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Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Sam Raskin

Talk Title: The notion of singular support in DAG and its applications III

Date: 2 / 7 / 19 **Time:** 2 : 30 am / pm (circle one)

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THE NOTION OF SINGULAR SUPPORT IN DAG AND ITS APPLICATIONS III

SAM RASKIN

1. Recap

We discussed that for lci X, $\mathbf{T}_X$ has amplitude in $[0,1]$.

We defined $\mathrm{Sing}(X) = \mathrm{Spec}_{X^{\mathrm{red}}} \mathrm{Sym}_{\mathcal{O}_X^{\mathrm{red}}}(H^1(\mathbf{T}_X|_{X^{\mathrm{red}}}))$.

For $\mathcal{N} \subset \mathbf{P}\mathrm{Sing}(X)$, we defined $\mathrm{IndCoh}_{\mathcal{N}}(X) \subset \mathrm{IndCoh}(X)$.

The picture to have in mind: for usual quasicoherent sheaves, the support comes from the action of central elements, namely the functions. Here the central elements were in degree 2, hence are "infinitesimal".

Example 1.1. For $X = 0 \times_V 0 = \mathrm{Spec}\ \mathrm{Sym}(V^\vee[1]))$. There is a Koszul duality

$$\mathrm{IndCoh}(X) \cong \mathrm{Sym}(V[-2]) - \mathrm{mod}.$$

Why? We have the object $k := \iota_* \mathcal{O}_0 \in \mathrm{IndCoh}(X)$ for $\iota \colon 0 \to X$. You can compute its self-Ext, and find that it's $\mathrm{Sym}(V[-2])$. The equivalence sends $k \mapsto \mathrm{Sym}(V[-2])$.

In this case $\mathrm{Sing}(X) = V^\vee$. The singular support of $\mathcal{F} \in \mathrm{Coh}$ is the same as the support of $H^*(KD(\mathcal{F}))$ under the action of $\mathrm{Sym}\, V$. (In this setup, $H^*(KD(\mathcal{F}))$ will be a finitely generated graded module.)

2. Drinfeld's characterization

Let $X = Y \times_V 0$ for Y smooth and V a vector space. Then $\mathrm{Sing}(X) \subset X^{\mathrm{red}} \times V^\vee$. Then for $\mathcal{F} \in \mathrm{Coh}(X)$, we have $(x \in X, \lambda \in V^\vee - 0) \notin \mathrm{ssupp}(\mathcal{F})$ if and only if $\iota_{\lambda *}\mathcal{F} \in \mathrm{Perf}(Y')$ in a neighborhood of x, where

$$\iota_\lambda \colon X \hookrightarrow Y' := \{\lambda = 0\} \subset Y.$$

So the singular support measures the directions in which $\mathcal{F}$ is not perfect.

3. Functorial behavior

For $f \colon X \to Y$ a map between lci DG schemes.

We have a diagram

$$\begin{tikzcd} \mathrm{Sing}(Y) \times_{Y^{\mathrm{red}}} X^{\mathrm{red}} \arrow[r, "\delta"] \arrow[d, "\pi"] & \mathrm{Sing}(X) \\ \mathrm{Sing}(Y) & \end{tikzcd}$$

Date: February 5, 2019.

(The situation is formally similar to that for cotangent bundles, because we're using H^{-1} instead of H^0.)

For $\mathcal{N}_X \subset \mathbf{P}\operatorname{Sing}(X)$,

$$f_*(\mathcal{N}_X) := (\pi\delta^{-1}(\mathcal{N}_X^{\mathrm{aff}}) - Y)/\mathbf{G}_m$$

This looks overcomplicated because of annoyances of adding and removing the 0 section, but informally you just "pull and push".

