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Rationality for geometrically rational
3-folds

Speaker: Brendan Hassett

(Joint work w/ Pirutka, Tschinkel)

f Goals

Let k be a field, X a smooth
projective 3-fold / k .

Assume $X_{\mathbb{E}}$ is rational.

Develop criteria for deciding whether
 X is rational over k .

Ex. 1) If X is rational, then $X(k) \neq \emptyset$.

2) If $k = \mathbb{R}$ then

X rational $\Rightarrow X$ is connected.

Prototype: when X is a surface,
question is answered by Enriques-
Fano-Iskovski classification

Eg. if X is a del Pezzo / k
with $K_X^2 \leq 4$ and X is minimal

$\Rightarrow X$ is not rational.

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1. To the extent of the ...

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Dim $X = r$ again.

Inputs: basic geometry \mathcal{O}_X of X
+ associated Galois structure.

§ Classical construction (char $k \neq 2$)

Let $X = \{Q_0 = Q_1 = \dots = Q_{r-1} = 0\} \subset \mathbb{P}^n$,
a smooth complete intersection
of quadrics, ($Q_i \in k[x_0, \dots, x_n]_2$ for all i)

Fact: if there is a $\mathbb{P}^{r-1} \subset X \subset \mathbb{P}^n$,
then projection from \uparrow

$\pi_{\mathbb{P}^{r-1}}: X \xrightarrow{\quad} \mathbb{P}^{n-r}$ is birational.

(Generalization of projection from a point for

~~quadric~~ conic curves)

Necessary (Lefschetz):

$$\dim p^{r-1} \leq \frac{1}{2} \dim X, \text{ i.e. } r-1 \leq \frac{1}{2} \dim X$$

Since $\dim X = n - r$, $3r - 2 \leq n$.

Not sufficient. For $X \subset \mathbb{P}^7$,

$$n = 7 \text{ \& } r = 3,$$

$\mathbb{P}^2 \subset X$ is a codim 3 condition
in moduli.

Ex: $X = \{Q_0 = Q_1 = 0\} \subset \mathbb{P}^4_{\mathbb{R}}$

can be rational w/o a line over \mathbb{R} .

Ex: $X \subset \mathbb{P}^7$, $r=3$ over \mathbb{C} .

Theorem (Hassett-Prutka-Tschinkel)

There exists countably infinite collection of codim 3 loci in moduli that parametrize rational X . The very general $X \subset \mathbb{P}^7$ is not (stably) rational.

Main example

$X \subset \mathbb{P}^5$, smooth complete
intersection of quadrics
 $\{Q_0 = Q_1 = 0\}$

Theorem (Hassett-Tschinkel, Wittenberg)

If X is rational then X contains
a line $l \subset X$ over k .

Basic geometry: ~~(1)~~

i) over k , after change of coordinates

$$X = \{x_0^2 + \dots + x_5^2 = A_0 x_0^2 + \dots + A_5 x_5^2 = 0\}$$

~~(1)~~ $(\mathbb{Z}/2\mathbb{Z})^5 \subset \text{Aut}(X_{\mathbb{Z}})$.

2) over k , have

$$\det(t_0 Q_0 + t_1 Q_1) \in k[t_0, t_1]_6$$

a square-free homogeneous sextic.

Azeoli-Indriestno: use (1) & (2) to compute Pic of the moduli stack of such X .

