5/8/2019

Degenerations of K3 surfaces and 24 points on the sphere

Speaker: Valery Alexeev

<u>Slogan</u>: Degenerations and complete moduli of abelian varieties are "easy".

<u>Want</u>: to do the "same" for K3s.

Reasons for slogan:

(1) Torelli is very explicit.

(2) Degenerations are toric.

<u>Reasons for want</u>: (1) Torelli is mysterious.

(2) Degenerations are toric.

§1. Complete moduli for K-trivial varieties.

<u>Moduli functor</u>: $S \mapsto$ flat families $f: (\mathcal{X}, \varepsilon B) \to S$

such that

(i) $\omega_{\mathcal{X}/S} \cong f^* \mathcal{O}_S$, (ii) B rel ample Cartier divisor.

(3) every fiber (X, B) is semi-log-canonical

here $0 < \varepsilon \leq 1$, can depend on fiber. one can show $\varepsilon = \varepsilon(\mathrm{vol}\, B)$ (depends on volume)

Easy: $\exists$ moduli space $\mathcal{M}^{slc}$, but components might not be proper.

Principally polarized abelian varieties

$$\mathcal{A}_g \hookrightarrow \mathcal{M}^{slc}$$

$$(A, \lambda) \longleftrightarrow (X, \varepsilon\Theta)$$

Can take closure $\overline{\mathcal{A}}_g^{slc} \subset \mathcal{M}^{slc}$.

K3 surfaces:

Look @ pairs (X, L), X a K3 surface, L a primitive line bundle on X, $L^2 = 2d$.

Moduli of

$$\begin{tikzcd} F_{2d} \arrow[r, hook] \arrow[dr, hook] & \mathcal{M}^{slc} \quad (X, \varepsilon L) \\ & \overline{F}_{2d}^{slc} \arrow[u, hook] \end{tikzcd}$$

~~involves choosing $B \in |NL|$~~

Involves making a canonical choice $B \in |NL|$

~~Theorem (Alexeev, Engel, Thompson)~~

<u>Theorem (Alexeev '02)</u>

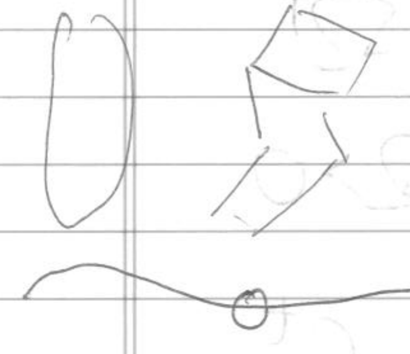
$\overline{A_g^{slc}}$ is proper. If

$\left(\overline{A_g^{slc}}\right)^\nu$ = normalization of

then $\left(\overline{A_g^{slc}}\right)^\nu = \overline{A}_g^{tor}$ for $\mathcal{F}^{vor}$, the 2nd Voronoi fan.

$\mathcal{F}^{vor}$ = moduli space for tropical PPAVs.

§ Degenerations of PPAVs



$$\begin{array}{ccccc} \mathcal{X}^0 & \subset & \mathcal{X} & \supset & \mathcal{X}_0 \\ \downarrow & & \downarrow & & \downarrow \\ C^0 & \subset & C & \ni & 0 \end{array}$$

~~The case~~ w/ $\mathcal{X}$ smooth, $\mathcal{X}_0$ a normal crossing union of smooth varieties, $\omega_{\mathcal{X}/S} \simeq \mathcal{O}_X$. C a smooth curve, $C^0 = C \setminus \{0\}$. These are called <u>Kulikov degenerations</u>

<u>Fact</u>: ~~...~~ there are infinitely many Kulikov degenerations, but the divisor B gives <u>uniqueness</u>:

There's a unique limit with B_0 not containing any strata of X_0. <u>But</u> $\mathcal{X}$ may be not smooth, $\mathcal{X}_0$ may be not snc.

