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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Yuri Tschinkel

Talk Title: Rationality problems

Date: 2 / 6 / 19 **Time:** 11 : 45 [✓ **am**] **/ pm (circle one)**

Please summarize the lecture in 5 or fewer sentences:

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

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- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

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☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Rationality problems

CALC II: INTEGRATION

- $\mathbb{A}^1 \subset \mathbb{P}^1$
- $H_p(x) = \max(1, |x|_p)$, $x \in \mathbb{A}^1(\mathbb{Q}_p)$
- $$U(0) := \{x \,|\, |x|_p \leq 1\}$$
  $$U(j) := \{x \,|\, |x|_p = p^j\}, \quad \mathrm{vol}(U(j)) = p^j(1 - \frac{1}{p})$$

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$$\begin{aligned} \textstyle\int_{\mathbb{Q}_p} H(x_p)^{-s}\mathrm{d}x_p &= \textstyle\int_{U(0)} H(x_p)^{-s}\mathrm{d}x_p + \sum_{j\geq 1} \int_{U(j)} H(x_p)^{-s}\mathrm{d}x_p \\ &= 1 + \textstyle\sum_{j\geq 1} p^{-js}\mathrm{vol}(U(j)) \\ &= \tfrac{1-p^{-s}}{1-p^{-(s-1)}} \end{aligned}$$

LEADING CONSTANT

Put $s = 2$:

$$\int \ldots = (1 + \frac{1}{p}) = \frac{\#\mathbb{P}^1(\mathbb{F}_p)}{p}$$

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We interpret this as a volume with respect to a natural measure.

HEIGHTS

- $F/\mathbb{Q}$ number field
- $X = X_F$ projective algebraic variety over F
- $X(F)$ its F-rational points
- $\mathcal{L} = (L, \|\cdot\|)$ adelically metrized very ample line bundle
- $H_{\mathcal{L}} : X(F) \to \mathbb{R}_{>0}$ associated height,
  depends on the metrization (choice of norms)

TAMAGAWA NUMBERS / PEYRE (1995)

Let X be a smooth projective Fano variety over F of dimension d. Assume that $-K_X$ is equipped with an adelic metrization.

For $x \in X(F_v)$ choose local analytic coordinates $x_1, \ldots, x_d$, in a neighborhood U_x. In U_x, a section of the canonical line bundle has the form $\mathsf{s} := \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$. Put

$$\tau_v = \tau_{X,v} := \|\mathsf{s}\|_v \mathrm{d}x_1 \cdots \mathrm{d}x_d,$$

where $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ is the standard normalized Haar measure on F_v^d. It globalizes to $X(F_v)$.

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where $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ is the standard normalized Haar measure on F_v^d. It globalizes to $X(F_v)$. For almost all v, and Zariski open $U \subset X$,

$$\int_{U(F_v)} \tau_v = \int_{X(F_v)} \tau_v = \int_{X(\mathfrak{o}_v)} \tau_v = \sum_{\tilde{x} \in X(\mathbb{F}_q)} \int_{\pi^{-1}(\tilde{x})} \tau_v = \frac{X(\mathbb{F}_q)}{q^d}.$$

BIRATIONAL CALABI-YAU (BATYREV 1997)

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- $$\int_{X(F_v)} \tau_v = \frac{X(\mathbb{F}_q)}{q^n} \quad \forall' v$$
- If $X \supset U \subset Y$, then

$$\frac{X(\mathbb{F}_q)}{q^n} = \int_{X(F_v)} \tau_v = \int_{U(F_v)} \tau_v = \int_{Y(F_v)} \tau_v = \frac{Y(\mathbb{F}_q)}{q^n}, \quad \forall q$$

DERIVED EQUIVALENT K3 SURFACES

LIEBLICH-OLSSON 2011

Let X and Y be derived equivalent K3 surfaces over $k = \mathbb{F}_q$. Then

$$|X(k)| = |Y(k)|.$$

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Let X and Y be derived equivalent K3 surfaces over $k = \mathbb{F}_q$. Then

$$|X(k)| = |Y(k)|.$$

Can this be viewed as an identity of p-adic integrals?

