MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Akshay Venkatesh

Talk Title: (Derived) moduli of local systems in number theory

Date: 2 / 8 / 19 **Time:** 2 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
    - **Overhead**: Obtain a copy or use the originals and scan them
    - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

(DERIVED) MODULI OF LOCAL SYSTEMS IN NUMBER THEORY

AKSHAY VENKATESH

1. General paradigm

I would like to talk about moduli spaces of local systems on a "curve" Z in four contexts.

(1) Let Z be a (smooth projective) curve over $\mathbf{C}$.
(2) Let Z be a curve over $\overline{\mathbf{F}}_p$.
(3) Let Z be a curve over $\mathbf{F}_p$.
(4) Let Z be Spec $\mathbf{Z}$, or more generally Spec of a ring of integers localized at some primes.

2. Context (1)

We start with the space of n-dimensional local systems over $\mathbf{C}$, which informally is

$$\{\pi_1(Z(\mathbf{C})) \to \mathrm{GL}_n(\mathbf{C})\}/\text{conjugacy}.$$

Now, $\pi_1(Z(\mathbf{C}))$ has a nice presentation:

$$\pi_1(Z(\mathbf{C})) = \langle \alpha_i, \beta_i \colon 1 \leq i \leq g, \prod[\alpha_i, \beta_i] = e\rangle$$

So the space of local systems is

$$\{a_1, \ldots, a_g, b_1, \ldots, b_g \in \mathrm{GL}_n(\mathbf{C}) \colon \prod[a_i, b_i] = e\}/\,\mathrm{PGL}_n(\mathbf{C}).$$

We let X be the subset parametrizing *irreducible* local systems. This is an open condition (it could be phrased as saying that some elements generate the matrix algebra). This X is a smooth $\mathbf{C}$-variety. Indeed, $x \in X_{\mathbf{C}}$ is identified with $\rho_x \colon \pi_1 \to \mathrm{GL}_n(\mathbf{C})$, and the tangent complex is

$$\mathbf{T}^i_x X_{\mathbf{C}} = H^{i-1}(Z, \operatorname{End}\rho := \rho \otimes \rho^\vee).$$

The H^0 and H^2 vanish when ρ_x is irreducible.

We can construct X over $\mathbf{Z}$, and then ask for any ring R what $X(R)$ is. The answer is pretty close to the set of absolutely irreducible representations

$$\{\pi_1(Z_{\mathbf{C}}) \to \mathrm{GL}_n(R)\}/\text{conj}.$$

For the R we'll discuss, this will be an isomorphism.

Warm-up: how much of X (the thing over $\mathbf{Z}$) can we see purely algebraically?

Date: February 8, 2019.

You can try to instead think about vector bundles with connection, but then you lose the integral structure. We really want to be able to talk about integral coefficients.

We can replace $\pi_1(Z(\mathbf{C}))$ with $\pi_1^{\text{ét}}(Z/\mathbf{C})$. This is the profinite completion, so it has the same maps to any finite group. We can still define a functor $\widehat{X}$ which takes R to the set of (conjugacy classes of) irreducible continuous representations of $\pi_1^{\text{ét}}$ with coefficients in R.

If $R = \mathbf{C}$, this isn't a good approximation, because any such representation will have finite image, and you can't approximate an arbitrary representation by one with finite image.

But $\widehat{X}$ does "see" some of the geometry of X. Roughly, you can recover stuff that can be described by maps into finite or profinite rings, e.g. $X(\mathbf{F}_\ell) = \widehat{X}(\mathbf{F}_\ell)$. You can also recover the formal neighborhood of x in X, i.e. $\mathcal{O}_{X,x}$ (the completed local ring of X at x) which is isomorphic to $\mathbf{Z}_\ell[[t_1, \ldots, t_?]]$. How? For any finite local ring $A \to \mathbf{F}_\ell$,

$$\mathrm{Hom}(\mathcal{O}_{X,x}, A) = \text{fiber of } X(A) \to X(\mathbf{F}_\ell) \text{ over } x.$$

Since these are finite, they are seen by $\widehat{X}$.

Picture: you see the (finitely many) points of X over various $\mathbf{F}_p$, and their formal neighborhoods.

What we'll see in (2)-(4) is that you have the same kind of information. That is: instead of a moduli space over Spec $\mathbf{Z}$ we have a bunch of points and their formal neighborhoods. But we'll explain that in each case there are hints of a more satisfactory geometric picture.

3. Context (2)

Let $Z/\overline{\mathbf{F}}_p$, coming from $Z/\mathbf{F}_p$. Again we can define $\widehat{X}$ from $\pi_1^{\text{ét}}(Z/\overline{\mathbf{F}}_p)$.

Note that because Z came from $\mathbf{F}_p$, there is an action of Frob on $\widehat{X}$. More precisely, there is an exact sequence

$$\pi_1^{\text{ét}}(Z/\overline{\mathbf{F}}_p) \to \pi_1^{\text{ét}}(Z/\mathbf{F}_p) \to \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p) \tag{3.1}$$

so $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ acts on $\pi_1^{\text{ét}}(Z/\overline{\mathbf{F}}_p)$.

We'll describe a theorem of Drinfeld that strongly suggests this thing has a global geometric structure.

Theorem 3.1 ($n = 2$ Drinfeld '80s, $n > 2$ H. Yu)**.** *The number of fixed points of* Frob^k *on* $\widehat{X}(\overline{\mathbf{Q}}_\ell)$ *is*

$$\sum_{i=1}^{I} m_i \gamma_i^k$$

for some $m_i \in \mathbf{Z}$ *and* $\gamma_i \in \overline{\mathbf{Q}}$ *(which are actually* p*-Weil numbers).*

Qualitatively, this behaves like the Lefschetz trace formula for counting points on an actual geometric space. If it were the case $k = 0$ would recover the Euler characteristic of the space.

