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Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Chenyang Xu

Talk Title: The uniqueness of K-polystable Fano degeneration

Date: 01 / 31 / 19 **Time:** 2 : 00 am / pm (circle one)

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THE UNIQUENESS OF K-POLYSTABLE FANO DEGENERATION

CHENYANG XU

1. Moduli of K-semistable Fano varieties

There are families of Fano varieties over a punctured disk which are all isomorphic over the punctured disk, but different over the special point. This made people think that there isn't a good moduli theory for such objects. But a conjecture has emerged that there is a good moduli space if you restrict your attention to "K-semistable" Fano varieties. ("I'll let you imagine what the K stands for".)

Conjecture 1.1. *Fix a dimension n. Consider n-dimensional, volume V, K-semistable $\mathbf{Q}$-Fano varieties X.*

1. *This forms an Artin stack $\mathcal{M}^{kss}_{n,V}$ of finite type.*
2. *$\mathcal{M}^{kss}_{n,V}$ admits a good moduli space $\mathcal{M}^{kps}_{n,V}$ ("K-polystable") in the sense of Alper (meaning it looks locally like a GIT quotient.)*
3. *$\mathcal{M}^{kps}_{n,V}$ is separable, proper, and projective.*

The points on $\mathcal{M}^{kps}_{n,V}$ parametrize "k-polystable Fano varieties".

Definition 1.2 (Tian, Donaldson)**.** Let X be a Fano variety. Using $-rK_X$ (for sufficiently divisible r), we make a map

$$(X, -rK_X) \xrightarrow{|-rK_X|} \mathbf{P}^N.$$

There is an action of $\mathrm{PGL}(N+1)$ on $\mathbf{P}^N$. We make an action of $\mathbf{C}^* \hookrightarrow \mathrm{PGL}(N+1)$ on X into a family $(X \times \mathbf{C}^*, \mathcal{L}) \to \mathbf{C}^* \subset \mathbf{C}$. Then we consider the special fiber (X_0, L_0). It will be the case that

$$h^0(X_0, L_0^k) = a_0 k^n + a_1 k^{n-1} + \dots$$

Now, $\mathbf{C}^*$ acts on $H^0(X_0, L_0^k)$. If you look at equivariant cohomology, you get a total weight of

$$b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

We define $Fwt(\mathcal{X}, \mathcal{L}) = \frac{b_0 a_1 - a_0 b_1}{a_0^2}$.

We say that X is *K-semistable* iff $Fwt(\mathcal{X}, \mathcal{L}) \geq 0$ for all possible r and $\mathbf{C}^* \subset \mathrm{PGL}(N+1)$. For fixed r this is similar to a standard GIT problem, but we are allowing r to vary.

Evidently it is difficult to check this condition!

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2. Another definition of K-semistability

Next we give a new (but essentially equivalent definition), due to Bouchson-Hisamoto-Jonsson. Instead of degenerations look at the valuations on X. Fujita defined a β-invariant to a divisor $D \to X$:

$$\beta(D) = A_X(D)(-K_X)^n - \int_0^\infty \mathrm{Vol}(-k_X - tD)\, dt$$

where $A_X(D)$ is a "log discrepancy function".

Theorem 2.1 (Fuita, Li)**.** *X is K-semistable iff $\beta(D) \geq 0$ for all D.*

This turns the $n+1$-dimensional problem into an n-dimensional problem, since this is phrased only in terms of X itself.

Definition 2.2 (Fujita, Li)**.** Let X be $\mathbf{Q}$-Fano.

(1) X is K-semistable iff $\beta(D) \geq 0$ for all D.
(2) X is K-stable iff $\beta(D) > 0$ for all D.
(3) X is K-polystable iff X is K-semistable and if X specializes to a K-semistable X_0 then $X \cong X_0$.

Conjecturally, stable is equivalent to uniformly k-stable.

We can now come back to the K-moduli conjecture. When we consider all smoothable Fano X, we know the conjecture (perhaps except for the projectivity) [Li-Wang-Xu]. The proof heavily lies on analytic geometry – the Yau-Tian-Donaldson Conjecture, which was solved in the smooth case by Chen-Donaldson-Sun, and Tian.

There are a couple drawbacks. First, it is analytic whereas we would like an algebraic proof. Second, we need to assume the smoothability to get into the smooth case.

3. What is known?

The boundedness is known by Jiang (after Birkar).

Our main theorem:

Theorem 3.1 (Blum-Xu)**.** *Let $X \to C \leftarrow X'$ be two families of $\mathbf{Q}$-fano varieties over C, such that*

(1) Over $C^0 := C - \{0\}$, $X \times_C C^0 \cong X' \times_C C^0$,
(2) X_0, X'_0 are K-semistable.

Consider a section $s \in H^0(-mK_{X_0})$. Define a (decreasing) filtration $F^\bullet$ on $H^0(-mK_{X_0})$ as follows: $s \in F^r H^0(-mK_{X_0})$ iff there exists an extension $\widetilde{s} \in H^0(-mK_{X/C})$ such that $\mathrm{ord}_{X'_0}(\widetilde{s}) \geq r$. In other words, look at all possible extensions of X and look at the maximum possible vanishing order.

Lemma 3.2. *We have*

$$\bigoplus_m \bigoplus_r \mathrm{gr}^r H^0(-mK_{X_0}) \cong \bigoplus_{m'} \bigoplus_{r'} H^0(-mK_{X'_0})$$

sending the (m,r) *summand to the* (m,r') *summand where* $r' = m(a+a') - r$, $a = A_{X,X_0}(X_0')$ *and* $a' = A_{X',X_0}(x_0)$.

We want to take Proj, but we don't know if it's finitely generated. How do we show it? And what does this have to do with K-semistability?

Let

$$\beta(F) = a(-K_{X_0})^n - \int \operatorname{vol}(F^t)\,dt$$

where

$$\operatorname{vol}(F^t) = \lim_{m\to\infty} \frac{\dim F^{tm}(H^0(-mK_{X_0}))}{m^n/n!}.$$

Since this is positive, $\beta(F) + \beta(F') = 0 \implies \beta(F) = \beta(F') = 0$. But this is not enough.

Theorem 3.3 (Li-Wang-Xu, Blum-Xu)**.** *Let* X *be a* K*-semistable Fano variety,* D *such that* $\beta(D) = 0$*. Then*

$$\bigoplus_n \bigoplus_m H^0(-mK_X - nD)$$

is finitely generated.