Resolution by weighted blowing up

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Recent Progress in Moduli Theory

MSRI, May 7, 2019

Where are the moduli?

Where are the moduli?

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Birational Geometry and Moduli Spaces

János Kollár

The aim of the program in birational geometry and moduli spaces is to deepen our understanding of the following two problems.

> **Main Question 1.** Given an algebraic variety X, is there another variety X^m such that X and X^m are "similar" to each other, while the global geometry of X^m is the "simplest" possible?

> **Main Question 2.** Given a family of algebraic varieties $X \to S$, is there another family $X^c \to S^c$ such that $X \to S$ and $X^c \to S^c$ are "similar" to each other, while the global geometry of $X^c \to S^c$ is the "simplest" possible?

Much of algebraic geometry in the last two centuries has been devoted to answering these questions; frequently, a key point was to reach an agreement on what the words "variety," "similar," or "simplest" should mean.

For one-dimensional varieties, that is, for algebraic curves, the answer to the first question was given by Riemann (1857): the "simplest" algebraic curves are the smooth, compact ones. As for the "simplest"

How to resolve a curve

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(1) find a singular point $x \in C$,

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Fact

p_a gets smaller.

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To resolve a singular surface S one wants to

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Fact

This in general does not get better.

Example: Whitney's umbrella

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(2) In the z chart we get
$x = x_3 z,\ y = y_3 z,$ giving
$x_3^2 z^2 - y_3^2 z^3 = 0,$ or $z^2(x_3^2 - y_3^2 z) = 0.$

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Classical solution:

(a) Remember exceptional divisors (this is OK)

(b) Remember their history (this is a pain)

Main result

Nevertheless:

Theorem ($\aleph$-T-W, MM, "weighted Hironaka", characteristic 0)

There is a procedure F associating to a singular subvariety $X \subset Y$ embedded with pure codimension c in a smooth variety Y, a center $\bar{J}$ with blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\mathrm{maxinv}(X') < \mathrm{maxinv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n smooth.

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Here

procedure

means

a functor for smooth surjective morphisms:

if $f : Y_1 \twoheadrightarrow Y$ smooth then $J_1 = f^{-1}J$ and $Y_1' = Y_1 \times_Y Y'$.

Preview on invariants

For $p \in X$ we define
$$\mathrm{inv}_p(X) \in \Gamma \subset \quad \mathbb{Q}_{\geq 0}^{\leq n},$$
with Γ well-ordered, and show

Proposition

- *it is lexicographically upper-semi-continuous, and*
- $p \in X$ *is smooth* $\Leftrightarrow \mathrm{inv}_p(X) = \min \Gamma$.

We define $\mathrm{maxinv}(X) = \max_p \mathrm{inv}_p(X)$.

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Remark

These invariants have been in our arsenal for ages.

Preview of centers

If $\mathrm{inv}_p(X) = \mathrm{maxinv}(X) = (a_1, \ldots, a_k)$ then, locally at p

$$J = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

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Remark

$\bar{J}$ has been staring in our face for a while.

Example: blowing up Whitney's umbrella $x^2 = y^2 z$

The blowing up $Y' \to Y$ makes $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$ principal. Explicitly:

- The z chart has $x = w^3 x_3, y = w^2 y_3, z = w^2$ with chart

$$Y' \;=\; [\,\mathrm{Spec}\,\mathbb{C}[x_3, y_3, w] \;/\; (\pm 1)\,],$$

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In fact, X has begged to be blown up like this all along.

Definition of $Y' \to Y$

Let $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Define the graded algebra

$$\mathcal{B}_{\bar{J}} \quad \subset \quad \mathcal{O}_Y[T]$$

as the integral closure of the image of

$$\begin{aligned} \mathcal{O}_Y[X_1, \ldots, X_n] &\longrightarrow \mathcal{O}_Y[T] \\ X_i \quad &\longmapsto x_i T^{w_i}. \end{aligned}$$

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Let

$$S_0 \subset \operatorname{Spec}_Y \mathcal{B}_{\bar{J}}, \quad S_0 = V((\mathcal{B}_{\bar{J}})_{>0}).$$

Then

$$Bl_{\bar{J}}(Y) \; := \; \mathcal{P}roj_Y \mathcal{B}_{\bar{J}} \; := \quad \left[(\operatorname{Spec} \mathcal{B}_{\bar{J}} \smallsetminus S_0) \; / \; \mathbb{G}_m \right].$$

Description of $Y' \to Y$

- **Charts:** The x_1-chart is

$$[\operatorname{Spec} k[u, x_2, \ldots, x_n] \;/\; \boldsymbol{\mu}_{w_1}],$$

with $x_1 = u^{w_1}$ and $x_i = u^{w_i} x_i'$ for $2 \leq i \leq k$, and induced action:

$$(u, x_2, \ldots, x_n) \;\mapsto\; (\zeta u\,,\, \zeta^{-w_2} x_2\,,\, \ldots\,,\, \zeta^{-w_k} x_k,\; x_{k+1}, \ldots, x_n).$$

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- **Toric stack:** Y' corresponds to the star subdivision $\Sigma := v_{\bar{J}} \star \sigma$ along

$$v_{\bar{J}} \ = \ (w_1, \ldots, w_k, 0, \ldots, 0),$$

with the cone

$$\sigma_i = \langle v_{\bar{J}}, e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle$$

endowed with the sublattice $N_i \subset N$ generated by the elements

$$v_{\bar{J}}, e_1, \ldots, \hat{e}_i, \ldots, e_n,$$

for all $i = 1, \ldots, k$.

