systems on M. By [PTVV] $\mathrm{Bun}_G(M)$ carries a natural $2-d$-shifted symplectic structure ω, and so if $k \geq 1$, it follows that ω is exact.

Higher Chern-Simons functionals (ii)

Theorem: [KPTVV] The tangential foliation for the restriction morphism

$$\mathrm{res}^{+} : \mathrm{Bun}_G(M) \to \mathrm{Bun}_G(M^{+})$$

can be equipped with a natural isotropic structure h which depends only on the orientation data of M and the shifted symplectic form on BG.

Hence we can find a shifted function

$$f : \left(\mathrm{Bun}_G(M) \,/\mathrm{Bun}_G(M^{+})\right)_{DR} \to \mathbb{A}^1[2-2k]$$

and a symplectic map

$$s : (\mathrm{Bun}_G(M), \omega) \to \left(\mathbf{Rcrit}(f), \omega_{\mathbf{Rcrit}(f)}\right) .$$

Potentials in non-abelian Hodge theory (i)

Let M be a smooth projective variety with $\dim_{\mathbb{C}} M = d$ and consider the derived stack of rank n local systems on M:

$$X := \mathbf{Loc}_n(M) = \mathrm{Map}_{\mathbf{dSt}_{\mathbb{C}}}(M, BGL_n).$$

From **[PTVV]** we know that X is equipped with a natural $(2-2d)$-shifted symplectic structure ω_X. This symplectic structure comes with natural refinements:

- $\mathbb{T}_X$ has a natural Hodge filtration.
- (X, ω_X) is the general fiber of a $\mathbb{C}^\times$ twisted symplectic family $(\mathscr{X}, \omega_{\mathscr{X}/\mathbb{A}^1}) \to \mathbb{A}^1$ of moduli of λ-connections, and on tangent complexes this gives the standard Hodge filtration.

Potentials in non-abelian Hodge theory (ii)

This implies

Claim: The natural map $\Theta_{\omega_X} : \mathbb{T}_X \to \mathbb{L}_X[2-2d]$ given by ω_X is a filtered quasi-isomorphism for the Hodge filtrations.

As a consequence in the middle degree one gets:

Theorem: When $d = 2k+1$, the natural map

$$F^{k+1}\mathbb{T}_X \to \mathbb{T}_X$$

admits a canonical structure of a Lagrangian foliation. In particular (X, ω_X) (and the corresponding Higgs moduli) are identified with the critical locus of a shifted potential.

Potentials in non-abelian Hodge theory (iii)

Remarks:

(1) The foliation $F^{k+1}\mathbb{T}_X \to \mathbb{T}_X$ is the tangential foliation for the map $\mathrm{res}^{\leq k+1} : X \to \mathbf{Loc}_n^{\leq k}(M)$.

derived moduli stack of dg modules over $\left(\Omega_M^{\leq k}, d\right)$ which are locally free of rank n

Potentials in non-abelian Hodge theory (iii)

Remarks:

(1) The foliation $F^{k+1}\mathbb{T}_X \to \mathbb{T}_X$ is the tangential foliation for the map $\mathrm{res}^{\leq k+1} : X \to \mathbf{Loc}_n^{\leq k}(M)$.

(2) If $k = 0$, then $\mathbf{Loc}_n^{\leq 0}(M) = \mathrm{Bun}_n(M)$ and the map $\mathrm{res}^{\leq 0} : X \to \mathrm{Bun}_n(M)$ is a twisted cotangent bundle.

(3) If $k \geq 1$, then the map induces an isomorphism of truncations $t_{\geq -k}\mathbf{Loc}_n(M) \to t_{\geq -k}\mathbf{Loc}_n^{\leq k}(M)$.

(4) The full untruncated stack $X = \mathbf{Loc}_n(M)$ is recovered as a critical locus of a shifted function on $\left(\mathbf{Loc}_n(M) \,/ \mathbf{Loc}_n^{\leq k}(M)\right)_{DR}$ which cna be checked again comes from an element $f \in H^{2-2k}\left(\mathbf{Loc}_n^{\leq k}(M), \mathcal{O}\right)$.