Torelli description:

$$X = \mathbb{C}^g / \left[I \,\Omega \right] \qquad \Omega = \Omega^t$$
$$\operatorname{Im}\Omega > 0$$

$$= (\mathbb{C}^*)^g / \left[c_{ij} \right] \qquad C = C^t$$

Family $X_t = (\mathbb{C}^*)^g / \left[c_{ij} t^{q_{ij}} \right]$

$Q \in \mathrm{Mat}_{g \times g}(\mathbb{Z})$

$Q = Q^t$, $Q \geq 0$

$c_{ij}(t)$ invertible

$g=2$ | types of degenerations

I) $Q = 0$: abelian

II) $\operatorname{rk} Q = 1$: <u>mixed</u>

III) $Q > 0$: <u>toric</u>

Set $M = \mathbb{Z}^g$; the <u>pairing</u>

$Q : M \times M \to \mathbb{Z}$ equivalent to

$i_Q : M \to M^\vee =: N$

$\mathrm{Vor}_Q \subset M_{\mathbb{R}} \supset M$

obtained from Voronoi

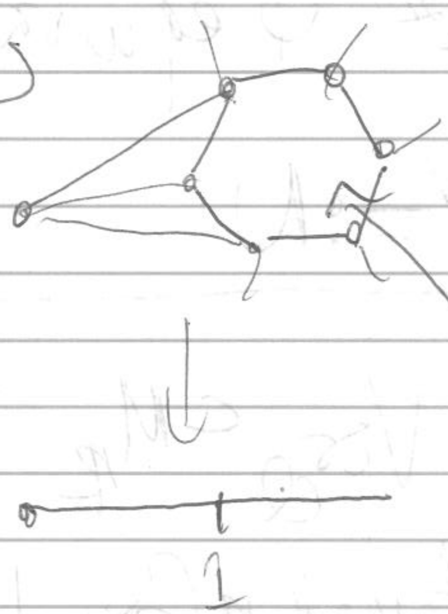
decomposition (see slides).

$$\frac{N_{\mathbb{R}}}{M} = \frac{\mathbb{R}^g}{\mathbb{Z}^g}; \quad \frac{L_Q(\mathrm{Var}_Q)}{L_Q(M)} = \Theta_{trop}$$

tropical theta divisor.

<u>Picture</u>

$\mathbb{R} \oplus N_{\mathbb{R}}$,



$$\begin{array}{c} \text{Kulikov model} \longrightarrow \widetilde{\mathcal{X}}/M \\ \quad \downarrow \text{resolution} \\ (\mathbb{C}^\times)^g/M \subset \mathcal{X}/M \supset \mathcal{X}_0/M \\ \downarrow \\ 0 \neq t \in \mathbb{A}^1 \ni 0 \end{array}$$

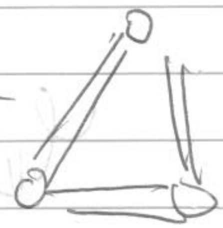
to <u>resolve</u>, <u>triangulate</u> this.

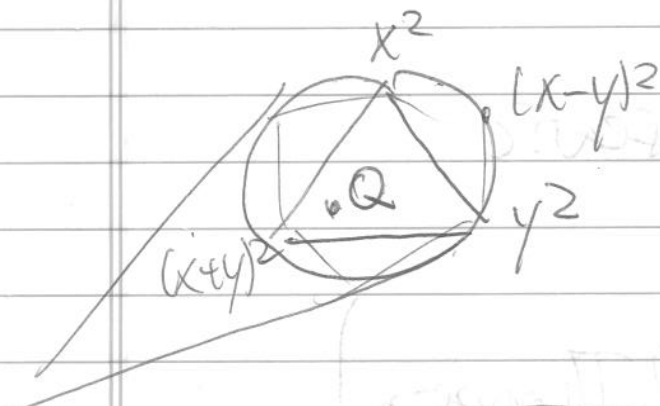
Analogy:

$\underbrace{\mathbb{R}^g/\mathbb{Z}^g}_{\text{tropical PPAVs}}$ $\qquad$ $\underbrace{S^2 + 24 \text{ singular points}}_{\text{"tropical" K3}}$