IGUSA INTEGRALS: LOCAL THEORY

Let $U := X \setminus D$, with

$$D = \cup_{\alpha \in \mathcal{A}} D_\alpha, \quad -K_X = \sum \rho_\alpha D_\alpha,$$

where D_α are geometrically irreducible, smooth, and intersecting transversally.

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$D_A \subset X$ is smooth, of codimension $\#A$ (or empty).

LOCAL HEIGHTS

Let

$$H_\alpha : U(F_v) \to \mathbb{R}_{\geq 0}$$

be the v-adic distance to the boundary component D_α.

Example: $U = \mathbb{P}^1 \setminus \{0, \infty\}$,

$$H_0(x) := \max(1, |x_0|/|x_1|), \quad H_\infty := \max(1, |x_1|/|x_0|)$$

HEIGHT INTEGRALS

$$Z_v(\mathbf{s}) := \int_{U(F_v)} \prod_{\alpha \in \mathcal{A}} H_\alpha(x)^{-s_\alpha} \mathrm{d}\tau_v$$

Local computations

In charts, via partition of unity: in a neighborhood of $x \in D_A^\circ(F)$ it takes the form

$$\int \prod_{\alpha \in A} |x_\alpha|_v^{s_\alpha - \rho_\alpha} \, \mathrm{d}\tau_v$$

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Essentially, this is a product of integrals of the form

$$\int_{|x|_v \leq 1} |x|_v^{s-1} \mathrm{d}x_v.$$

DENEF'S FORMULA

For almost all v one has:

$$Z_v(\mathbf{s}) = \sum_A \frac{\# D_A^\circ(\mathbb{F}_q)}{q^{\dim X}} \prod_{\alpha \in A} \frac{q-1}{q^{s_\alpha - \rho_\alpha + 1} - 1}.$$

APPLICATIONS

The integral

- is an invariant under blowups,
- encodes information about singularities of X,
- plays a central role in analytic/spectral approches to Manin's conjectures, volume asymptotics, etc.

BASIC QUESTIONS

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- How much geometry can be read off from arithmetic?

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(S) stably rational: if $X \times \mathbb{P}^n$ is rational, for some n

(U) unirational: if $\mathbb{P}^n \dashrightarrow X$, for some n

CLASSICAL RESULTS, OVER $\mathbb{C}$

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- $\dim(X) = 3$: (U) $\not\Rightarrow$ (R)
  - Iskovskikh-Manin: quartic in $\mathbb{P}^4$ via birational rigidity
  - Clemens-Griffiths: cubic in $\mathbb{P}^4$ via intermediate Jacobians
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- $\dim(X) = 3$: (S) $\not\Rightarrow$ (R)
  - Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer: via universal torsors

SPECIALIZATION OF (STABLE) RATIONALITY

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- Nicaise–Shinder (2017): $\mathrm{K}_0(Var_k)/\mathbb{L}$, $\mathrm{char}(k) = 0$
- Kontsevich–T. (2017): $\mathrm{Burn}(k)$, $\mathrm{char}(k) = 0$

SPECIALIZATION OF (STABLE) RATIONALITY

- **Larsen–Lunts (2003):** $\mathrm{K}_0(Var_k)/\mathbb{L}$ is isomorphic to the free abelian group spanned by classes of algebraic varieties over k, modulo stable rationality. Key: Bittner's presentation of $\mathrm{K}_0(Var_k)$: smooth proper varieties + blowup relations, and the Weak Factorization theorem for birational maps.
- **Nicaise–Shinder (2017):** motivic reduction – formula for homomorphism

  $$\mathrm{K}_0(Var_K)/\mathbb{L} \to \mathrm{K}_0(Var_k)/\mathbb{L}, \quad K = k((t)),$$

  inspired by motivic integration as in Denef–Loeser, , ...
- **Kontsevich–T. (2017):** Same formula for

  $$\mathrm{Burn}(K) \to \mathrm{Burn}(k),$$

  the free abelian group spanned by classes of varieties over the corresponding field, modulo rationality.

