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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Charles Godfrey **Email/Phone:** cgodfrey@uw.edu

Speaker's Name: Aaron Bertram

Talk Title: Hilbert polynomials, stability conditions on derived categories, moduli and birational geometry

Date: 5 / 6 / 2019 **Time:** 11 : 00 **am / pm (circle one)**

Please summarize the lecture in 5 or fewer sentences: The talk begins with prototype stability conditions on complexes of sheaves on P^n are defined in terms of Hilbert polynomials. Then Bridgeland stability conditions on surfaces are discussed in detail. Lastly, perverse Serre t-structures are introduced.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [X] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [X] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- [X] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [X] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [ ] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Stability Conditions

MSRI

May 6, 2019

Stability Conditions and Birational Geometry

1. The Prototype

Stability Conditions and Birational Geometry

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Regularity

For each $k \in \mathbb{Z}$, consider the **category** $\mathcal{A}_k(\mathbb{P}^n)$ of complexes:

$$(*)\ \mathcal{O}_{\mathbb{P}^n}(-k-n) \otimes V_{-n} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-k) \otimes V_0$$

where V_{-p} are finite-dimensional vector spaces over $\mathbb{C}$.

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A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is (Castelnuovo-Mumford) k-regular:

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if and only if $\mathcal{F}$ is an object of $\mathcal{A}_k(\mathbb{P}^n)$, in which case (Beilinson):

$$V_{-p} = \mathrm{H}^0(\mathbb{P}^n, \mathcal{F} \otimes \Omega^p(p+k))$$

Complexes

The **scheme** of framed complexes $(*)$ is affine, a disjoint union of:

$$Y_k(r_{-n}, \ldots, r_0) \subset \times_{p=1}^{n} \mathrm{Hom}(\mathbb{C}^{r_{-p}}, \mathbb{C}^{r_{-p+1}} \otimes W)$$

(where $W = \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$) with an action of $G = \prod GL(r_{-p})$.

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A **stability condition** in this context is a set of complex numbers:

$$Z(\sigma_p) = z_p \in \mathbb{H} = \{z \in \mathbb{C} \mid 0 < \arg(z) \leq 1\}$$

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defining a linear map $Z : K(\mathbb{P}^n) \to \mathbb{C}$ with $Z(\mathcal{A}_k(\mathbb{P}^n)) \subset \mathbb{H}$. Thus,

$$Z(E) = \sum_{p=0}^{n} z_p r_{-p} \text{ for } E \in C_k(r_{-n}, \ldots, r_0)$$

King's Theorem

Theorem (King). Each Z gives polarized GIT quotients:

$$\mathcal{M}_{\mathbb{P}^n}(k; r_n, ..., r_0) = Y_k(r_{-n}, ..., r_0)//GL(\underline{r})$$

such that $E \in Y_k$ has a semi-stable orbit if and only if:

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Finiteness Corollary. Each subcategory

$$P(\phi) = \{E \in \mathcal{A}_k(\mathbb{P}^n) \mid \phi_Z(E) = \phi \text{ and } E \text{ is semi-stable}\}$$

is Artinian (in particular, Abelian) and each complex $E \in \mathcal{A}_k(\mathbb{P}^n)$ has a unique finite Harder-Narasimhan filtration.

Incorporating the Hilbert Polynomial.

Corollary. (Altavilla,B,Petkovic,Mu). For $s \in \mathbb{R}$, let:

$$Z_s(E) := h'_E(s) + i \cdot h_E(s) \in \mathbb{C}$$

where $h_E(s)$ is the **Hilbert polynomial** of E. Then Z_s defines a stability condition on $\mathcal{A}_k(\mathbb{P}^n)$ for $k = \lceil s \rceil$.

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Proof. $h_{\mathcal{O}_{\mathbb{P}^n}}(s)$ has *simple zeroes* at $-n, ..., -1$, therefore $Z_s(\mathcal{O}_{\mathbb{P}^n})$ winds around the origin, crossing the x-axis at $s = -n, ..., -1$. It follows that $Z_s(\sigma_p) = (-1)^p Z_s(\mathcal{O}_{\mathbb{P}^n}(-p)) \in \mathbb{H}$ for $s \in (-1, 0]$.

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Question. Every coherent sheaf $\mathcal{F}$ is eventually k-regular.
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Note. If $\mathcal{F}$ "comes on board" it is for $s > \text{reg}(\mathcal{F}) - 1$.

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$\mathcal{O}_E(p)$ is 1-regular, and s-stable for $s > 0.7$ via $\mathcal{O}_\Lambda \to \mathcal{O}_E(p)$ and at that point, s-moduli is the moduli of Gieseker-stable sheaves.

(The indicated morphisms are *injective* morphisms of complexes).

Chern Classes and (Hilbert) Polynomials

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be the Hilbert polynomial. Both maps factor through finitely generated free abelian subgroups Γ of the respective rational vector spaces. We will also see a simplified polynomial:

$$f : K(X) \to \mathbb{Q}[s];\;\; f_E(s) = \deg(e^{sH}\mathrm{ch}(E))$$

and we'll write $K(X) \to \Gamma$ to mean any one of these three.