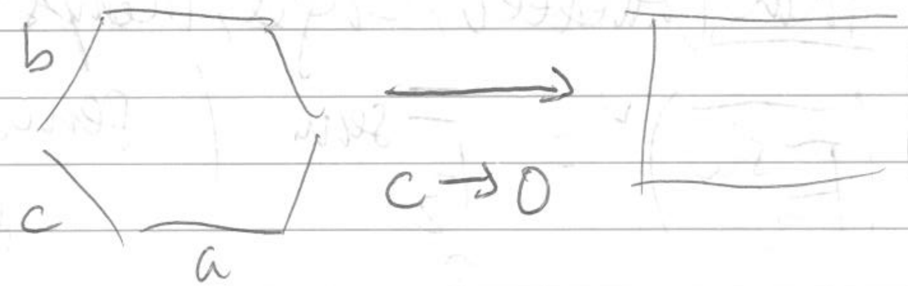
Integral-affine.

For $g=2$:

$F^{Vor} = F^{cox}$ (coxeter fan) for 



$$Q = ax^2 + by^2 + c(x+y)^2$$



or $a \to 0$: 

or $a \to 0$
$c \to 0$: 

Recall: $F_2 = \{\underbrace{(X,L)}_{K3}, L^2 = 2\} \hookrightarrow \overline{F_2}^{slc}$

$$\overline{F_2}^{slc} = \overline{M}^{Hacking}(\mathbb{P}^2, C_6)$$

$$\begin{array}{ccc} \varphi_{|L|} : X & \xrightarrow[2:1]{} & \mathbb{P}^2 \\ \cup & & \cup \\ 3L \sim R & & C_6 \text{ sextic} \end{array}$$

Thm (Alexeev, Engel, Thompson)

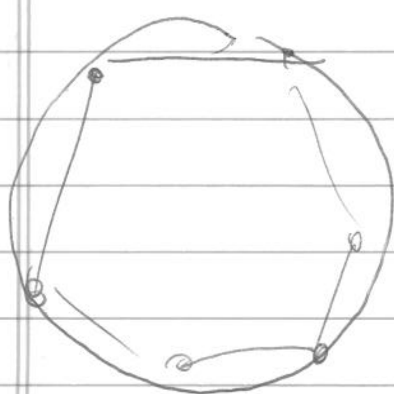
$(\overline{F_2}^{slc})^\nu = \overline{F_2}^{semi}$ (semi for F^{semi} semi fan

$\uparrow$

$\overline{F}^{tor}$ for $F^{cox} < F^{semi}$

Here F^{semi} = moduli space for integral-affine (S^2, R_{top})

F^{cot} = ↑ + additional structure.



$W = \langle w_r \mid r^2 = -2 \rangle$

$N = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \oplus E_8(-1)^2 \oplus \langle -2 \rangle$

$(1, 18)$

(see slides)

<u>cones</u> of F^{cot} $\longleftrightarrow$ ADE

and max $\widetilde{ADE}$ subdiag. of G_{cot}

type II

cones of F^{semi} $\longleftrightarrow$ same as ↑ but without irrelevant components.

(types of stable pairs)

$\boxed{\text{See slides for cool illustrations}}$

Construction: Given

$a_i = V \cdot r_i \in \mathbb{Z}^{24}_{\geq 0}$,

$(a_i) \rightsquigarrow$ (1) integral-affine $(S^2, \mathbb{R})$

(2) Kulikov models

(3) All stable models

Again, see slides for visuals.

Question: How would one guess all this?

Answer: Mirror symmetry.

Q] Ingredients of proof?

A] • Monodromy theorem of Fengel and Friedman

• Friedman's Torelli theorem for Kulikov models.

Questions from audience

See video.