SPECIALIZATION (KONTSEVICH-T. 2017)

- Let $\mathfrak{o} \simeq k[[t]]$, $K \simeq k((t))$, $\mathrm{char}(k) = 0$.
- Let X/K be a smooth proper (or projective) variety of dimension n, with function field $L = K(X)$.
- Choose a regular model

  $$\pi : \mathcal{X} \to \mathrm{Spec}(\mathfrak{o}),$$

  such that π is proper and the special fiber $\mathcal{X}_0$ over $\mathrm{Spec}(k)$ is a simple normal crossings (snc) divisor:

  $$\mathcal{X}_0 = \cup_{\alpha \in \mathcal{A}} \, d_\alpha D_\alpha, \quad d_\alpha \in \mathbb{Z}_{\geq 1}.$$

- Put

  $$\rho_n([L/K]) := \sum_{\emptyset \neq A \subseteq \mathcal{A}} (-1)^{\#A-1} [D_A \times \mathbb{A}^{\#A-1}/k] \in \mathrm{Burn}(k),$$

EQUIVARIANT BIR. TYPES (KONTSEVICH-T. 2019)

- G - finite abelian group, $A = G^\vee = \mathrm{Hom}(G, \mathbb{G}_m)$
- X - smooth projective variety, with G-action
- $$\beta : X \mapsto \sum_\alpha [F_\alpha, [\ldots]], \quad X^G = \sqcup F_\alpha.$$

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- $$\beta : X \mapsto \sum_\alpha [F_\alpha, [\ldots]], \quad X^G = \sqcup F_\alpha.$$

Let $\tilde{X} \to X$ be a G-equivariant blowup. Consider relations

$$\beta(\tilde{X}) - \beta(X) = 0.$$

BIRATIONAL TYPES $\mathcal{B}_n(G)$

Fix an integer $n \geq 2$. Consider the $\mathbb{Z}$-module

$$\mathcal{B}_n(G) \quad \text{generated by} \quad [a_1, \ldots, a_n], \quad a_i \in A$$

such that $a_1, \ldots, a_n$ generate A, i.e., $\sum_i \mathbb{Z}a_i = A$, and

(S) for all $\sigma \in \mathfrak{S}_n$, $a_1, \ldots, a_n \in A$ we have

$$[a_{\sigma(1)}, \ldots, a_{\sigma(n)}] = [a_1, \ldots, a_n],$$

(B) for all $2 \leq k \leq n$, all $a_1, \ldots, a_k \in A$, $b_1, \ldots, b_{n-k} \in A$ such that

$$\sum_i \mathbb{Z}a_i + \sum_j \mathbb{Z}b_j = A$$

we have

$$[a_1, \ldots, a_k, b_1, \ldots b_{n-k}] =$$

$$= \sum_{1 \leq i \leq k,\ a_i \neq a_{i'}, \forall i' < i} [a_1 - a_i, \ldots, a_i, \ldots, a_k - a_i, b_1, \ldots, b_{n-k}]$$

MODULAR/MOTIVIC TYPES $\mathcal{M}_n(G)$

Fix an integer $n \geq 2$. Consider the $\mathbb{Z}$-module

$$\mathcal{M}_n(G) \quad \text{generated by} \quad \langle a_1, \ldots, a_n \rangle, \quad a_i \in A$$

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we have

$$\langle a_1, \ldots, a_k, b_1, \ldots b_{n-k} \rangle =$$

$$= \sum_{1 \leq i \leq k} \langle a_1 - a_i, \ldots, a_i, \ldots, a_k - a_i, b_1, \ldots, b_{n-k} \rangle$$

BIRATIONAL TYPES

KONTSEVICH-T. 2019

The class

$$\beta(X) \in \mathcal{B}_n(G)$$

is a well-defined G-equivariant birational invariant.

