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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Kathryn Hess

Talk Title: Topological Hochschild homology and topological cyclic homology: from classical to modern - I

Date: 2 / 6 / 19 **Time:** 9 : 00 am / pm (circle one)

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THH AND TC: CLASSICAL TO MODERN

KATHRYN HESS

1. Motivation and overview

This will be be a leadup from a classical perspective on topological Hochschild homology and cyclic homology to the recent work of Nikolaus-Scholze.

The ultimate goal is to compute algebraic K-theory: $K(R)$.

Why would we want to do this? For one thing, it contains a lot of important number-theoretic information. Furthermore, if R comes from geometry, e.g the group ring of π_1, then it encodes important geometric information as well.

But there is a big problem: it is really hard to compute! So we need tools to help us do this.

The method of attack that I'm going to describe is the following:

- We will construct and (hopefully) compute various approximations to algebraic K-theory, and
- determine how good these approximations are.

Notation: $\mathbf{T}$ will denote the circle group S^1.

2. Classical approximations

Let A be a flat $\mathbf{Z}$-algebra (perhaps simplicial, or differential graded...).

2.1. Hochschild homology. We will look at a certain simplicial construction to build a simplicial abelian group $\mathrm{HH}_*(A) \in \mathrm{sAb}$, with $\mathrm{HH}_n(A) = A^{\otimes(n+1)}$. The face maps

$$d_i \colon \mathrm{HH}_n(A) \to \mathrm{HH}_{n-1}(A)$$

is given by

$$a_0 \otimes \ldots \otimes a_n \mapsto \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & 0 \le i \le n \\ a_n a_0 \otimes a_1 \otimes \ldots a_{n-i} & i = n \end{cases}.$$

Evidently there is an action of C_{n+1} on $\mathrm{HH}_n(A)$. These cyclic actions realize to an action of $\mathbf{T}$ on $|\mathrm{HH}_*(A)|$. This leads to other invariants, including cyclic homology.

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2.2. **Cyclic homology.** We define

$$\mathrm{HC}(A) := \mathrm{HH}(A)_{h\mathbf{T}}.$$

We will later see an explicit and nice chain model for this.

We also have *negative cyclic homology*

$$\mathrm{HC}^{-}(A) := \mathrm{HH}(A)^{h\mathbf{T}}.$$

2.3. **Goodwillie's Theorem.** Anyway, we have these various invariants. But how good are they as approximations to algebraic K-theory?

Theorem 2.1 (Goodwillie '86)**.** *There is a natural map*

$$\mathrm{tr}\colon K_*(A) \to \mathrm{HC}^{-}_*(A)$$

for all simplicial rings A such that for all $f\colon A \to B$ which induce surjections on π_0 with nilpotent kernel,

$$\begin{tikzcd} K_*(A)\otimes\mathbf{Q} \arrow[r] \arrow[d] & \mathrm{HC}^{-}(A\otimes\mathbf{Q}) \arrow[d] \\ K_*(B)\otimes\mathbf{Q} \arrow[r] & \mathrm{HC}^{-}(B\otimes\mathbf{Q}) \end{tikzcd}$$

This means intuitively that the "difference" between $K_*(A)\otimes\mathbf{Q}$ and $K_*(B)\otimes\mathbf{Q}$ is the same as the "difference" between $\mathrm{HC}^{-}(A\otimes\mathbf{Q})$ and $\mathrm{HC}^{-}(B\otimes\mathbf{Q})$. To make this precise, one finds

$$K_*f\otimes\mathbf{Q} \cong \mathrm{HC}^{-}_*(f\otimes\mathbf{Q}).$$

Terminology: the composition $K_*(A) \xrightarrow{\mathrm{tr}} \mathrm{HC}^{-}_*(A) \to \mathrm{HH}_*(A)$ is called the Dennis trace.

Remark 2.2. You can think of the K-theory as some sort of higher version of the determinant. Under that analogy, what we're trying to do is understand determinants via traces.

Question: can we generalize Goodwillie's result to the non-$\mathbf{Q}$ case?

Answer: yes, but we'll need to enter the world of "Brave New Algebra". The slogan is that we replace ordinary commutative algebra over $\mathbf{Z}$ by "commutative ring spectra over the sphere spectrum $\mathbf{S}$".

3. Brave new algebra approximations

Let A be an associate ring spectrum. We'll discuss spectra more later; for now we just say that a spectrum is a sequence of spaces X_n, with maps $\Sigma X_n \to X_{n+1}$. It has a symmetric monoidal structure $X \wedge Y$, which you can think of as "tensor product over the sphere spectrum".