DT invariants of abelian 3-folds (i)

A - a 3 dimensional complex abelian variety;
M - a component of the moduli stack of coherent sheaves on A.

Note: Such M's have a symmetric perfect obstruction theory (which can be refined to a (-1)-shifted symplectic structure) but the associated Donaldson-Thomas invariants often vanish (due to deformation invariance).

DT invariants of abelian 3-folds (ii)

[BOPY'2015]: To get meaningful counts modify the obstruction theory by removing two dual pieces in the tangent complex: the piece controlling the obstructions to deforming the Chern classses to Hodge classes, and the piece controlling the deformations coming from the translation action of A.

DT invariants of abelian 3-folds (ii)

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J. Bryan, G. Oberdieck, R. Pandharipande, Q. Yin

DT invariants of abelian 3-folds (ii)

[BOPY'2015]: To get meaningful counts modify the obstruction theory by removing two dual pieces in the tangent complex: the piece controlling the obstructions to deforming the Chern classses to Hodge classes, and the piece controlling the deformations coming from the translation action of A.

[BOPY'2015]: The procedure results in a reduced symmetric obstruction theory on $[M/A]$ and gives rise to new reduced DT invariants of A, computed in terms of Jacobi forms.

DT invariants of abelian 3-folds (iii)

Interpretation: The reduced obstruction theory comes from a (-1)-shifted symplectic structure which is a symplectic reduction of the standard (-1)-shifted symplectic structure on the stack of perfect complexes.

DT invariants of abelian 3-folds (iii)

Interpretation: The reduced obstruction theory comes from a (-1)-shifted symplectic structure which is a symplectic reduction of the standard (-1)-shifted symplectic structure on the stack of perfect complexes.

$M = t_0(X)$ where X is the corresponding component of $MAP(A, \mathrm{Perf})$ and X. X comes equipped with a (-1) shifted symplectic structure ω. The **[BOPY'2015]** theorem can be repackaged in the following statements:

DT invariants of abelian 3-folds (iv)

- The A-action on (X, ω) is Hamiltonian and has an A-equivariant moment map $\mu : X \to \mathfrak{a}^{\vee}[-1]$.
- μ is equal to zero on the truncation $M = t_0 X$, and so $R\mu^{-1}(0)$ is M with a different derived structure in which the three dimensional space of obstructions is killed.
- The reduced symmetric obstruction theory on $[M/A]$ is the symmetric obstruction theory corresponding to the (-1)-shifted symplectic structure on $[R\mu^{-1}(0)/A]$ coming from the symplectic reduction.

DT invariants of abelian 3-folds (iv)

- The A-action on (X, ω) is Hamiltonian and has an A-equivariant moment map $\mu : X \to \mathfrak{a}^\vee[-1]$.
- μ is equal to zero on the truncation $M = t_0 X$, and so $R\mu^{-1}(0)$ is M with a different derived structure in which the three dimensional space of obstructions is killed.
- The reduced symmetric obstruction theory on $[M/A]$ is the symmetric obstruction theory corresponding to the (-1)-shifted symplectic structure on $[R\mu^{-1}(0)/A]$ coming from the symplectic reduction.

Note: Explicitly $[R\mu^{-1}(0)/A]$ is the derived intersection of two Lagrangians in $[\mathfrak{a}^\vee[-1]/A] = T^\vee_{BA}$: the zero section and $\mu : [X/A] \to [\mathfrak{a}^\vee[-1]/A]$.

Note: The same construction is expected to work for the classical reduced obstruction theory on a K3 surface S: it should be the symplectic reduction of the (-1) shifted symplectic structure on the stack of perfect complexes on $S \times E$ symplectically reduced by the action of the elliptic curve E.

