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Rationality for geometrically rational 3-folds

Speaker: Brendan Hassett

(Joint work w/ Pirutka, Tschinkel)

§ Goals

Let $k$ be a field, $X$ a smooth projective 3-fold $/k$.

Assume $X_{\bar{k}}$ is rational.

Develop criteria for deciding whether $X$ is rational over $k$.

__Ex.__ 1) If $X$ is rational, then $X(k) \neq \emptyset$.

2) If $k \subset \mathbb{R}$ then

$X$ rational $\Rightarrow X$ is connected.

__Prototype__: when $X$ is a __surface__, question is answered by Enriques–Fano–Iskovski classification

E.g. if $X$ is a del Pezzo $/k$ with $K_X^2 \leq 4$ __and__ $X$ is minimal

$\Rightarrow X$ is __not__ rational.

Dim $X = 3$ again.

Inputs: basic geometry of $X$
+ associated Galois structure.

§ Classical construction (char $k \neq 2$)

Let $X = \{Q_0 = Q_1 = \cdots = Q_{r-1} = 0\} \subset \mathbb{P}^n$,
a smooth complete intersection
of quadrics ($Q_i \in k[x_0, \ldots, x_n]_2$ for all $i$)

Fact: if there is a $\mathbb{P}^{r-1} \subset X \subset \mathbb{P}^n$,
then projection from $\mathbb{P}^{r-1}$

$$\pi_{\mathbb{P}^{r-1}} : X \dashrightarrow \mathbb{P}^{n-r} \text{ is birational.}$$

(Generalization of projection from a point for

~~quadrics~~ conic curves)

Necessary (Lefschetz):

$\dim \mathbb{P}^{r-1} \leq \frac{1}{2} \dim X$, i.e. $r-1 \leq \frac{1}{2} \dim X$

Since $\dim X = n-r$, $3r-2 \leq n$.

Not sufficient. For $X \subset \mathbb{P}^7$,

$n=7$ & $r=3$,

$\mathbb{P}^2 \subset X$ is a codim 3 condition in moduli.

Ex: $X = \{Q_0 = Q_1 = 0\} \subset \mathbb{P}^4_{\mathbb{R}}$

can be rational w/o a line over $\mathbb{R}$.

Ex: $X \subset \mathbb{P}^7$, $r = 3$ over $\mathbb{C}$.

Theorem (Hassett–Pirutka–Tschinkel)

There exists a countably infinite collection of codim 3 loci in moduli that parametrize rational $X$. The very general $X \subset \mathbb{P}^7$ is not (stably) rational.

## Main example

$X \subset \mathbb{P}^5$, smooth complete intersection of Quadrics

$\{Q_0 = Q_1 = 0\}$

<u>Theorem</u> (Hassett-Tschinkel, Wittenberg)

If $X$ is rational then $X$ contains a line $l \subset X$ over $k$.

<u>Basic geometry</u>:

1) over $\bar{k}$, after change of coordinates

$$X = \{ y_0^2 + \cdots + y_5^2 = A_0 y_0^2 + \cdots + A_5 y_5^2 = 0 \}$$

so, $(\mathbb{Z}/2\mathbb{Z})^5 \subset \mathrm{Aut}(X_{\bar{k}})$.

2) over $k$, have

$$\det(t_0 Q_0 + t_1 Q_1) \in k[t_0, t_1]_6$$

a square-free homogeneous sextic.

Ascoli-Indiostro: use (1) & (2) to compute Pic of the moduli stack of such $X$.

(3) Considering again the pencil $t_0 Q_0 + t_1 Q_1$, get

$$\begin{array}{cl} Bl_X \mathbb{P}^5 = \{t_0 Q_0 + t_1 Q_1 = 0\} \subset \mathbb{P}^5 \times \mathbb{P}^1 & \\ \downarrow \pi_2 & \text{a quadric 4-fold fibration} \\ \mathbb{P}^1 & \text{w/ degenerate fibers over} \end{array}$$

$\{\det(t_0 Q_0 + t_1 Q_1) = 0\} \subset \mathbb{P}^1$

Look at max'l isotropic subspaces in fibers:

$$\begin{tikzcd} F_2(\pi_2) \arrow[d] \arrow[r] & C \arrow[dl] \\ \mathbb{P}^1 & \end{tikzcd}$$

Stein factorization

étale $\mathbb{P}^3$-bundle

double cover branched over $\{\det(t_0 Q_0 + t_1 Q_1) = 0\}$

$$\begin{tikzcd} \mathrm{Conics}(X) \simeq F_2(\pi_2) \arrow[d] \\ C \end{tikzcd}$$

, +

$\mathrm{Lines}(X) \cong$ Principal homogeneous space over $\mathrm{Jac}(C)$

(Bhargava + Gross, Wang)

Picture:

$$\begin{array}{ccc}
\mathrm{Sym}^2(\mathrm{lines}(X)) \to & \text{Reducible conics} \subset & \mathrm{Conics}(X) \\
\text{(Kummer surface)} & & \downarrow \ \mathbb{P}^3\text{-bundle} \\
\searrow & \mathrm{Pic}^1(C) \supset & C
\end{array}$$

$$2 \cdot [\mathrm{lines}(X)] = \mathrm{Pic}^1(C)$$

as principal homogeneous spaces over $\mathrm{Jac}(C)$.

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Idea of proof: Suppose $X$ is rational. ~~Rlee~~ Pick a factorization

$$\begin{tikzcd} & Z \arrow[dl, "\varphi"'] \arrow[dr] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\sim"] & & X \end{tikzcd}$$

blowup up along smooth centers.

Given a curve $D \subset W$, $W$ a 3fold, have

Intermediate Jacobians:

$$IJ(Bl_D W) = IJ(W) \oplus Jac(D)$$

and $$CH^2(Bl_D W) = CH^2(W) \oplus CH^1(D)$$

Also: $IJ(X) = Jac(C)$.

So Roughly: only 1 center of $\varphi$ ~~blob~~ survives in $X$, call it $D$.

$$CH^2(X) = CH^1(D).$$

Torelli: $Jac(C) \simeq Jac(D)$

$$\Rightarrow C \simeq D.$$

Now $[\text{lines}(X)] = [\text{Pic}^1(D)]$

& since genus$(C) = 2$, $2[\text{Pic}(C)] = [\text{Pic}^2(C)]$
$= 0$

and as $D \simeq C$,

$\Rightarrow \text{lines}(X)(k) \neq \emptyset$.

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Remark: new even over $\mathbb{R}$.

Nine isotopy classes of smooth complete intersections of 2 quadrics over $\mathbb{R}$ (Krasnov)

5 contain lines
1 empty ($X(\mathbb{R}) = \emptyset$)
1 disconnected
2 connected without lines.

$\Big\}$ not rational.

Audience questions

Q] ~~TRC~~ over $\mathbb{R}$ the locus of rational $X$'s ended up being a union of isotopy classes. Is that to be expected?

A] Not clear a priori.