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Extension to $\mathcal{D}^b(X)$. The Z-semi-stable objects of $\mathcal{D}^b(E)$ are:

$$E[p] \text{ for } E \in P(\phi) \text{ and } p \in \mathbb{Z}$$

with $\phi_Z(E[p]) = \phi_Z(E) + p$.

Significance

Theorem (Bridgeland) With a natural topology on the locus Stab(X) of stability conditions (factoring through Γ) the map:

$$\mathsf{Stab}(X) \to \mathsf{Hom}(\Gamma, \mathbb{C}) = \mathbb{C}^{\mathsf{rk}(\Gamma)}$$

is a local homeomorphism, making Stab(X) a complex manifold. (i.e. locally, the t-structure and $\mathcal{A}$ deform along with Z).

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• Moduli of semi-stable objects are equipped with "determinant" polarizations (Bayer-Macrí) that vary continuously, resulting in wall-crossing and (in some surface cases) determining ample, movable etc cones of moduli spaces of Gieseker-stable sheaves.

• Families of varieties (or triangulated categories) support family stability conditions, and families of moduli of semi-stable objects.

Curves

Let C be a curve. Then $f_E(s) = s \cdot \mathrm{rk}(E) + \deg(E)$ and let:

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A *tilt* creates $\mathcal{A} = Coh^Z(C)$ mapping to $\mathbb{H}$ via:

$$\mathcal{A} = \langle \{E \in P(\phi) \; | \phi < 0\}, \{F[1] \in P(\phi+1) | \phi \leq 0\} \rangle$$

K3 Surfaces

Theorem. (Bridgeland) Let S be a K3 surface and

$$g_E(s) = \deg(e^{sH}\mathrm{ch}(E)\sqrt{\mathrm{td}(S)})$$

and let $Z_{s,t}(g) = ig(s - it) = \Im(g(s + it)) + i\Re(g(s + it))$.

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Polynomial facts.

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(i) If $f : \mathbb{R} \to \mathbb{R}$, then $\Re(f(s + it))$ has no complex roots when $t > \max\{\Im(z) \mid f(z) = 0\}$

(ii) Then $f(s + it)$ winds around the origin (with s), and so does:

$$\frac{d}{ds}\Re f(s + it) + i \cdot \Re f(s + it) =: Z_{s,t}(f)$$

(there isn't an important distinction when $\deg(f) = 2$).

Surfaces

Geometric Facts. Let S be any surface and

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Bogomolov. If E is a Mumford-stable torsion-free sheaf, then:

$$\Delta(f_E) = (H \cdot c_1(E))^2 - 2H^2 c_0(E) c_2(E) \geq 0$$

and f_E has only real roots.

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The key point is that $\Re(f_E(s+it))$ winds around the origin when E is Mumford (equivalently $Z'_{s,t}$) stable, so that the intersection of the parabola $Z_{s,t}(f_E)$ with the y-axis has the right sign.

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Remark. For fixed Chern class c, the "walls" for moduli are semi-circles. In particular, there can (in principle) be "dead zones" where there are no $Z_{s,t}$-stable objects with invariant c.

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(ii) If X is $\mathbb{P}^3$, Fano of Picard rank one, abelian, $E \times \mathbb{P}^2$,... then (Bayer,Macrí, Stellari), the points $s + it$ for which $Z_{s,t}(f_E)$ moves in the "wrong direction" are dead zones for $Z'_{s,t}$-moduli. Like Bogomolov, this is a **quadratic** condition on the coefficients. (formulation is due to Bayer,Macrí, Stellari). It is what we need.

The Heart of the Matter

The succesive approximations by tilting produce a series of hearts:

$Coh(X) = \mathcal{A}_0$ with $Z^{(n-1)}$ mapping to the right half plane.

$Coh^{Z^{(n-1)}}(X) = \mathcal{A}_1$ with $Z^{(n-2)}$ mapping to the right half plane.

$\vdots$

$Coh^{\cdots,Z}(X) = \mathcal{A}_n$ with Z mapping to $\mathbb{H}$. Stability condition.

These are succesive approximations to a stability condition in the sense that coherent sheaves supported in codimension $> i+1$ are in $\mathcal{A}_i$ and map to zero under $Z^{(n-i-1)}$. But the categories $\mathcal{A}_i$ seem to be a series of "perversier" t-structures in the following sense:

More and More Perverse

For all coherent sheaves $\mathcal{F}$:

(0) $Coh(X)$ contains $\mathcal{F}$ and $Hom(\mathcal{F}, \mathcal{O}_X) = RHom(\mathcal{F}, \mathcal{O}_X)_{\geq 0}$.

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Call a heart satisfying property (n) a perverse **Serre** t-structure (unless it's already been named).

Challenge. To directly find Serre t-structures on varieties X. (e.g. Calabi-Yau 3-folds). Can be done when $\mathcal{D}^b(X)$ has strong full exceptional collections.