BIRATIONAL TYPES → MODULAR TYPES

Consider the map

$$\mu : \mathcal{B}_n(G) \to \mathcal{M}_n(G)$$

$$
\begin{array}{lll}
(\boldsymbol{\mu}_0) & [a_1, \dots, a_n] \mapsto \langle a_1, \dots, a_n \rangle, & \text{if all } a_1, \dots, a_n \neq 0, \\
(\boldsymbol{\mu}_1) & [0, a_2, \dots, a_n] \mapsto 2\langle 0, a_2, \dots, a_n \rangle, & \text{if all } a_2, \dots, a_n \neq 0, \\
(\boldsymbol{\mu}_2) & [0, 0, a_3, \dots, a_n] \mapsto 0, & \text{for all } a_3, \dots, a_n,
\end{array}
$$

and extended by $\mathbb{Z}$-linearity.

BIRATIONAL TYPES → MODULAR TYPES

KONTSEVICH–T. 2019

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This would follow from:

For any integer $N \geq 2$,

$$[0,0,1] \in \mathcal{B}_3(\mathbb{Z}/N\mathbb{Z})$$

is a torsion element of order a power of 2.

BIRATIONAL TYPES: $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$

Assume that

$$G = \mathbb{Z}/p\mathbb{Z} \simeq A.$$

Then $\mathcal{B}_2(G)$ is generated by symbols $[a_1, a_2]$ such that

$$a_1, a_2 \in \mathbb{Z}/p\mathbb{Z}, \quad \gcd(a_1, a_2, p) = 1,$$

and

- $[a_1, a_2] = [a_2, a_1]$,
- $[a_1, a_2] = [a_1, a_2 - a_1] + [a_1 - a_2, a_2]$, where $a_1 \neq a_2$,
- $[a, a] = [a, 0]$, for all $a \in \mathbb{Z}/p\mathbb{Z}$, $\gcd(a, p) = 1$.

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- $\langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle$,
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and

- $\langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle$,
- $\langle a_1, a_2 \rangle = \langle a_1, a_2 - a_1 \rangle + \langle a_1 - a_2, a_2 \rangle$, for all a_1, a_2.

The only difference: $[a, a] = [a, 0], \quad \langle a, a \rangle = 2\langle a, 0 \rangle$.

HECKE OPERATORS ON $\mathcal{M}_n(G)$

The modular groups carry (commuting) Hecke operators:

$$T_{\ell,r} : \mathcal{M}_n(G) \to \mathcal{M}_n(G) \quad 1 \leq r \leq n-1$$

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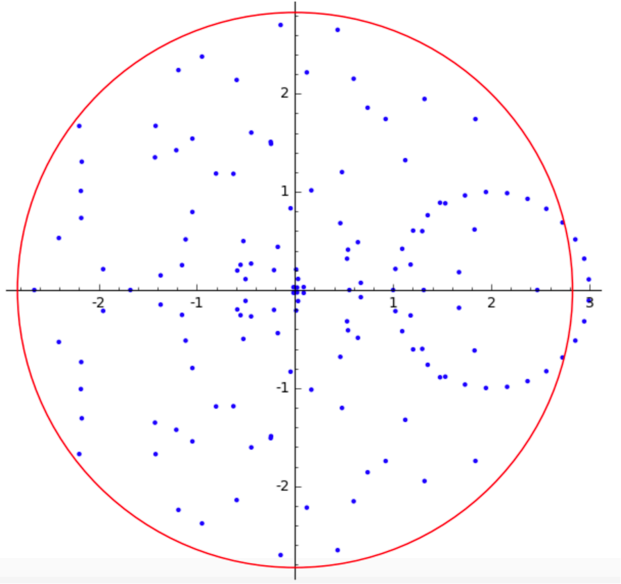
Example:

$$T_2(\langle a_1, a_2\rangle) = \langle 2a_2, a_2\rangle + \big(\langle a_1 - a_2, 2a_2\rangle + \langle 2a_1, a_2 - a_1\rangle\big) + \langle a_1, 2a_2\rangle.$$