3.1. **Topological Hochschild homology.** Let A be an associative ring spectrum. The idea is to make a simplicial object in spectra $\mathrm{THH}_*(A)$, which will look like

$$\dots A \wedge A \rightrightarrows A$$

We have action of various cyclic groups levelwise, and when we take the geometric realization of $\mathrm{THH}_*(A)$ they will induce an action of $\mathbf{T}$-action on the spectrum $|\mathrm{THH}_*(A)|$.

Remark 3.1. There are some technical issues. The construction $A \rightsquigarrow \mathrm{THH}(A)$ is not going to be homotopy invariant unless we restrict to cofibrant A, for one thing. To get something homotopy invariant, we should replace A by an equivalent construction due to Bökstedt.

Bökstedt's model for $\mathrm{THH}(A)$ has the property that $\mathrm{THH}(A)^{C_{p^n}}$ is also homotopically meaningful (i.e. homotopy invariant in A).

3.2. **Brave new cyclic homology.** It turns out that $\mathrm{THH}(A)$ is a $\mathbf{T}$-spectrum (informally, a spectrum with an action of $\mathbf{T}$) together with "Frobenius maps". In other words, $\mathrm{THH}(A)$ is a "cyclotomic spectrum". This is used to construct the Brave New version of HC. (One doesn't just take homotopy fixed points or homotopy orbits – the cyclotomic structure is really used in a deep way.)

Brave New cyclic homology [Bökstedt-Hsian-Madsen '93]

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These F and R maps commute.

We'll use this to build something which is related to THH, but a better approximation to K-theory.

Definition 3.2. Fix a prime p. Set $\mathrm{TR}(A;p) := \underleftarrow{\mathrm{holim}}_{R^n} \mathrm{THH}(A)^{C_{p^n}}$. Then we define the "$p$-typical topological cyclic homology"

$$\mathrm{TC}(A;p) = \underleftarrow{\mathrm{holim}} \left(\mathrm{TR}(A;p) \underset{F}{\overset{\mathrm{Id}}{\rightrightarrows}} \mathrm{TR}(A;p) \right).$$

We can enhance this by "putting all the primes together". Let $\mathbf{P}$ be the set of all primes. Define

$$\begin{array}{ccc} \mathrm{TC}(A) & \longrightarrow & \mathrm{THH}(A)^{h\mathbf{T}} \\ \downarrow & & \downarrow \\ \prod_{p \in \mathbf{P}} \mathrm{TC}(A;p)^\wedge_p & \longrightarrow & \left(\prod_{p\in\mathbf{P}} \mathrm{THH}(A)^\wedge_p\right)^{h\mathbf{T}} \end{array}$$

where $\mathrm{THH}(A)^\wedge_p$ denotes the "p-completion of $\mathrm{THH}(A)$".

Recall that the whole point of this was to get a non-rational version of Goodwillie's result. This will work out, with HC^- replaced by TC.

Theorem 3.3 (Dundas-Goodwillie-McCarthy '13)**.** *"The difference between K and* TC *is locally constant." More precisely, there exist natural trace maps*

$$K(A) \xrightarrow{\mathrm{trc}} \mathrm{TC}(A) \to \mathrm{THH}(A)$$

(trc is the "cyclotomic trace" and the composition is called the "Dennis trace") such that for all morphisms of ring spectra $f\colon A \to B$ with $\pi_0(f)$ a surjection with nilpotent kernel, the diagram

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is homotopy cartesian.

4. ∞-categorical innovations

First of all, why ∞-categories (as opposed to model categories)? There are situations that cannot be expressed cleanly, or at all, in the world of model categories. For example, Nikolaus-Scholze significantly cleaned up the theory of cyclotomic spectra using this formalism.

[Blumberg-Gepner-Tabuada '13]

- Connective algebraic K-theory is the "universal additive invariant". (By additive, we mean it inverts Morita equivalences, preserves filtered colimits, and sends split exact sequences to cofiber sequences.) The universal property means that any additive invariant receives a map from algebraic K-theory. This is useful for producing maps out of K-theory.
- Nonconnective algebraic K-theory is the "universal localizing invariant" (meaning it is additiev and sends *all* exact sequences to cofiber sequences). It turns out that THH, TC are also localizing, so the universal property gives trace maps $K \to \mathrm{TC}$ and $K \to \mathrm{THH}$.

What is [Nikolaus-Scholze '13] about?

- They provide an elegant ∞-categorical approach to cyclotomic spectra. It is *much* simpler and cleaner than the usual approach. This leads to a very clean description of TC.
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5. A few computations

The Nikolaus-Scholze approach makes it easier to do some computations of THH and TC that were otherwise out of reach. This is explained in [Hesselholt-Nikolaus]

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