Degenerations of K3 surfaces and 24 points on the sphere

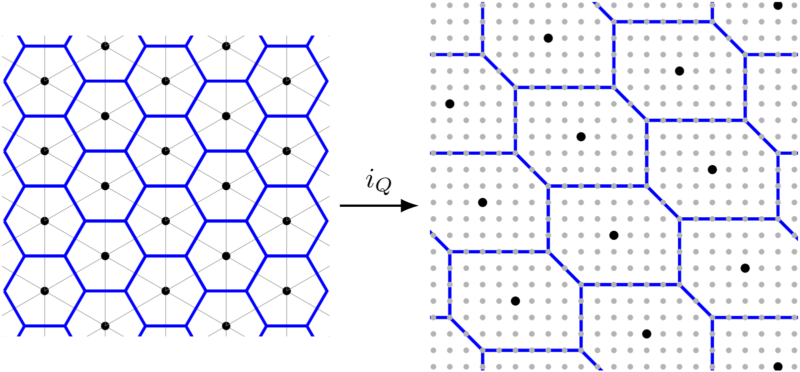
Valery Alexeev

Based on joint works with Philip Engel and Alan Thompson

MSRI, *Recent Progress In Moduli Theory*, May 8, 2019

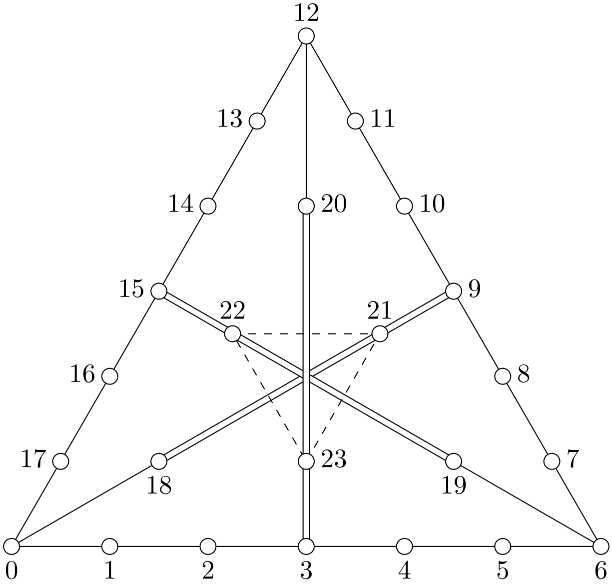
Tropical PPAV

$M \simeq \mathbb{Z}^g \qquad Q\colon M \times M \to \mathbb{Z} \iff i_Q\colon M \to M^* = N$

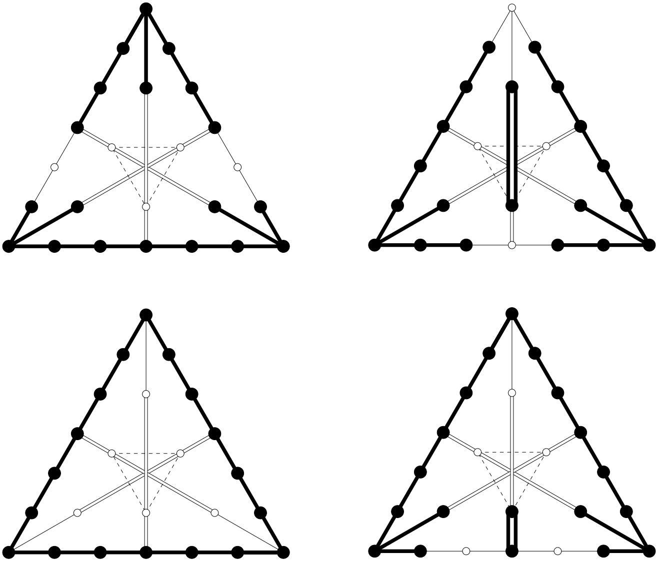


$\mathit{Vor}_Q \subset M_{\mathbb{R}} \supset M \qquad i_Q(\mathit{Vor}_Q) \subset N_{\mathbb{R}} \supset N \supset i_Q(M)$

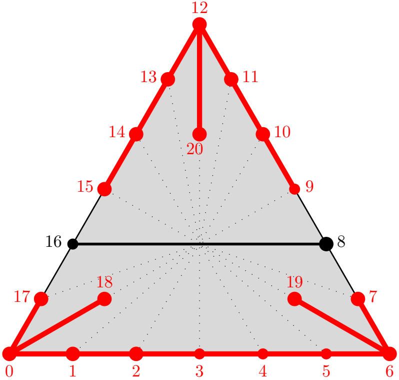
Coxeter diagram of the hyperbolic lattice $H \oplus E_8^2 \oplus A_1$

