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Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Karl Schwede

Talk Title: Birational algebraic geometry in positive characteristic

Date: 2 / 5 / 19 **Time:** 3 : 30 am / [✓ pm] (circle one)

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BIRATIONAL ALGEBRAIC GEOMETRY IN POSITIVE CHARACTERISTIC

KARL SCHWEDE

1. Recap

We begin by recalling a prototypical argument from last time.

Let X be a normal CM variety over a field of characteristic $p > 0$. Let $H \subset X$ be a reduced Cartier divisor. We have $F^e_* \omega_X \to \omega_X$ (dual to $\mathcal{O}_X \to F^e_* \mathcal{O}_X$). This induces

$$0 \to \omega_X \to \omega_X(H) \to \omega_H \to 0.$$

Let L be an ample line bundle on X. We have an exact sequence

$$\begin{tikzcd} H^0(\omega_X(L)) \arrow[r] & H^0(\omega_X(H+L)) \arrow[r] & H^0(\omega_H(L|_H)) \arrow[r] & H^1(\omega_X(L)) \end{tikzcd}$$

but since we are not in characteristic 0, "we don't know that the map $H^0(\omega_X(H+L)) -> H^0(\omega_H(L|_H))$ is surjective.

However, we have a diagram

$$\begin{tikzcd} H^0(\omega_X(L)) \arrow[r] & H^0(\omega_X(H+L)) \arrow[r] & H^0(\omega_H(L|_H)) \arrow[r] & H^1(\omega_X(L)) \\ H^0(F^e_*\omega_X(p^eL)) \arrow[r] \arrow[u] & H^0(F^e_*\omega_X(H+p^eL)) \arrow[r] \arrow[u] & H^0(F^e_*\omega_H(p^eL|_H)) \arrow[r] \arrow[u] & H^1(F^e_*\omega_X(p^eL)) = 0 \arrow[u] \end{tikzcd}$$

If X is Frobenius-split, then the map $H^0(F^e_*\omega_H(p^eL|_H)) \to H^0(\omega_H(L|_H))$ is surjective, so we get the desired surjectivity.

If $F^e_*\omega_H \to \omega_H$ is not surjective, then the sections of $H^0(\omega_H(L|_H))$ not in the image of $H^0(F^e_*\omega_H)$ cannot be lifted via our argument.

Today we will discuss local obstructions to this surjectivity.

2. Frobenius splitting

First we discuss some non-obvious ways of getting Frobenius splittings.

Theorem 2.1 (Kunz). *Let Y be a variety over $k = k^p$. Then Y is regular if and only if $F^e_*\mathcal{O}_Y$ is locally free for some (equivalently, all) $e > 0$.*

Suppose you have a surjective projection (not necessarily a Frobenius splitting) $F^e_*\mathcal{O}_{Y,y} \twoheadrightarrow \mathcal{O}_{Y,y}$. There is a map $F^e_*\mathcal{O}_{Y,y} \to F^e_*\mathcal{O}_{Y,y}$ sending $1 \mapsto r$. Then we have the map $\mathcal{O}_{Y,y} \to F^e_*\mathcal{O}_{Y,y}$ sending $1 \mapsto 1$.

Conclusion: any map $F^e_*\mathcal{O}_{Y,y} \twoheadrightarrow \mathcal{O}_{Y,y}$ gives a Frobenius splitting.

Date: February 4, 2019.

3. Frobenius divisor

Definition 3.1. Let R be a normal local ring of characteristic $p > 0$. Frobenius is finite. Suppose you have a non-zero map $\varphi \in \operatorname{Hom}(F^e_* R, R) = F^e_* \omega_R^{\otimes(1-p^e)} = F^e_* R((1-p^e)K_R)$. The map φ determines $D_\varphi \sim (1-p^e)K_R$. Normalize with respect to p, e:

$$\Delta_\varphi := \frac{1}{p^e - 1} D_\varphi \sim_{\mathbf{Q}} -K_R$$

i.e. $K_R + \Delta_\varphi \sim_{\mathbf{Q}} 0$.