Azumaya property of quantizations (i)

X - a smooth scheme over a perfect field k of characteristic $p > 0$.
$S = T^\vee_X[n]$ - the n-shifted cotangent bundle of X.
$\mathscr{A}$ - the shifted quantization of $\mathcal{O}_S$.

Conjecture: [Hablicsek,Haugseng, …] Consider the Frobenius twist S' of S and the zero section $i : X' \to S'$. Then the algebra $\mathscr{A}$ can be regarded as an E_{n+1}-algebra over $\mathcal{O}_{S'}$ so that:
[Weak Morita equivalence:] The $(\infty, n+1)$-category of coherent $i^*\mathscr{A}$-modules is equivalent to the $(\infty, n+1)$-category of coherent $\mathcal{O}_{S'}$-modules ($\mathcal{O}_{S'}$ is viewed as an E_{n+1}-algebra).
[Weak Azumaya property:] Étale locally over X, the $(\infty, n+1)$-category of coherent $\mathscr{A}$-modules is equivalent to the $(\infty, n+1)$-category of coherent $\mathcal{O}_{S'}$-modules.

Azumaya property of quantizations (ii)

Consider $S = T_X^\vee[1]$. In this case $\mathscr{A}$ has an explicit model - the crystalline Hochschild cosimplicial complex.

Remark: [Hablicsek] The pullback $i^*\mathscr{A}$ is the $\mathcal{O}_{X'}$-linear Hochschild cosimplicial complex of polydifferential operators $\mathscr{D}iff_{\mathcal{O}_{X'}}(\mathcal{O}_X^\bullet, \mathcal{O}_X)$ which is **not** Morita equivalent to $\mathcal{O}_{X'}$.

Nevertheless we have

Theorem: [Hablicsek] If we view $\mathcal{O}_{X'}$ as an E_2 algebra, then the category of coherent $\mathcal{O}_{X'}$-modules is equivalenct to the full thick subcategory of coherent $\mathscr{D}iff_{\mathcal{O}_{X'}}(\mathcal{O}_X^\bullet, \mathcal{O}_X)$-modules generated by $\mathcal{O}_X$.

Tangent complex

$X \in \mathrm{dSt}_{\mathbb{C}}$, $x : \mathrm{Spec}(\mathbb{C}) \to X$ a point

$$\left(\begin{array}{l}\text{Stalk } \mathbb{T}_{X,x} \text{ of the}\\ \text{tangent complex}\end{array}\right) = \left(\begin{array}{l}\text{normalized chain complex}\\ \text{of the homotopy fiber of}\\ X(\mathbb{C}[\varepsilon]) \to X(\mathbb{C}) \text{ over } x\end{array}\right)$$

simplicial abelian group

Tangent complex

$X \in \mathrm{dSt}_{\mathbb{C}}$, $x : \mathrm{Spec}(\mathbb{C}) \to X$ a point

$$\begin{pmatrix}\text{Stalk } \mathbb{T}_{X,x} \text{ of the} \\ \text{tangent complex}\end{pmatrix} = \begin{pmatrix}\text{normalized chain complex} \\ \text{of the homotopy fiber of} \\ X(\mathbb{C}[\varepsilon]) \to X(\mathbb{C}) \text{ over } x\end{pmatrix}$$

When X is a moduli stack:

$H^{-1}(\mathbb{T}_{X,x}) =$ infinitesimal automorphisms of x;

$H^{0}(\mathbb{T}_{X,x}) =$ infinitesimal deformations of x;

$H^{1}(\mathbb{T}_{X,x}) \supseteq$ obstructions of x.