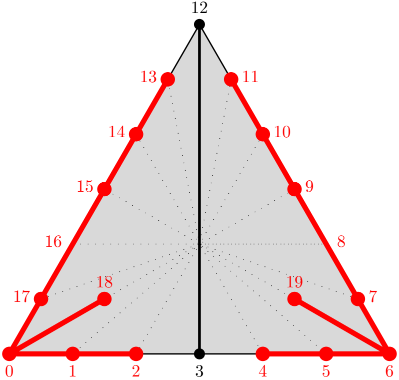


Type II rays of Coxeter fan: $\widetilde{D}_{10}\widetilde{E}_7$, $\widetilde{E}_8^2\widetilde{A}_1$, $\widetilde{A}_{17}$, $\widetilde{D}_{16}\widetilde{A}_1$

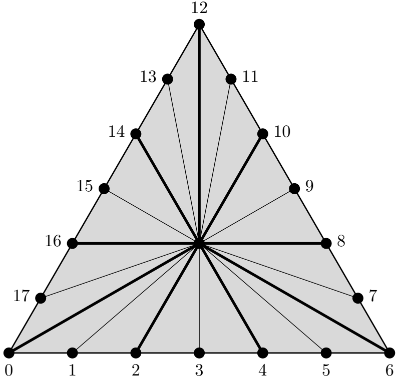


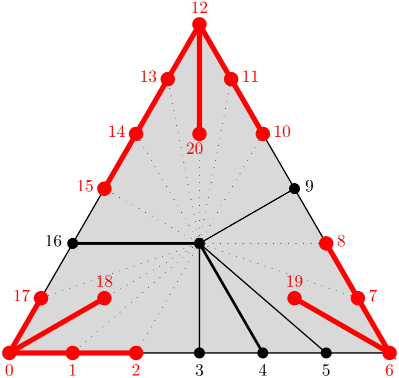
Type II degenerations of $(\mathbb{P}^2, \mathcal{O}(6))$: $\widetilde{D}_{10}\widetilde{E}_7$ and $\widetilde{E}_8^2$



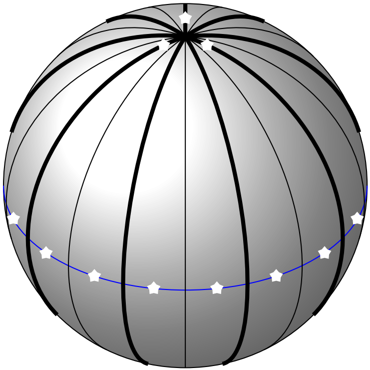


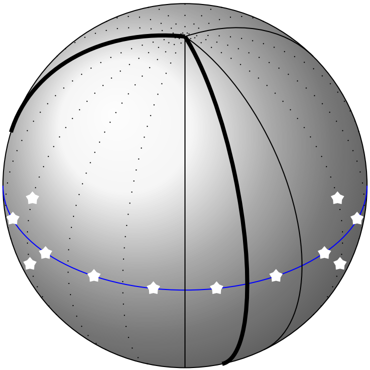
Type III degenerations of $(\mathbb{P}^2, \mathcal{O}(6))$: A_0^{18} and $D_5 A_0^2 A_4' E_7$



