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17 Gauss Way   Berkeley, CA 94720-5070   p: 510.642.0143   f: 510.642.8609   www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Charles Godfrey **Email/Phone:** cgodfrey@uw.edu

Speaker's Name: Melody Chan

Talk Title: Tropical moduli spaces and the cohomology of M_{g,n}

Date: 5 / 10 / 2019 **Time:** 9 : 30 **am / pm (circle one)**

Please summarize the lecture in 5 or fewer sentences: Studies the top weight rational cohomology of M_{g,n}, including its S_n-equivariant Euler characteristic, using boundary complexes, tropical moduli spaces, and a new cellular homology theory for dual complexes (more generally, a new class of objects called symmetric delta complexes).

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [X] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [X] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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  - **Handouts**: Obtain copies of and scan all handouts
- [X] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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- [ ] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

TROPICAL MODULI SPACES AND THE COHOMOLOGY OF $M_{g,n}$

MELODY CHAN

Links

- The S_n-equivariant top weight Euler characteristic of $M_{g,n}$, https://arxiv.org/abs/1904.06367 [CFGP19]
- Topology of moduli spaces of tropical curves with marked points, https://arxiv.org/abs/1903.07187 [CGP19]
- Tropical curves, graph homology, and top weight cohomology of M_g, https://arxiv.org/abs/1805.10186 [CGP18]

References

[CFGP19] Melody Chan, Carel Faber, Soren Galatius, and Sam Payne, *The s_n-equivariant top weight euler characteristic of m_ {g, n}*, arXiv preprint arXiv:1904.06367 (2019).

[CGP18] Melody Chan, Soren Galatius, and Sam Payne, *Tropical curves, graph homology, and top weight cohomology of m_g*, arXiv preprint arXiv:1805.10186 (2018).

[CGP19] ______, *Topology of moduli spaces of tropical curves with marked points*, arXiv preprint arXiv:1903.07187 (2019).

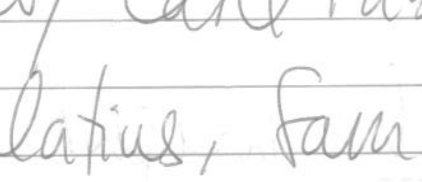
Date: May 10, 2019.

Tropical moduli spaces and cohomology of $M_{g,n}$

Speaker: Melody Chan

(Joint work w/ Carel Faber, Søren Galatius, Sam Payne)

Work over $\mathbb{C}$.

$M_{g,n} = \{$  $\}$ moduli space of curves of type g,n

<u>Goal</u>: study $H^*(M_{g,n};\mathbb{Q})$ using boundary combinatorics of $M_{g,n} \subset \overline{M}_{g,n}$ (DM compactification by stable curves)

<u>Thm 1</u> (CGP) $g \geq 2$

$\dim H^{4g-6}(M_g; \mathbb{Q}) \gg \beta^g$
for any $\beta < \beta_0 \simeq 1.32$
(real root of $t^3 - t - 1$)

<u>Recall</u>: weight filtration (Deligne)

$H^j(M_{g,n}; \mathbb{Q}) = W_{2j} H^j(M_{g,n}; \mathbb{Q})$
$\supset \cdots \supset W_0 H^j(M_{g,n}; \mathbb{Q})$

$$Gr_\ell^W H^j(M_{g,n}; \mathbb{Q}) = \frac{W_\ell H^j(M_{g,n}; \mathbb{Q})}{W_{\ell-1} H^j(M_{g,n}; \mathbb{Q})}$$

Thm actually shows
$\dim Gr_{2d}^W \gg \beta^g$, w/ $d = 3g-3+n$

$Gr^W_{2d} H^i(\mathcal{M}_{g,n};\mathbb{Q})$ = "top weight coho"

<u>Thm 2</u> (CFGP)

Top weight S_n-equivariant Euler characteristic of $\mathcal{M}_{g,n}$ can be computed as fol:

Write $Gr^W_{2d} H^i(\mathcal{M}_{g,n};\mathbb{Q})$

$$= \bigoplus_{\lambda \vdash n} c^i_\lambda \underbrace{V_\lambda}_{\text{irrep cor to } \lambda}$$

<u>Def</u>: (Frobenius character)

$$Z_g = \sum_{n \geq 0} \sum_{\lambda \vdash n} \left(\sum_{i \geq 0} (-1)^i c^i_\lambda \right) s_\lambda$$

$$\in \hat{\Lambda} = \varprojlim_n \mathbb{Q}[x_1, \ldots, x_n]^{S_n}$$

here s_λ = Shur function of shape λ.