Examples

- $X = BG = [\mathrm{pt}\,/\,G] \Rightarrow \mathbb{T}_{X,\mathrm{pt}} = \mathfrak{g}[1]$.
- $X =$ derived intersection $L_1 \times^h_M L_2 = (L_1 \cap L_2, \mathcal{O}_{L_1} \otimes^L_{\mathcal{O}_M} \mathcal{O}_{L_2})$ of smooth subvarieties $L_1, L_2 \subset M$ in a smooth $M \Rightarrow$
  $\mathbb{T}_{X,x} = [T_{L_1,x} \oplus T_{L_2,x} \to T_{M,x}]$,
  $H^0(\mathbb{T}_{X,x}) =$ $T_{L_1 \cap L_2,x}$;
  $H^1(\mathbb{T}_{X,x}) =$ failure of transversality.
- $X =$ moduli of vector bundles E on a smooth projective $Y \Rightarrow$ $\mathbb{T}_{X,E} = R\Gamma(Y, \mathrm{End}(E))[1]$.
- $X =$ moduli of maps f from C to $Y \Rightarrow \mathbb{T}_{X,f} = R\Gamma(C, f^* T_Y)$.
- $X =$ moduli of local systems $\mathbb{E}$ on a compact manifold $Y \Rightarrow$ $\mathbb{T}_{X,\mathbb{E}} = R\Gamma(Y, \mathrm{End}(\mathbb{E}))[1]$.

Cotangent complex

$A \in \mathrm{cdga}_{\mathbb{C}}$, $X = \mathbb{R}\mathbf{Spec}(A) \in \mathrm{dSt}_{\mathbb{C}}$,
$A' \to A$ a cofibrant (semifree) replacement

$$\begin{pmatrix} \text{cotangent} & \text{complex} \\ \mathbb{L}_X = \mathbb{L}_A & \end{pmatrix} = \begin{pmatrix} \text{Kähler} & \text{differentials} \\ \Omega^1_{A'} \text{ of } A' & \end{pmatrix}$$

If $X \in \mathrm{dSt}_{\mathbb{C}}$ is a general derived Artin stack, then $X = \mathrm{hocolim}\{\mathbb{R}\mathbf{Spec}\, A \to X\}$ (in the model category $\mathrm{dSt}_{\mathbb{C}}$) and

$$\mathbb{L}_X = \mathrm{holim}_{\mathbb{R}\mathbf{Spec}\, A \to X}\, \mathbb{L}_A$$

Note:

- $\mathbb{L}_X \in L_{qcoh}(X)$ - the dg category of quasi-coherent $\mathcal{O}_X$ modules.
- X is locally of finite presentation iff $\mathbb{L}_X$ is perfect. In this case $\mathbb{T}_X = \mathbb{L}_X^{\vee} = \mathrm{Hom}(\mathbb{L}_X, \mathcal{O}_X)$.

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p-forms

$A \in \mathrm{cdga}_{\mathbb{C}}, \quad X = \mathbb{R}\mathbf{Spec}(A) \in \mathrm{dSt}_{\mathbb{C}}$,
$A' \to A$ a cofibrant (semifree) replacement. Then:

$\oplus_{p\geq 0} \wedge^p_A \mathbb{L}_A = \oplus_{p\geq 0}\Omega^p_{A'}$ - a fourth quadrant bicomplex with

vertical differential $d : \Omega^{p,i}_{A'} \to \Omega^{p,i+1}_{A'}$ induced by $d_{A'}$, and

horizontal differential $d_{DR} : \Omega^{p,i}_{A'} \to \Omega^{p+1,i}_{A'}$ given by the de Rham differential.

Hodge filtration: $F^q(A) := \oplus_{p>q}\Omega^p_{A'}$: still a fourth quadrant bicomplex.

(shifted) closed p-forms

Motivation: If X is a smooth scheme$/\mathbb{C}$, then $\Omega_X^{p,cl} \cong \left(\Omega_X^{\geq p}, d\right)$.
Use $(\Omega_X^{\geq p}, d)$ as a model for closed p forms in general.