Similarly for $\mathcal{N}_Y \subset \mathbf{P}\operatorname{Sing}(Y)$, define

$$f^*(\mathcal{N}_Y) \subset \mathbf{P}\operatorname{Sing}(X).$$

Theorem 3.1 (Arinkin-Gaitsgory)**.** *For $f\colon X \to Y$ a map between lci DG schemes.*

(1) Let $\mathcal{N}_X \subset \mathbf{P}\operatorname{Sing}(X)$. Then

$$f_*^{\mathrm{IndCoh}}\colon \operatorname{IndCoh}(X) \to \operatorname{IndCoh}(Y)$$

maps

$$\operatorname{IndCoh}_{\mathcal{N}_X}(X) \to \operatorname{IndCoh}_{f_*\mathcal{N}_X}(Y).$$

(2) If f is proper and surjective, then the essential image of $f_*^{\mathrm{IndCoh}}(\operatorname{IndCoh}_{\mathcal{N}_X}(X))$ generates $\operatorname{IndCoh}_{f_*\mathcal{N}_X}(Y)$ under colimits.

(3) $f^!\colon \operatorname{IndCoh}(Y) \to \operatorname{IndCoh}(X)$ maps $\operatorname{IndCoh}_{\mathcal{N}_Y}(Y)$ to $\operatorname{IndCoh}_{f^*\mathcal{N}_Y}(X)$ and generates if f is an lci morphism and affine.

Remark 3.2. Roughly, f_*^{IndCoh} is like pushforward for quasicoherent sheaves.

Example 3.3. (1) f is an lci morphism if and only if $\delta\colon \operatorname{Sing}(Y) \times_Y X \to \operatorname{Sing}(X)$ is a closed embedding, which is equivalent to $f_*\emptyset = \emptyset$. In this case, the theorem says $f_* \operatorname{Perf}(X) \subset \operatorname{Perf}(Y)$ if f is proper (this is just to allow us to ignore the question of what f_*^{IndCoh} is). The reason is that f_* has finite Tor dimension for lci f.

(2) Consider $y\colon X = \mathrm{pt} \hookrightarrow Y$. Then $y_*(\emptyset) = \mathbf{P}\operatorname{Sing}(Y)_y$. The theorem says that $y_*(k)$ has maximal singular support.

(3) $f^*(\emptyset) = \emptyset$. The theorem says that $f^!$ always preserves $\operatorname{QCoh} \subset \operatorname{IndCoh}$ always. This is not a surprise, as lci implies Gorenstein, which implies $\omega_Y \in \operatorname{QCoh}(Y) \subset \operatorname{IndCoh}$. Furthermore, it is clear that this generates in the affine case.

What's the use of this stuff? The motivation seems to be that lci singularities are ubiquitous, and the relevant homological algebra is similarly ubiquitous.

4. Role in Geometric Langlands

Let X be a smooth projective curve over k. Define $\operatorname{LocSys}_n(X)$ to be the stack parametrizing $\{(\mathcal{E}, \nabla)\}$ where $\mathcal{E}$ is a rank n vector bundle on X with connection ∇.

An idea of Beilinson-Drinfeld is that there should be a kind of equivalence between coherent sheaves on $\operatorname{LocSys}_n(X)$ and D-modules on $\operatorname{Bun}_n(X)$.

For $n = 1$, they proved that on the nose

$$\operatorname{QCoh}(\operatorname{LocSys}_1(X)) \cong D(\operatorname{Bun}_1).$$

It was also known that the obvious generalization of this to GL_2 could not be true. The reason is that for LocSys_2, the Eisenstein functor

$$\mathrm{QCoh}(\mathrm{LocSys}_1(X)^2) \to \mathrm{QCoh}(\mathrm{LocSys}_2(X))$$

should preserves compact objects.

Arinkin-Gaitsgory realized that the "correct" LHS of geoemtric Langlands is the minimal subcategory of $\mathrm{IndCoh}(\mathrm{LocSys}_2(X))$ generated by QCoh and objects come from the Eisentein functor. [AG] found that this category can be described by a singular support condition. Here

$$\mathrm{Sing}(\mathrm{LocSys}_n) = \{(\mathcal{E}, \nabla) + \varphi\colon (\mathcal{E}, \nabla) \to (\mathcal{E}, \nabla) \text{ preserving the connection}\}.$$

The condition [AG] found is that φ is nilpotent.