Examples: Defining J

(1) Consider $X = V(x^5 + x^3y^3 + y^8)$ at $p = (0,0)$; write $\mathcal{I} := \mathcal{I}_X$.

- Define $a_1 = \operatorname{ord}_p \mathcal{I} = 5$. So $J_{\mathcal{I}} = (x^5,\ y^{\star}\ )$.

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(2) If instead we took $X = V(x^5 + x^3y^3 + y^7)$, then since $7 < 15/2$ we would use

$$J_{\mathcal{I}} = (x^5, y^7) \quad \text{and} \quad \bar{J}_{\mathcal{I}} = (x^{1/7}, y^{1/5}).$$

Examples: describing the blowing up

(1) Considering $X = V(x^5 + x^3y^3 + y^8)$ at $p = (0,0)$,

- the x-chart has $x = u^3, y = u^2y_1$ with $\boldsymbol{\mu}_3$-action, and equation

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(1) Considering $X = V(x^5 + x^3y^3 + y^7)$ at $p = (0,0)$,

- the x-chart has $x = u^7, y = u^5y_1$ with $\boldsymbol{\mu}_7$-action, and equation
$$u^{35}(1 + uy_1^3 + y_1^7)$$
with smooth proper transform.
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Coefficient ideals

We must restrict to $x_1 = 0$ the data of all

$$\mathcal{I},\ \mathcal{D}\mathcal{I},\ \ldots,\ \mathcal{D}^{a_1-1}\mathcal{I}$$

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We combine these in

$$C(\mathcal{I}, a_1)\ :=\ \sum f\left(\mathcal{I}, \mathcal{D}\mathcal{I}, \ldots, \mathcal{D}^{a_1-1}\mathcal{I}\right),$$

where f runs over monomials $f = t_0^{b_0} \cdots t_{a_1-1}^{b_{a_1-1}}$ with weights

$$\sum b_1(a_1 - i) \geq a_1!.$$

Define $\mathcal{I}[2] = C(\mathcal{I}, a_1)|_{x_1=0}$.

Defining $J_{\mathcal{I}}$

Definition

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$$\mathrm{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left(a_1, \frac{\mathrm{inv}_p(\mathcal{I}[2])}{(a_1 - 1)!}\right)$$

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Example

(1) for $X = V(x^5 + x^3y^3 + y^8)$ we have $\mathcal{I}[2] = (y)^{180}$, so $J_{\mathcal{I}} = (x^5, y^{180/24}) = (x^5, y^{15/2})$.

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What is J?

Definition (Temkin)

Consider the Zariski-Riemann space $\mathbf{ZR}(X)$ with its sheaf of ordered groups Γ, and associated sheaf of rational ordered group $\Gamma \otimes \mathbb{Q}$.

- A valuative $\mathbb{Q}$-ideal is

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A center is in particular a valuative $\mathbb{Q}$-ideal.

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Indeed, on Y' the center J becomes E^{ℓ}, in particular principal.

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Proposition

J *is* $\mathcal{I}$*-admissible if and only if* $J^{(a_1-1)!}$ *is* $C(\mathcal{I}, a_1)$*- admissible.*

This is a consequence of the following technical, but known, lemma.

Structure of coefficient ideals

Lemma

If $\operatorname{ord}_p(\mathcal{I}) = a_1$ *and* x_1 *a corresponding maximal contact, then in* $\mathbb{C}[\![x_1,\ldots,x_n]\!]$ *we have*

$$C(\mathcal{I},a) = (x_1^{a!}) + (x_1^{a!-1}\tilde{\mathcal{I}}_{a!-1}) + \cdots + (x_1\tilde{\mathcal{I}}_1) + \tilde{\mathcal{I}}_0,$$

where

$$\mathcal{I}_0 \subset (x_2,\ldots,x_n)^{a!} \subset k[\![x_2,\ldots,x_n]\!],$$

where $\mathcal{I}_{j+1} := \mathcal{D}^{\leq 1}(\mathcal{I}_j)$ *satisfy* $\mathcal{I}_{a!-k}\mathcal{I}_{a!-l} \subset \mathcal{I}_{a!-(k+l)}$, *and* $\tilde{\mathcal{I}}_j = \mathcal{I}_j k[\![x_1,\ldots,x_n]\!]$.

The lemma and proposition are proven by looking at monomials.

The key theorems

Theorem

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Proof.

Apply induction! ♠

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Proof.

Indeed, on the x_1-chart the first term $x_1^{a_1}$ becomes exceptional with $C' = (1)$. On the x_i-chart we have by induction that $\text{inv}_p((\mathcal{I}_0)') < (a_2, \ldots, a_k)$, which means that $\text{inv}_p((x_1^{a_1} + \mathcal{I}_0)') = inv_p((x_1'^{\,a_1}) + (\mathcal{I}_0)') < (a_1, a_2, \ldots, a_k)$, implying the claim. ♠

The end

Thank you for your attention

Questions:

Moduli applications?

Semistable reduction. Trying to make things functorial forced us to look at weighted blowups.

What to do with the resulting stack?

~~End up~~ Start with $X_0 \subset Y_0$ and end up with
singular smooth

$X_n \subset Y_n$
smooth stack, smooth stack

Now apply Bergh's destackification to X_n

What about log resolutions and keeping track of exceptional divisors?

Should work; thesis problem?

What about logarithmic (log structures of Fontaine Illusie) analogues?

<u>Same</u>.

Characteristic p? Have to think about it. But it's an evil problem!