Definition:

- **complex of closed p-forms on $X = \mathbb{R}\mathbf{Spec}\, A$:**
  $\mathbf{A}^{p,cl}(A) := \mathrm{tot}^{\prod}(F^p(A))[p]$
- **complex of n-shifted closed p-forms on $X = \mathbb{R}\mathbf{Spec}\, A$:**
  $\mathbf{A}^{p,cl}(A;n) := \mathrm{tot}^{\prod}(F^p(A))[n+p]$
- **Hodge tower:**
  $\cdots \to \mathbf{A}^{p,cl}(A)[-p] \to \mathbf{A}^{p-1,cl}(A)[1-p] \to \cdots \to \mathbf{A}^{0,cl}(A)$

(shifted) closed p-forms (ii)

Explicitly an n-shifted closed p-form ω on $X = \mathbb{R}\mathbf{Spec}\, A$ is an infinite collection

$$\omega = \{\omega_i\}_{i \geq 0}\,, \qquad \omega_i \in \Omega_A^{p+i,n-i}$$

satisfying

$$d_{DR}\omega_i = -d\omega_{i+1}.$$

Note: The collection $\{\omega_i\}_{i \geq 1}$ is the key closing ω.

p-forms and closed p-forms

Note:

- The complex $\mathbf{A}^{0,cl}(A)$ of closed 0-forms on $X = \mathbb{R}Spec\, A$ is exactly Illusie's derived de Rham complex of A.
- There is an **underlying p-form map**
  $$\mathbf{A}^{p,cl}(A; n) \to \wedge^p \mathbb{L}_{A/k}[n]$$
  inducing
  $$H^0(\mathbf{A}^{p,cl}(A)[n]) \to H^n(X, \wedge^p \mathbb{L}_{A/k}).$$
- The homotopy fiber of the underlying p-form map can be very complicated (complex of **keys**): being closed is *not* a property but rather a list of coherent data.

Functoriality and gluing:

- the *n*-shifted *p*-forms ∞-functor $\mathcal{A}^{p}(-;n) : \mathbf{cdga}_{\mathbb{C}} \to \mathbf{SSets} : A \mapsto |\,\Omega^{p}_{QA}[n] \simeq (\wedge^{p}_{A}\mathbb{L}_{A})[n]\,|$
- the *n*-shifted closed *p*-forms ∞-functor $\mathcal{A}^{p,\mathrm{cl}}(-;n) : \mathbf{cdga}_{\mathbb{C}} \to \mathbf{SSets} : A \mapsto |\,\mathbf{A}^{p,cl}(A)[n]\,|$
- $\mathcal{A}^{p}(-;n)$ and $\mathcal{A}^{p,\mathrm{cl}}(-;n)$ are derived stacks for the étale topology.
- underlying *p*-form map (of derived stacks)

$$\mathcal{A}^{p,\mathrm{cl}}(-;n) \to \mathcal{A}^{p}(-;n)$$

Notation: $|-|$ denotes $Map_{\mathbb{C}-dgMod}(\mathbb{C},-)$ i.e. Dold-Kan applied to the ≤ 0-truncation [dg-modules have cohomological differential]

global forms and closed forms

For a derived Artin stack X (locally of finite presentation $/\mathbb{C}$) we have

Definition:

- $\mathcal{A}^p(X) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^p(-))$ is the space of p-forms on X;
- $\mathcal{A}^{p,\mathrm{cl}}(X) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^{p,\mathrm{cl}}(-))$ is the space of closed p-forms on X;
- the corresponding n-shifted versions :
  $\mathcal{A}^p(X; n) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^p(-; n))$
  $\mathcal{A}^{p,\mathrm{cl}}(X; n) := Map_{\mathbf{dSt}_{\mathbb{C}}}(X, \mathcal{A}^{p,\mathrm{cl}}(-; n))$
- an n-shifted (resp. closed) p-form on X is an element in $\pi_0\mathcal{A}^p(X; n)$ (resp. in $\pi_0\mathcal{A}^{p,\mathrm{cl}}(X; n)$)

global forms and closed forms (ii)

Note:

- If X is a smooth scheme there are no negatively shifted forms.
- If $X = \mathbb{R}Spec\, A$ then there are no positively shifted forms.

