5/7/2019

Positivity of the Chow-Mumford line bundle in families of K-stable Fano varieties

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Work over algebraically closed field k of characteristic 0.
Sometimes $k = \mathbb{C}$.

Big picture:

alg. var $\longrightarrow$ MMP $\longrightarrow$ (log) Fano / weak (log) CY / (log-) Canonical model

$\longrightarrow$ moduli theory $\leadsto$ Projective moduli spaces for building blocks.

K-stable moduli $\longleftrightarrow$ $\exists$ singular Kahler-Einstein metric

$K_X < 0$

II K-stability

- X Fano, projectively normal variety. $(-K_X$ ample, X klt$)$
- X is K-stable iff the singularities of

$$|-K_X|_{\mathbb{Q}} = \left\{ \frac{D}{m} \;\middle|\; D \in |-mK_X|, m>0 \right\}$$

are mild.

- D is a q-basis type divisor if there is a basis

$$s_1, \ldots, s_r \in H^0(X, \mathcal{O}_X(-qK_X))$$

so $$D = \sum_{i=1}^{r} \frac{(s_i = 0)}{q\, h^0(-qK_X)}$$

- for a divisor E on X, the <u>log canonical threshold</u> of E is

$$\mathrm{lct}(X,E) = \sup\{t \geq 0 \mid (X, tE) \text{ is klt}\}$$

- <u>Definition</u>:

$$\delta_q(X) = \min\{\mathrm{lct}(X,D) \mid D \text{ } q\text{-basis type}\}$$

$$\delta(X) = \lim_{q\to\infty} \delta_q(X). \quad \left(\begin{array}{l}\text{limit exists}\\ \text{by Blum-Jonsson}\end{array}\right)$$

<u>Def</u> X is K-sstable $\iff \delta(X) \geq 1$

X is uniformly K-stable $\iff \delta(X) > 1$

Example: X smooth Del Pezzo

X K-sstab iff $(-K_X)^2 \neq 7, 8$ or $X = \mathbb{P}^1 \times \mathbb{P}^1$

iff $\mathrm{Aut}(X)$ is reductive

State of the art:

$$\exists \text{ sing. K-E metric} \xrightarrow{\text{Berman}} \boxed{\text{K-sstab}}$$

$$\exists \text{ sing. K-E metric} \underset{\substack{\text{conj YTD} \\ \text{known smoothable}}}{\Longleftrightarrow} \boxed{\text{K-polystable}}$$

$$\boxed{\text{unif K-stab}} \Longrightarrow \exists \text{ sing. K-E metric}$$

[Li Tian Wang]

III CM line bundle

$f : X \longrightarrow T$ flat, proper,

n-dim'l <u>normal</u> fibers

$K_{X/T}$ $\mathbb{Q}$-Cartier, X_t Fano for all t

<u>Definition</u>: ~~the Chow-Mumford line bundle~~

$$\lambda_f := -f_*\left((-K_{X/T})^{n+1}\right)$$ as <u>cycles</u> on T.

<u>Thm</u> (Knudsen-Mumford)

$$c_1\left(f_* \mathcal{O}_X(-qK_{X/T})\right) = -\lambda_f \frac{q^{n+1}}{(n+1)!} + O(q^n)$$

$\boxed{IV}$ Moduli Conjecture

Fix $n = \dim$, $V = (-K_X)^n$. There are

$\boxed{a}$ $\mathcal{M}^{K\text{-}ss}_{n,V}$ Artin stack of finite type/k

$\boxed{b}$ $M^{Kps}_{n,V}$ proper good moduli space of such that λ descends.

$\boxed{C}$ $M^{Kps}_{n,V}$ projective using λ

$\Updownarrow$ via Nakai-Moishezon

$\boxed{C1}$ λ_f is nef in families of K-ss Fanos

$\boxed{C2}$ λ_f big in max'ly varying families of K-pstab Fanos

V Theorem (Codogni–Patakfalvi)

C1, C2 if general fiber is uniformly K-stable.