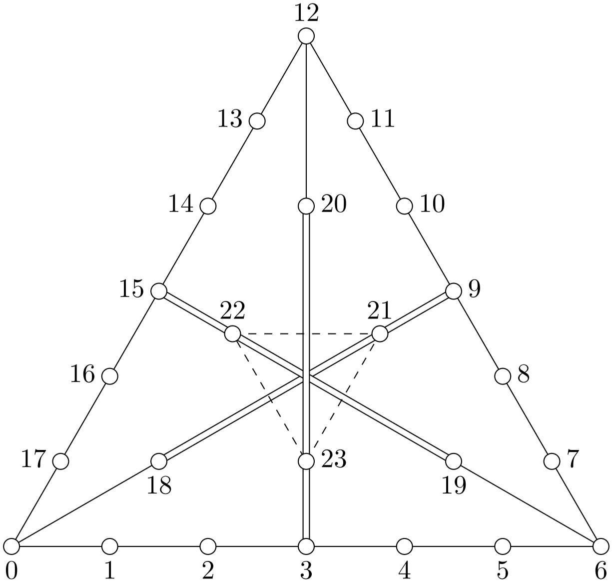


Type III degenerations of K3 surfaces: A_0^{18} and $D_5 A_0^2 A_4' E_7$

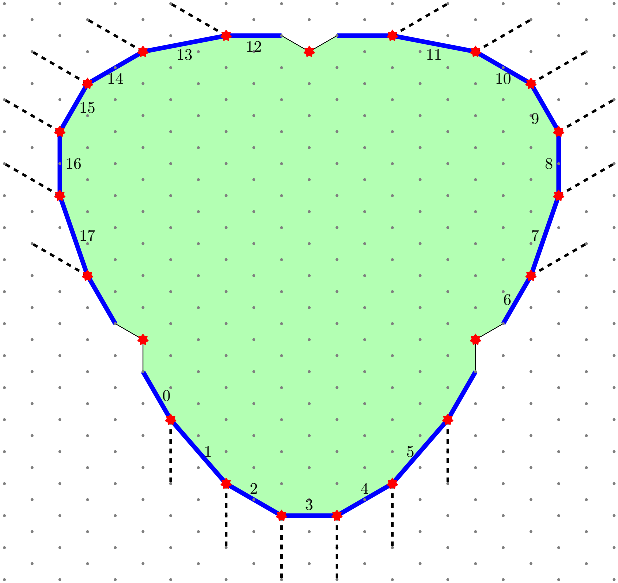




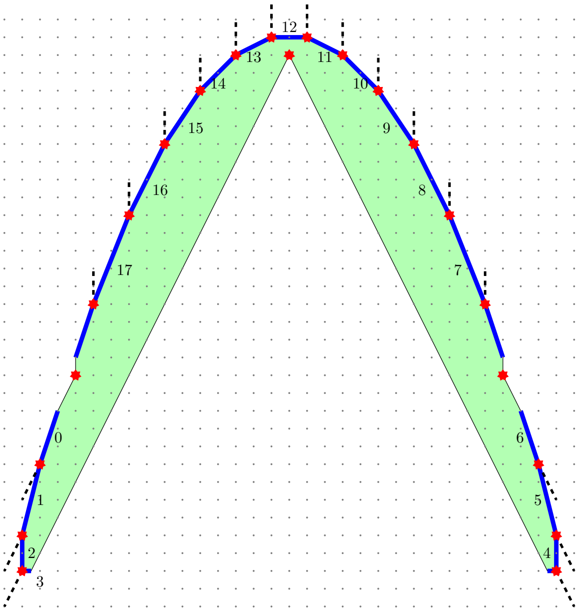
Recall: the Coxeter diagram of $W(H \oplus E_8^2 \oplus A_1)$



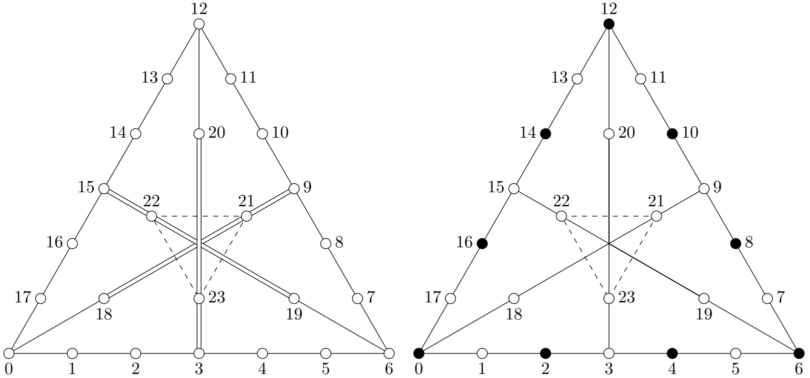
Integral-affine structure on $S^2 = D \cup D^{\mathrm{opp}}$



A second way to glue the same IAS^2



Exceptional curves on mirror K3 surface S and on $T = S/\mathbb{Z}_2$



$$E_i^2 = -2$$

$$F_i^2 = \begin{cases} -4, & \text{black vertex} \\ -1, & \text{white vertex} \end{cases}$$

The A'_{18} ray