For a general X shifted forms potentially exist for any $n \in \mathbb{Z}$.

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global forms and closed forms (ii)

Consider the **underlying p-form** map (of simplicial sets):

$$\mathcal{A}^{p,\mathrm{cl}}(X;n) \to \mathcal{A}^{p}(X;n),$$

then:

- This map is not a monomorphism for general X, its homotopy fiber at a given p-form ω_0 is the space of **keys** of ω_0.
- If X is a smooth and proper scheme then this map is a mono (homotopy fiber is either empty or contractible) $\Rightarrow$ no new phenomena in this case.
- **Theorem (PTVV):** For X derived Artin, $\mathcal{A}^p(X;n) \simeq \mathrm{Map}_{\mathrm{L}_{\mathrm{qcoh}}(X)}(\mathcal{O}_X, (\wedge^p \mathbb{L}_X)[n])$ (smooth descent)
- in particular a n-shifted p-form on X is an element in $H^n(X, \wedge^p \mathbb{L}_X)$

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Examples (i):

(1) If $X = \mathrm{Spec}(A)$ is an usual (underived) smooth affine scheme, then

$$\mathcal{A}^{p,cl}(X;n) = (\tau_{\leq n}(\underset{0}{\Omega_A^p} \xrightarrow{d_{DR}} \underset{1}{\Omega_A^{p+1}} \xrightarrow{d_{DR}} \cdots))[n],$$

and hence

$$\pi_0\mathcal{A}^{p,cl}(X;n) = \begin{cases} 0, & n < 0 \\ \Omega_A^{p,cl}, & n = 0 \\ H_{DR}^{n+p}(X), & n > 0 \end{cases}$$

e.g. if $X = \mathbb{C}^\times$, then $dz/z \in \pi_0\mathcal{A}^{1,cl}(X;0)$ and also $dz/z \in \pi_0\mathcal{A}^{0,cl}(X;1)$.

Examples (ii):

(2) If X is a smooth and proper scheme, then $\pi_i \mathcal{A}^{p,cl}(X;n) = F^p H^{n+p-i}_{DR}(X)$.

(3) If X is a (underived) higher Artin stack, and $X_\bullet \to X$ is a smooth affine simplicial groupoid presenting X, then $\pi_0 \mathcal{A}^p(X;n) = H^n(\Omega^p(X_\bullet), \delta)$ with $\delta =$ Čech differential.
In particular if G is a complex reductive group, then

$$\pi_0 \mathcal{A}^p(BG;n) = \begin{cases} 0, & n \neq p \\ \left(\mathrm{Sym}^\bullet \mathfrak{g}^\vee\right)^G, & n = p. \end{cases}$$

Examples (iii):

(4) Similarly

$$\mathcal{A}^{p,cl}(BG;n) = \left| \prod_{i\geq 0} \left(\mathrm{Sym}^{p+i}\,\mathfrak{g}^{\vee}\right)^{G}[n+p-2i] \right|,$$

and so

$$\pi_0\mathcal{A}^{p,cl}(BG;n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \left(\mathrm{Sym}^{p}\,\mathfrak{g}^{\vee}\right)^{G}, & \text{if } n \text{ is even.} \end{cases}$$

Examples (iv):

(5) If $X = \mathrm{Rzero}(s)$ for $s \in H^0(L, E)$ on a smooth L, then

$$\begin{tikzcd} \Omega^1_X = & E^\vee_{|Z} \arrow[r, "(\nabla s)^\flat"] & \Omega^1_{L|Z}, \\ & -1 & 0 \end{tikzcd}$$

and if we choose ∇ local flat algebraic connection on E we can rewrite Ω^1_X as a module over the Koszul complex:

$$\begin{tikzcd} \cdots \arrow[r] & \wedge^2 E^\vee \otimes \Omega^1_L \arrow[r, "s^\flat"] & E^\vee \otimes \Omega^1_L \arrow[r, "s^\flat"] & \Omega^1_L \arrow[r] & \Omega^1_{L|Z} & 0 \\ \cdots \arrow[r] & \wedge^2 E^\vee \otimes E^\vee \arrow[r, "s^\flat"] \arrow[u] & E^\vee \otimes E^\vee \arrow[r, "s^\flat"] \arrow[u] & E^\vee \arrow[r] \arrow[u, "{[\nabla, s^\flat]}"'] & E^\vee_{|Z} \arrow[u, "(\nabla s)^\flat"'] & -1 \end{tikzcd}$$