Notation:

$p_i = x_1^i + x_2^i + \cdots$, & $P_i = 1 + p_i$

Remark: Z_g is a sum of Laurent ~~poly~~ monomials in P_i of degree $1-g$ with Bernoulli coefficients.

~~Thm 2.6 (i)~~

$$\cancel{Z_g = \sum_{k,m,r,s,a,d} \frac{(-1)^{k-r}(k-1)!\, B_r\, m^{r-1}}{r!} \prod_{p \mid m, d,}}$$

$Z_g =$

Corollary:

$$\chi^{top}(\mathcal{M}_{g,n}) = (-1)^{n+1} \frac{(g+n-2)!}{g!} B_g$$

Compare with (Harer-Zagier '86)

$$\chi^{orb}(\mathcal{M}_{g,n}) = (-1)^n (2g-1) \frac{(2g-n-3)!}{(2g)!} B_{2g}$$

Gorsky S_n-equivariant Euler char. of $\mathcal{M}_{g,n}$.

Context for theorem

Harer '86: virtual chomological dimension of $M_g = 4g-5$

Church-Farb-Putman, Morita-Sakasai-Suzuki:

$H^{4g-5}(M_g; \mathbb{Q}) = 0$ too.

CFP '14 conjecture "stabilization of unstable cohomology": fix $i>0$.

$H^{4g-5-i}(M_g; \mathbb{Q}) = 0$ for $g \gg 0$.

Implied by Kontsevich '93 ~~conjecture~~ conjecture as observed by MSS.

~~The~~ The techniques

boundary complexes

I) Algebraic Geometry $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$
Deligne-Mumford-Knudson
compactification

II) Tropical geometry: $\Delta_{g,n}$ tropical
moduli space of curves
(Abramovich-Caporaso-Payne)

cellular homology

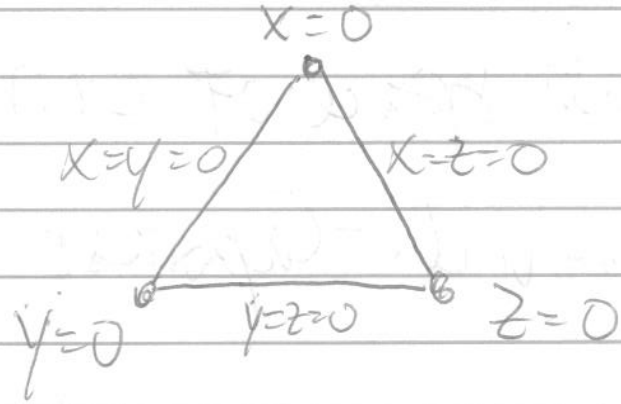
III Combinatorics: $K^{(g,n)}$ n-marked
Kontsevich graph complex '94.

X a smooth variety / DM stack

$U \subset X$ open, $D = X \setminus U$ normal crossing divisor

$\Delta(U \subset X)$ boundary complex

<u>Ex</u>: $\Delta((\mathbb{C}^\times)^2 \subset \mathbb{P}^2)$ looks like



<u>so</u>: vertices are divisors,
edges are intersections of divisors
etc.

Ex: $\Delta(M_{g,n} \subset \overline{M}_{g,n}) \cong \Delta_{g,n}$,

Thm (ACP)

where $\Delta_{g,n}$ is the tropical moduli space

Def: a tropical curve is (Γ, ℓ), where

$\Gamma = (G, m, w)$ dual graph of a stable curve of type g,n

graph

$m: \{1, \dots, n\} \to V(G)$

$w: V(G) \to \mathbb{Z}_{\geq 0}$

$\ell: E(G) \to \mathbb{R}_{>0}$, $\sum \ell(e) = 1$.