EIGENVALUES OF T_2 ON $\mathcal{M}_2(\mathbb{Z}/59\mathbb{Z})$



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Jumps at

$$p = 43, 59, 67, 83, ...$$

MODULAR SYMBOLS

$\mathbb{M}_2(\Gamma_1(N))$ is generated by symbols (c, d), $c, d \in \mathbb{Z}/N\mathbb{Z}$, $\gcd(c, d, N) = 1$, subject to relations

- $(c, d) + (d, -c) = 0$
- $(c, d) + (d, -c - d) + (-c - d, c) = 0$

$$\dim S_2(\Gamma_1(p)) = 1 + \frac{(p-1)(p+1)}{24} - \frac{(p-1)}{2}$$

MODULAR SYMBOLS

There is an involution on $\mathbb{M}_2(\Gamma_1(N))$

$$\iota : (c,d) \mapsto -(-c,d).$$

The (+)-eigenspace for ι contains the space

$$\mathbb{M}_2^+(\Gamma_1(N))'$$

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$$\mathbb{M}_2^+(\Gamma_1(N))' \simeq \mathcal{M}_2(\mathbb{Z}/N\mathbb{Z})$$

COMPUTATIONS: $\mathcal{B}_2(\mathbb{Z}/N\mathbb{Z})$

$$\Delta_{2,\ell}(\mathbb{Z}/p\mathbb{Z}) := \dim(\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})_{\mathbb{F}_\ell}) - \frac{p^2+23}{24}$$

varies; there are frequent jumps for $\ell \mid p \pm 1$, e.g.,

$$\Delta_{2,31}(\mathbb{Z}/61\mathbb{Z}), \Delta_{2,13}(\mathbb{Z}/79\mathbb{Z}), \Delta_{2,11}(\mathbb{Z}/89\mathbb{Z}) = 1$$

COMPUTATIONS: $\mathcal{B}_3(\mathbb{Z}/N\mathbb{Z})$

For p a prime,

$$\Delta_{3,\mathbb{Q}}(\mathbb{Z}/p\mathbb{Z}) := \dim(\mathcal{B}_3(\mathbb{Z}/p\mathbb{Z})_{\mathbb{Q}}) - \frac{(p-5)(p-7)}{24} = 0$$

for all primes up to 41, but

$$\Delta_{3,\mathbb{Q}}(\mathbb{Z}/p\mathbb{Z}) = 1, \text{ for } p = 43, 59, 79, \ldots \text{ and } \Delta_{3,\mathbb{Q}}(\mathbb{Z}/163\mathbb{Z}) = 10.$$

$\dim(\mathcal{B}_4(G)_{\mathbb{Q}})$ AND 2-TORSION JUMP

21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**
0	0	0	0	0	0	1	0	0	0	0	0	2	0	0	3	0	0	4	0	0	2	1	2	8	0	0	7	0	0
0	2	0	5	1	4	0	8	1	10	0	13	1	10	4	16	1	14	2	33	3	27	0	31	7	25	0	49	5	43

BIRATIONAL TYPES: SUMMARY

- Construction of groups related to $\mathcal{B}_n(G)$
- Definition of Hecke operators on these groups
- Nonabelian versions
- Refined G-equivariant birational invariants...
- Unexpected connection between the Cremona group and automorphic forms (cohomology of congruence subgroups)

SPECIALIZATION METHOD: FIRST APPLICATIONS

Very general varieties below are not stably rational:

- Quartic double solids $X \to \mathbb{P}^3$ with ≤ 7 double points (Voisin 2014)
- Quartic threefolds (Colliot-Thélène–Pirutka 2014)
- Sextic double solids $X \to \mathbb{P}^3$ (Beauville 2014)
- Fano hypersurfaces of high degree (Totaro 2015)
- Cyclic covers $X \to \mathbb{P}^n$ of prime degree (Colliot-Thélène–Pirutka 2015)
- Cyclic covers $X \to \mathbb{P}^n$ of arbitrary degree (Okada 2016)
- Quadric bundles and Fano hypersurfaces of low degree (Schreieder 2017-18)

CONIC BUNDLES OVER RATIONAL SURFACES

THEOREM (HASSETT–KRESCH–T. 2015)

Very general conic bundles over rational surfaces with discriminant of sufficiently large degree are not stably rational.