Examples (v):

In the same way we can describe Ω^2_X as a module over the Koszul complex

$$\begin{tikzcd}
\cdots \arrow[r] & \wedge^2 E^\vee \otimes \Omega^2_L \arrow[r] & E^\vee \otimes \Omega^2_L \arrow[r] & \Omega^2_L \arrow[r] & \Omega^2_{L|Z} & 0 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & (E^\vee \otimes \Omega^1_L)_{|Z} \arrow[u] & -1 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u] & E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u] & S^2 E^\vee \arrow[r] \arrow[u] & S^2 E^\vee | Z \arrow[u] & -2
\end{tikzcd}$$

Examples (v):

In the same way we can describe Ω^2_X as a module over the Koszul complex

$$\begin{tikzcd}
\cdots \arrow[r] & \wedge^2 E^\vee \otimes \Omega^2_L \arrow[r] & E^\vee \otimes \Omega^2_L \arrow[r] & \Omega^2_L \arrow[r] & \Omega^2_{L|Z} & 0 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u, two heads] & E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u, two heads] & E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u, two heads] & (E^\vee \otimes \Omega^1_L)_{|Z} \arrow[u, two heads] & -1 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u, two heads] & E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u, two heads] & S^2 E^\vee \arrow[r] \arrow[u, two heads] & S^2 E^\vee | Z \arrow[u, two heads] & -2
\end{tikzcd}$$

2 forms of degree -1

Examples (v):

In the same way we can describe Ω^2_X as a module over the Koszul complex

$$\begin{tikzcd}
\cdots \arrow[r] & \wedge^2 E^\vee \otimes \Omega^2_L \arrow[r] & E^\vee \otimes \Omega^2_L \arrow[r] & \Omega^2_L \arrow[r] & \Omega^2_{L|Z} & 0 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & E^\vee \otimes E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & E^\vee \otimes \Omega^1_L \arrow[r] \arrow[u] & (E^\vee \otimes \Omega^1_L)_{|Z} \arrow[u] & -1 \\
\cdots \arrow[r] & \wedge^2 E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u] & E^\vee \otimes S^2 E^\vee \arrow[r] \arrow[u] & S^2 E^\vee \arrow[r] \arrow[u] & S^2 E^\vee | Z \arrow[u] & -2
\end{tikzcd}$$

Note: The de Rham differnetial $d_{DR} : \Omega^1_X \to \Omega^2_X$ is the sum $d_{DR} = \nabla + \kappa$, where κ is the Koszul contraction

$$\kappa : \wedge^a E^\vee \otimes S^b E^\vee \to \wedge^{a-1} E^\vee \otimes S^{b+1} E^\vee .$$

Examples (vi):

Important Remark: **[Behrend]** If $E = \Omega_L^1$ and so s is a 1-form, then a 2-form of degree -1 corresponds to a pair of elements $\alpha \in (\Omega_L^1)^\vee \otimes \Omega_L^2$ and $\phi \in (\Omega_L^1)^\vee \otimes \Omega_L^1$ such that $[\nabla, s^\flat](\phi) = s^\flat(\alpha)$.

Take $\phi = \mathrm{id} \in (\Omega_L^1)^\vee \otimes \Omega_L^1$. Suppose the local ∇ is chosen so that $\nabla(\mathrm{id}) = 0$ (i.e. ∇ is torsion free). Then $[\nabla, s^\flat](\mathrm{id}) = ds$.