Why is Δ_φ interesting?

1. To get improved vanishing statements in characteristic 0.
2. To cook up sections of $\omega_X \otimes L$.
3. To keep track of canonical divisors are you change varieties.

Example 3.2. If you have a finite surjective map $f\colon Z \to Y$ between normal varieties in characteristic $p > 0$, and $\varphi\colon F^e_* \mathcal{O}_Y \to \mathcal{O}_Y$, we have

$$K_Z = f^* K_W + \text{ram}.$$

We can rewrite this as

$$K_Z + \underbrace{(f^*\Delta_\varphi - \text{ram})}_{=:\Delta_Z} = f^*(K_Y + \Delta_\varphi)$$

Assume further that f is generically separable. It turns out that Δ_Z corresponds to the unique extension of φ to $\varphi_Z\colon F^e_* \mathcal{O}_Z \to \mathcal{O}_Z$. (It exists if and only if $\Delta_Z \geq 0$.) This could be useful to getting a Frobenius splitting.

4. Log-canonical pairs

What if we just have a birational map? Suppose $\pi\colon \widetilde{Y} \to Y$ is a proper birational map. Then we can write

$$K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} = \pi^*(K_Y + \Delta_\varphi).$$

Definition 4.1. (Y, Δ_Y) is *log-canonical* if when I write

$$K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} = \pi^*(K_Y + \Delta_Y)$$

(for any birational map π), the coefficients of $\Delta_{\widetilde{Y}} \leq 1$.

It turns out that the condition that the coefficients of $\Delta_\varphi \leq 1$ is equivalent to φ being surjective in codimension 1.

Theorem 4.2 (Hara-Watanabe, Smith-Schwede)**.** *If R is F-split, then there exists $\Delta_\varphi \geq 0$ such that (R, Δ_φ) is log-canonical.*

Corollary 4.3. *If (R, Δ) is a pair over $\mathbf{Q}$ and $K_R + \Delta$ is $\mathbf{Q}$-Cartier, with the property that the pair reduced modulo p is F-pure for infinitely many p, then (R, Δ) is log-canonical.*

5. F-RATIONAL RINGS

Definition 5.1. We say R is *F-rational* if:

(1) $F^e_*\omega_R \to \omega_R$ has no stable submodules.
(2) R is Cohen-Macaulay.

Compare to the definition of rationality: R is Cohen-Macaulay and $\pi_*\omega_{\widetilde{Y}} = \omega_Y$.

Theorem 5.2 (Smith)**.** *F-rational implies rational in characteristic p.*

Proof. $\pi_*\omega_{\widetilde{Y}} \subset \omega_Y$ is stable under $F^e_*\omega_Y \to \omega_Y$. □

Theorem 5.3 (Hara-Mehta-Srinivas)**.** *R has rational singularities in characteristic 0 if and only if $R \pmod p$ has F-rational singularities for a Zariski-dense set of p.*

It turns out that we have some ways of checking whether varieties in characteristic p are F-pure, F-rational, F-split, etc.

Example 5.4. A cone over an F-split variety is F-pure.

Theorem 5.5 (Fedder)**.** *Let $R = S/I$ where $S = k[x_1, \dots, x_n]$. Then R is F-pure at a maximal ideal $\mathfrak{m}$ if and only if $I^{[p^e]} : I$ is not contained in $\mathfrak{m}^{[p^e]}$.*

Here $I^{[p^e]}$ is the ideal generated by p^eth powers of elements of I and

$$I : J := \{r \in R \colon rJ \subset I\}$$

Example 5.6. Consider $f = zy^2 - x(x - \lambda z)(x + z)$, which cuts out a cone over an elliptic curve. We want to know if $f^{p-1} \in (x^p, y^p, z^p)$? The issue is the coefficient of $x^{p-1}y^{p-1}z^{p-1}$, which is basically the Hasse invariant. So we recover the fact that the cone over an ordinary elliptic curve is F-pure; recall Example 5.4 and the fact that ordinary abelian varieties are F-split.