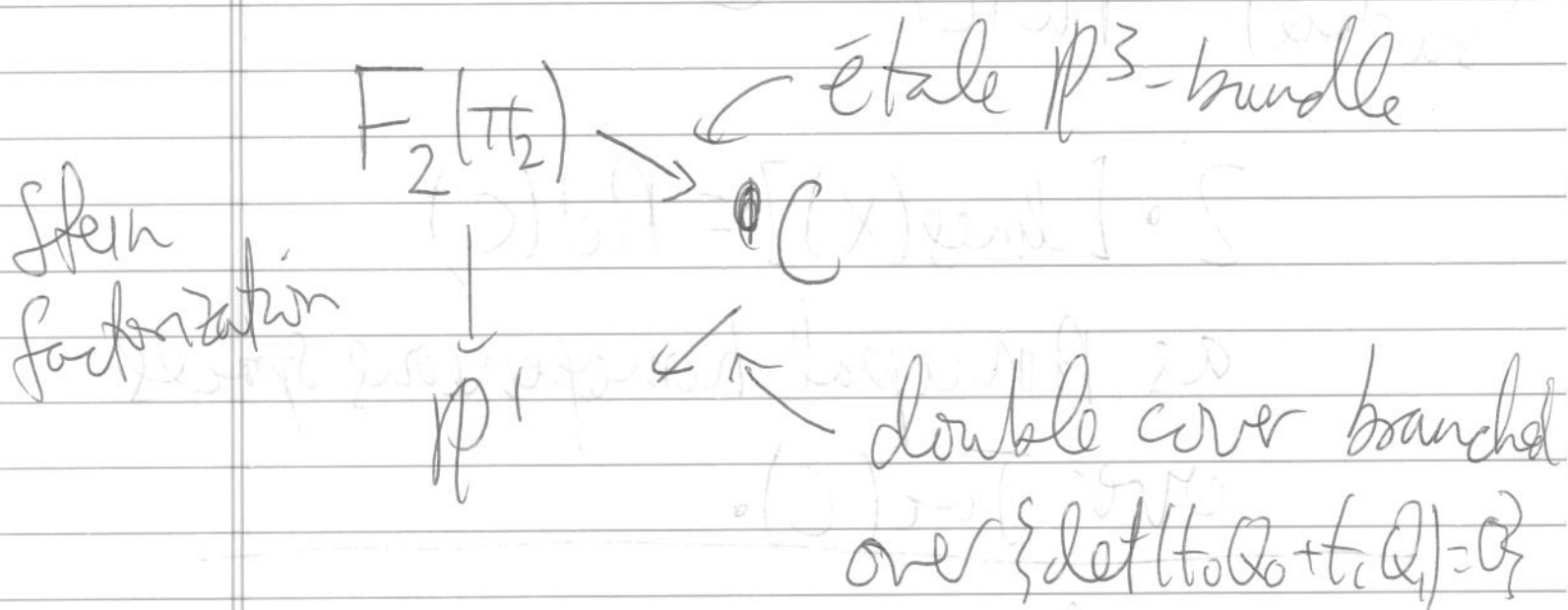
(3) Considering again the pencil $t_0 Q_0 + t_1 Q_1$, get

$$\text{Bl}_X \mathbb{P}^5 = \{t_0 Q_0 + t_1 Q_1 = 0\} \subset \mathbb{P}^5 \times \mathbb{P}^1$$

$\pi_2 \downarrow$
 \mathbb{P}^1 a quadric \mathbb{P}^4 -fold fibration
w/ degenerate fibers over

$$\left\{ \det \begin{pmatrix} t_0 Q_0 + t_1 Q_1 \\ \dots \\ \dots \end{pmatrix} = 0 \right\} \subset \mathbb{P}^1$$

Look at max'l isotropic subspaces
~~in~~ in fibers:



$$\text{Conics}(X) \cong F_2(\mathbb{P}^2), \quad +$$

$$\downarrow$$

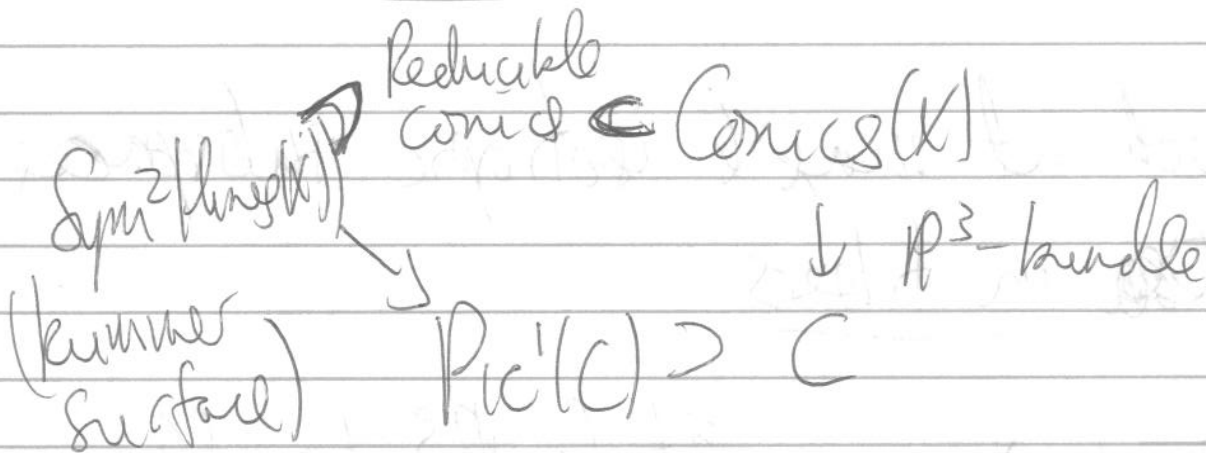
$$\text{Lines}(X) \cong \text{Pencil}$$

homogeneous space over

$$\text{Jac}(\mathbb{C})$$

(Bhargava + Gross, Wang)

Picture:

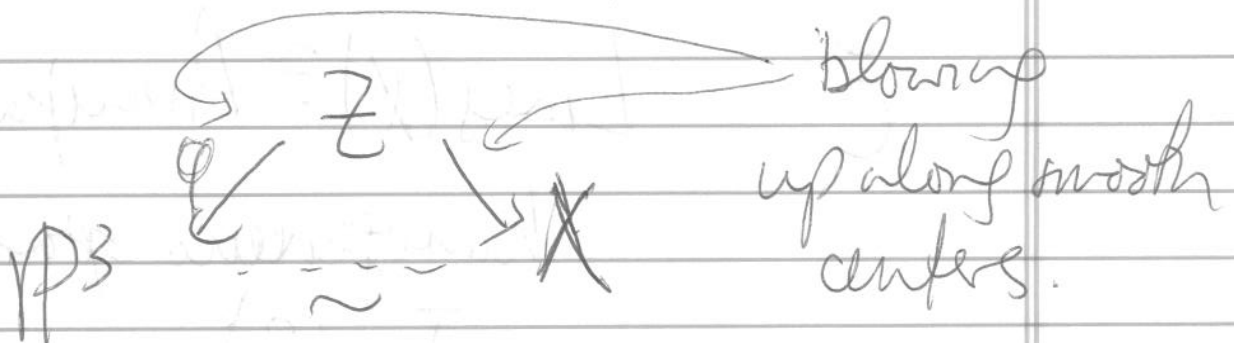


$$2 \cdot [\text{lines}(X)] = \text{Pic}^1(C)$$

as principal homogeneous spaces
over $\text{Jac}(C)$.

Idea of proof: Suppose X is rational.

~~Recall~~ Pick a factorization



Given a curve $D \subset W$, W a 3 fold,
have

Intermediate Jacobians:

$$IJ(\text{Bl}_D W) = IJ(W) \oplus \text{Jac}(D)$$

and $CH^2(\text{Bl}_D W) = CH^2(W) \oplus CH^1(D)$

Also: $IJ(X) = \text{Jac}(C)$.

So roughly: only 1 center of
 φ ~~survives~~ survives in X ,
call it D .

$$CH^2(X) = CH^1(D).$$

Torelli: $\text{Jac}(C) \cong \text{Jac}(D)$

$$\Rightarrow C \cong D.$$

Now $[\text{lines}(X)] = [P_{\mathbb{C}}^{-1}(D)]$

same genus $(C) = 2$, $2[P_{\mathbb{C}}(C)] = [P_{\mathbb{C}}^2(C)]$
 $= 0$

and as $D \approx C$,

$\Rightarrow \text{lines}(X)(k) \neq 0$.

Remark: new even over \mathbb{R} .

Nine isotopy classes of smooth
complete intersections of 2 quadrics
over \mathbb{R} (Krasnov)

5 contain lines

1 empty ($X(\mathbb{R}) = \emptyset$)

1 disconnected

2 connected without lines.

} not rational.

Audience questions

Q] ~~Q~~ over \mathbb{K} the locus of \bullet rational X 's ended up being a union of isotopy classes. Is that to be expected?

A] Not clear a priori.

Antennae structure

Antennae are the primary sense organs of insects.

They are used for touch, taste, smell, and hearing.

Antennae are divided into three segments: the base, the middle, and the tip.

The base is called the scape.

The middle segment is called the pedicel.

The tip is called the flagellum.