Conclusion: The pair (α, id) gives a 2-form of degree -1 iff $ds = s^\flat(\alpha)$, or equivalently $ds_{|Z} = 0$, i.e. is an almost closed 1-form on L.

Exercise: Suppose s is almost closed and let (α, id) be an associated 2-form of degree -1. Describe the complex of keys for (α, id) if it exists.

Twisted cotangent bundles (i):

Let M be a complex algebraic manifold and let (X, ω) be the cotangent bundle of M equipped with the standard symplectic form. This symplectic structure is uniquely characterized by the following

Properties:

- The natural projection $\pi : X \to M$ is a smooth Lagrangian fibration.
- For any locally defined one form α on M we have $t_{\alpha}^{*}\omega = \omega + \pi^{*}(d\alpha)$.

Twisted cotangent bundles are symplectic structures that are modeled on this geometry.

Twisted cotangent bundles (ii):

Definition: A **twisted cotangent bundle** over M is specified by data $(\pi_Y : Y \to M, \omega_Y)$, where

- $\pi_Y : Y \to M$ is a torsor over $T^\vee M$;
- ω_Y is an algebraic symplectic form on Y, and:
  - The projection $\pi_Y : Y \to M$ is a Lagrangian fibration for ω_Y.
  - For any locally defined one form α on M we have $t_\alpha^* \omega_Y = \omega_Y + \pi_Y^*(d\alpha)$.

Note: The $T^\vee M$-torsor structure is superfluous. It is uniquely determined from π_Y and ω_Y. Indeed, the infinitesimal action of a local one form α is given by the vector field $\Theta_{\omega_Y}^{-1}(\pi_Y^* \alpha)$.

Twisted cotangent bundles (iii):

Recall: Let $C^\bullet = \left[C^0 \stackrel{d}{\to} C^1 \right]$ be a complex of sheaves of $\mathbb{C}$-vector spaces on M concentrated in degrees 0 and 1. Then a **torsor over $C^\bullet$** is a pair (A, t), where A is a C^0-torsor and $t : A \to C^1$ is a trivialization of the associated C^1-torsor $d(A)$. Concretely t is a map of sheaves satisfying $t(a + c) = t(a) + d(c)$ for all $a \in A$, $c \in C^0$.

Twisted cotangent bundles (iii):

Recall: Let $C^\bullet = \left[C^0 \xrightarrow{d} C^1 \right]$ be a complex of sheaves of $\mathbb{C}$-vector spaces on M concentrated in degrees 0 and 1. Then a **torsor over $C^\bullet$** is a pair (A, t), where A is a C^0-torsor and $t : A \to C^1$ is a trivialization of the associated C^1-torsor $d(A)$.

Lemma: **[Beilinson-Berstein]** There is a canonical equivalence of groupoids

$$\begin{pmatrix} \text{twisted} & \text{cotangent} \\ \text{bundles over } M & \end{pmatrix} \leftrightarrow \left(\Omega_M^{\geq 1}[1]\text{-torsors} \right)$$

Twisted cotangent bundles (iv):

The equivalence of groupoids is described as follows:

→ Given a twisted cotangent bundle $(\pi_Y : Y \to M, \omega_Y)$ we define a $\Omega_M^{\geq 1}[1]$-torsor (A, c), where A is the sheaf of sections of π_Y, and $c : A \to \Omega_Y^{2,\mathrm{cl}}$ is given by $c(a) = a^* \omega_Y$.

← Conversely, given a $\Omega_M^{\geq 1}[1]$-torsor (A, c), define a twistwd cotangent bundle $(\pi_Y : Y \to M, \omega_Y)$ by taking $\pi_Y : Y \to M$ to be the total space of the Ω_M^1-torsor A, and ω_Y to be the unique form such that for every local section σ of π_Y, the associated isomorphism of $T^\vee M$-torsors $f_\sigma : Y \to T^\vee M$ satisfies $f_\sigma^* (\omega + \pi^*(c(\sigma))) = \omega_Y$.

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