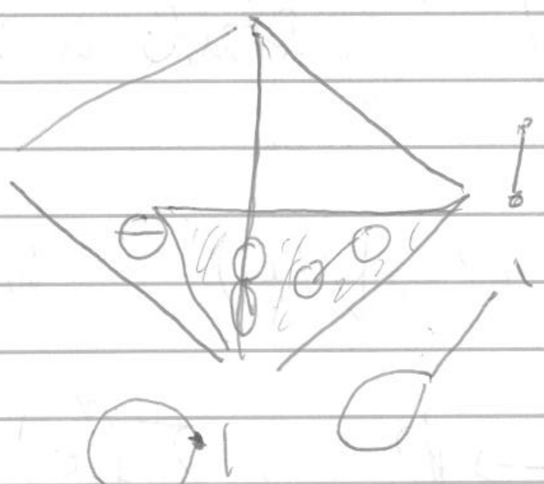
$\Delta_{g,n}$ moduli space of

(Boggi-Melo-Viviani, Caporaso, Culler-Vogtmann)

$\Delta_{g,n}$ has a stratification

$$\Delta_{g,n} = \coprod_{\Gamma \text{ type}(g,n)} \frac{\mathbb{R}_{>0}^{E(G)} \cap \{\sum \ell(e) = 1\}}{\operatorname{Aut} \Gamma}$$

<u>Ex</u>: $\Delta_{2,0}$ looks like



(see video for better picture)

Thm (dates back to Deligne)

$$\begin{tikzcd}[column sep=small] & H^{2d-i}(U;\mathbb{Q}) \arrow[dl] \arrow[dr, "\sim"] & \\ Gr^{W}_{2d} H^{2d-i}(U;\mathbb{Q}) \arrow[rr, phantom, "\simeq"] & & \widetilde{H}_{i-1}(\Delta(U\subset X);\mathbb{Q}) \end{tikzcd}$$

§ Symmetric Δ-complexes

Def: a symmetric Δ-complex is a functor

$$X : \mathrm{FinInj}^{op} \longrightarrow \mathrm{Set}$$

~~category~~ FinInj is the cat. of finite sets with injections.

[Compare with: a Δ-complex is a functor

$X: \mathrm{OrdInj}^{op} \to \mathrm{Set}$, where

OrdInj = finite ordered sets, with order preserving injections)

Ex: The half-segment

$X_0 = X(\{0\}) = \{v\}$

$X_1 = X(\{0,1\}) = \{e\}$, and

~~$X(\{0,1\} \xrightarrow{} \{0,1\})$~~

$X(\{0,1\} \xrightarrow{inv} \{0,1\}) = (\{e\} \xrightarrow{id} \{e\})$

Cellular chain complex $\mathcal{C}(X;\mathbb{Q})$

$$C_p = \mathbb{Q}X_p \Big/ x = \operatorname{sgn}\sigma \cdot \sigma x, \quad x \in X_p, \ \sigma \in S_{p+1}$$

Prop: $\mathcal{C}(X;\mathbb{Q})$ computes $\mathbb{Q}$-homology of $|X|$.

So $\Delta_{g,n}$ has $\mathbb{Q}$-homology computed by a chain complex $F^{g,n}$ where

$F_i^{(g,n)}$ is *spanned* by Γ dual graphs of stable curves type (g,n) with edges that are *alternating*
(Aut Γ acts on $E(G)$ via alt. perm.)

Prop (CGP)

$\{ (\Gamma, \ell) \mid w(v) > 0$ for some v or $m : \{1, \dots, n\} \to V(G)$ not inj $\}$

is contractible. Thus we may restrict to $K^{(g,n)}$ by requiring $w(v) = 0$ for all v and m injective.

Case where $n = 0$

$K^{(g,0)}$ is Kontsevich's 1994 complex

$$GC = \prod_{g \geq 2} \mathrm{Hom}(K^{(g,0)}, \mathbb{Q})$$

Willwacher '15: $H^0(GC) \cong \mathfrak{grt}_1$ Grothendieck where $\mathfrak{grt}_1$ is the Grothendieck-Teichmuller Lie algebra

Now use Theorem (F. Brown '12)

There is an <u>injection</u>

$$\underbrace{\hat{\mathbb{F}}_{Lie}(\sigma_3, \sigma_5, \sigma_7, \dots)}_{\substack{\text{free Lie alg. on generators} \\ \sigma_i \text{ of degree } i,\ i = 3, 5, 7, \dots}} \hookrightarrow \mathfrak{grt}_1$$

and $\dim \hat{\mathbb{F}}_{Lie}(\sigma_3, \sigma_5, \sigma_7, \dots) \geq 1.32^g$

using Poincaré-Birkhoff-Witt.