Va | Consequences to

$f: X \to \mathbb{P}^1$ K-stab gen fiber
fano

① $(-K_X)^n \leq 2(\dim X)^{\dim X}$

(Examples: $Bl_p\mathbb{P}^2 \to \mathbb{P}^1$
$Bl_L\mathbb{P}^3 \to \mathbb{P}^1$)

② $\exists \leq 4$ multiple irreducible fibers.

$\boxed{\text{VI}}$ | Proof of nefness.

$f: X \to T$ family of of K-sstab Fanos

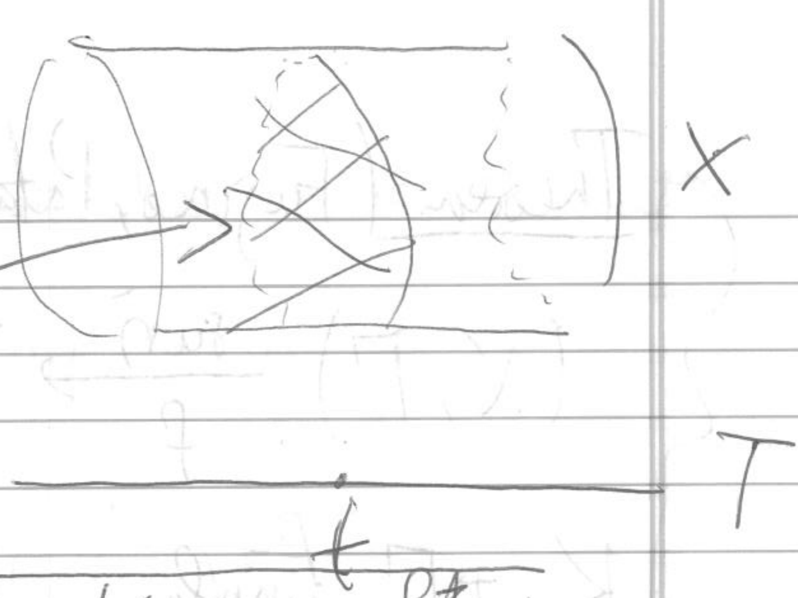
Want: λ_f nef.

Since λ_f is functorial, can reduce to: T smooth projective curve.

For simplicity: generic fiber unif K-ss ($\delta(X_{gen}) > 1$)

~~As~~ T a smth proj curve, so
λ_f nef $\iff \deg \lambda_f \geq 0$.

<u>Idea:</u>



q-basis type divisor on a fiber

$\boxed{\text{set } L = K_{X/T} - \varepsilon f^* H}$

<u>Translation to vector bundles:</u>

~~here~~ $\cancel{s_i \in H^0(X_t, \mathcal{O}_{X_t}(-qK_{X_t})) \simeq k(t) \otimes f_* \mathcal{O}_X(}$

<u>have</u> $s_i \in H^0(X_t, \mathcal{O}_{X_t}(qL_t)) \simeq k(t) \otimes f_* \mathcal{O}_X(qL)$

<u>lift</u> to $\tilde{s}_i \in H^0(T, f_* \mathcal{O}_X(qL))$

+ perturbation $F_q \subseteq f_* \mathcal{O}_X(qL)$ globally generated part.

<u>Enough</u>: $\lim_{q \to \infty} \frac{\mathrm{rk}\, F_q}{\mathrm{rk}\, f_* \mathcal{O}_X(qL)} = 1$.

Magic: replace f w/ its m-fold fiber product

$$\underbrace{X \times_T X \times_T \cdots \times_T X}_{m\text{-times}} \xrightarrow{f^{(m)}} T$$

now if $F_{q,m}$ is globally generated part of $f_* \mathcal{O}_X(-qK_{X/T})^{\otimes m}$,

$$\lim_{q \to \infty} \frac{\operatorname{rk} F_{q,m}}{\operatorname{rk} H^0(T, f_* \mathcal{O}_X(-qK_{X/T})^{\otimes m})} \longrightarrow 1.$$