For $k = 0$,

$$\sum_{i=1}^{I} m_i = \chi(X_{\mathbf{C}}).$$

In fact this is true, but both sides are 0 because there's a circle action. Deligne refines the conjecture after stripping out this action.

Conjecture 3.2 (Deligne)**.** *This is true after fixing determinant.*

Remark 3.3. There are similar phenomena for sheaves on $\widehat{X}$.

Problem: develop "geometry" for $\widehat{X}$ explaining this.

Picture: there should be a "vertical" structure on this union of punctured disks.

4. Context (3)

We will now explain a "horizontal" geometry. This is due to de Jong and Gaitsgory, with input from work of Drinfeld and Lafforgue.

Representations of $\pi_1(Z/\mathbf{F}_p)$ (which are irreducible when restricted to $\pi_1(Z/\overline{\mathbf{F}}_p)$) are the Frob fixed points of $\pi_1(Z/\overline{\mathbf{F}}_p)$. Refer to the exact sequence (3.1).

So $\widehat{X}$ in context (3) is essentially ($\widehat{X}$ in context (2))$^{\mathrm{Frob}}$. By the way, you might think it would be better to take "derived fixed points", but it turns out that there is no derived structure to be had here (i.e. that would give the same answer as naïve fixed points).

In fact $\widehat{X}^{\mathrm{Frob}}$ has a $\mathbf{Z}[1/p]$-structure. Take $x \in \widehat{X}^{\mathrm{Frob}}(\mathbf{F}_\ell)$. You then get a complete local ring $\mathcal{O}_{X,x}$. In previous contexts these were smooth, but now they need not be. We will construct an integral structure on it.

From x we get $\rho_x \colon \pi_1 \to \mathrm{GL}_n(\mathbf{F}_\ell)$. That lifts to $\widetilde{\rho}_x$ over the complete local ring $\mathcal{O}_{X^{\mathrm{Frob}},x}$. (It's basically by definition that $\mathcal{O}_{X^{\mathrm{Frob}},x}$ is the largest ring to which this lifts.) We define $S \subset \mathcal{O}_{X^{\mathrm{Frob}},x}$ in the following way.

We'll specify S as an explicit $\mathbf{Z}$-algebra. Each $\mathbf{F}_{p^n}$-point z of Z gives a conjugacy class $\mathrm{Frob}_z \in \pi_1$. Define $S \subset \mathcal{O}_{X^{\mathrm{Frob}},x}$ to be the $\mathbf{Z}$-algebra generated by $\mathrm{Tr}\,\widetilde{\rho}_x(\mathrm{Frob}_z)$. Amazingly, S is a finite free $\mathbf{Z}$-module, and $S \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \xrightarrow{\sim} \mathcal{O}_{X^{\mathrm{Frob}},x}$.

(We probably mean $\mathbf{Z}[1/p]$ instead of $\mathbf{Z}$.)

Although this definition works, we'd like to understand it in a more intrinsic way.

Let me explain some motivation for why it's a reasonable thing. Start with a $\mathbf{Z}_\ell$-valued point $x \in \widehat{X}(\mathbf{Z}_\ell)$. What does it mean for this point to extend to a "$\mathbf{Z}$-point"? One reasonable candidate: if $\rho_x \colon \pi_1 \to \mathrm{GL}_n(\mathbf{Z}_\ell)$ arises from the monodromy of a family of (smooth, projective) algebraic varieties $M \to Z$, then the trace of Frobenii would be integers (since they count points).

5. Context (4)

The overall picture is similar to (3), except you add derivedness.

Replace Z by $\mathbf{Z}$ or $\mathbf{Z}[1/7]$ or $\mathbf{Z}[\sqrt{3}]$, etc.

There is some serious issue which is the subject of p-adic Hodge theory, which we will try to suppress.

Now when you set up the deformation theory, you find that the local ring $\widehat{\mathcal{O}}_{X,x}$ are derived. You get $\pi_*\widehat{\mathcal{O}}_{X,x}$ as a graded commutative ring such that $\pi_0\widehat{O}_{X,x}$ is the "naïve" complete local ring.

Galatius and I propose that $\pi_*\widehat{\mathcal{O}}_{X,x}$ acts more or less freely on the homology of arithmetic groups with $\mathbf{Z}_\ell$-coefficients. The thing I want to emphasize is that this homology has an integral structure. That gives an integral structure on $\pi_*\widehat{\mathcal{O}}_{X,x}$. This suggests that $\pi_*\widehat{\mathcal{O}}_{X,x}$ has a $\mathbf{Z}$-structure as in Context (3).

I'll say one brief remark about what this $\mathbf{Z}$-structure means. The higher π_* of this ring are related to the fact that the deformation problem is obstructed, i.e. $H^2_{\text{ét}}(\operatorname{End}\rho)$ for suitable $\rho\colon \pi_1 \to \mathrm{GL}_n$. The $\mathbf{Z}$ structure suggests that you want to replace this by a motivic analogue, which then has a $\mathbf{Z}$-structure. But it turns out that the relevant motivic cohomology group vanishes. You can replace this H^2 by H^1 via duality, and then use motivic cohomology to give an integral structure on H^1. But this is a weird maneuver, and we would like to understand it in a less ad hoc manner.