THEOREM (KRESCH–T. 2017)

Same for Brauer-Severi surface bundles.

DEL PEZZO FIBRATIONS

THEOREM (HASSETT-T. 2016)

A very general fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ *in quartic del Pezzo surfaces which is not rational and not birational to a cubic threefold is not stably rational.*

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THEOREM (KRYLOV-OKADA 2017)

A very general del Pezzo fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ *of degree 1, 2, or 3 which is not rational and not birational to a cubic threefold is not stably rational.*

FANO THREEFOLDS

THEOREM (HASSETT-T. 2016)

A very general nonrational Fano threefold X which is not birational to a cubic threefold is not stably rational.

Idea of proof: Specialize to quartic del Pezzo fibrations over $\mathbb{P}^1$.

Conclusion: Stable rationality for very general rationally connected threefolds (with the exception of cubic threefolds) is settled, via degeneration to varieties admitting only rational double points as singularities.

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Conclusion: Stable rationality for very general rationally connected threefolds (with the exception of cubic threefolds) is settled, via degeneration to varieties admitting only rational double points as singularities. There are no obvious obstructions to stable rationality before specialization.

RATIONALITY IN FAMILIES

Let $\pi : \mathcal{X} \to B$ be a smooth family of rationally connected varieties and put

$$\operatorname{Rat}(\pi) := \{\, b \in B \mid \mathcal{X}_b \quad \text{is rational} \ \}.$$

$\operatorname{Rat}(\pi)$ is a countable union of closed subsets of B.

- Timmerscheidt (1981), de Fernex–Fusi (2012) in dim $= 3$,
- Kontsevich–T. (2017) in general.

Rat(π): HASSETT-PIRUTKA-T. (2016)

Rat(π) and its complement can be dense on the base.

Consider $(2, 2)$-hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$, over $\mathbb{C}$. Then

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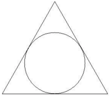
Idea of proof: via specialization to a quadric surface bundle degenerating along



- Computing $\mathrm{H}^2_{nr}(X, \mathbb{Z}/2)$: general approach by Pirutka (2016);
- Desingularization: by hand;
- Rationality: goes back to Hassett's 1999 treatment of cubic fourfolds with a plane.

FURTHER EXAMPLES

- Hassett–Pirutka–T. 2016: Quartic double covers of $\mathbb{P}^4$
- Hassett–Pirutka–T. 2017: Intersections of three quadrics in $\mathbb{P}^7$
- Schreieder 2017: general approach to quadric bundles in higher dimensions

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There exist nonrational forms of $\bar{\mathcal{M}}_{0,6}$ (Florence–Reichstein 2017).

FANO THREEFOLDS OVER $\mathbb{C}(t)$ (HASSETT-T. 2018)

TARGET THEOREM

Let X be a nontoric geometrically rational smooth Fano threefold, very general in its family. Then X is not stably rational over k.

FANO THREEFOLDS OVER $\mathbb{C}(t)$ (HASSETT–T. 2018)

Our favorite example:

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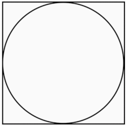

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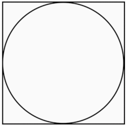

- Pirutka's computation of H^2_{nr}, singularities as before.

FANO THREEFOLDS OVER $\mathbb{C}(t)$ (HASSETT–T. 2018)

- It also admits a degeneration to a singular toric intersection of two quadrics, $\mathbb{P}^3$ blown up in 4 points:

$$x_0x_1 - x_2x_3 = x_2x_3 - x_4x_5 = 0.$$

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Toric degenerations of Fano threefolds have been worked out in connection with mirror symmetry. Those of interest to us admit (?) degenerations to singular toric varieties X_Σ with symmetries compatible with nontrivial

$$\mathrm{H}^1(\mathrm{Gal}, \mathrm{Pic}(\bar{X}_\Sigma)).$$