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# NOTETAKER CHECKLIST FORM

**(Complete one for each talk.)**

**Name:** Charles Godfrey **Email/Phone:** cgodfrey@uw.edu

**Speaker's Name:** Max Lieblich

**Talk Title:** Perfect curves on elliptic K3 surfaces

**Date:** 5 / 8 / 201 **Time:** 11 : 00 **am / pm (circle one)**

**Please summarize the lecture in 5 or fewer sentences:** A criterion for the existence of a purely inseparable multi-section of a super-singular elliptic K3 surface with Artin invariant 8, 9, or 10 is described. Consequences for Artin's unirationality conjecture are addressed.

# CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
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- [ ] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

# Perfect curves on elliptic K3 surfaces

Speaker: Max Lieblich

(Joint with Dan Bragg)

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$K = k(t)$, $k = \bar{k}$, char $k \geq 5$

$C/_K$ curve. $C(K^{perf}) = ?$

equiv: pts of $C$ with purely inseparable residue field $/K$.

Idea: $\begin{array}{c} C \subset X \\ \downarrow \\ \mathbb{P}^1 \end{array}$ perfect points correspond to purely inseparable multisections.

Picture

$$\begin{tikzcd} & X \arrow[d] \\ \mathbb{P}^1 \arrow[ur] \arrow[r, "F^e"'] & \mathbb{P}^1 \end{tikzcd}$$

some power of the Frobenius

**Theorem**: Fix $8 \leq s \leq 10$. For a very general supersingular K3 surface $X$ of artin invariant $s$, any elliptic structure

$$X \longrightarrow \mathbb{P}^1$$

admitting a perfect curve has a section.

Remark: Liedtke 2015, proof of
Artin's unirationality conjecture
(any supersing K3 is unirational)
Key step: any elliptic structure
$X \to \mathbb{P}^1$ has a perfect curve.

This is false.

(Lots of audience questions, see
video).

Status of unirationality conjecture:

known for artin invariant $\leq 2$.

maybe $\leq 4$

depending on characteristic.

Two parts: 1) Why is there one ~~example~~ example?

2) Why are there lots of examples?

Elliptic K3s:

start with Weierstrass equation

$$y^2 = x^3 + a(t)x + b(t) \text{ over } k(t).$$

$$\begin{array}{ll} \text{Get } X & \text{when is } X \text{ K3?} \\ \downarrow & \\ \mathbb{P}^1 & \sim \deg a = 8, \deg b = 12 \end{array}$$

(when fiber @ $\infty$ is multiplicative)

Classification of singular fiber
(Kodaira-Tate)

Determined by $a, b, d = 4a^3 + 27b^2$

Ex: II $\left\{ \right.$ cuspidal $a, b$ common root at $\alpha$

$r_\alpha(d) = 2$, $a(t) = (t-\alpha)\,\bar{a}$
$b(t) = (t-\alpha)\,\bar{b}$

If $a(t) \not\equiv 0$ : $\leq 8$ fibers of type II

$$y^2 = x^3 + (t-\alpha_1)\cdots(t-\alpha_8)x$$
$$+ \beta_0 (t-\alpha_1)\cdots(t-\alpha_8)$$
$$\cdot (t-\beta_1)\cdots(t-\beta_4)$$

Special divisor: (Frobenius-Tate divisor)

$$\Delta(f) = \sum V_x x, \quad \cancel{x \in \mathbb{P}}$$

Runs over $x \in \mathbb{P}^1$ pt, $X_x$ is additive

$V_x = V_x(d)$

Ex 8 II: $\Delta(f) = \sum_{i=1}^{8} 2\alpha_i \Leftrightarrow f(s,t) = \prod_{i=1}^{8} (t-\alpha_i s)^2$

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Supersingular K3s.

Def: A K3 surface $X$ is ~~supersingular~~ supersingular if $\operatorname{rk} \operatorname{Pic}(X) = 22$

Thm (Artin) If $X$ is a supersingular K3 then ~~disc~~

$$\operatorname{disc}(\operatorname{Pic}(X)) = p^{2\sigma_0}, \text{ where}$$

$1 \leq \sigma_0 \leq 10$. Moreover, the surfaces w/ $\sigma_0 = s$ come in families of dim $s-1$.

Ogus: developed moduli theory

$\sigma_0$ = Artin invariant.

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Br/

$$\begin{tikzcd} J \arrow[r, "f"] & \mathbb{P}^1 \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

Leray spectral sequence gives

$$\operatorname{Br}(J) = Ш(J_q), \; J_q \text{ the generic fiber.}$$

$H^2(J, \mathbb{G}_m) \longleftrightarrow$ étale forms of $J$

$X \longrightarrow \mathbb{P}^1$

$J$ K3 $\Rightarrow X$ K3.

Special property when $J$ is supersingular!

<u>Theorem</u> (Artin, Bragg-L, Fakhruddin, Ogus –)

(i) There is a class $\alpha \in \mathrm{Br}(J \times \mathbb{A}^1)$ such that for all $\beta \in \mathrm{Br}(J)$,

$$\exists\, s \in \mathbb{A}^1 \text{ such that } \beta = \alpha|_{J_s}.$$

(ii) $\alpha|_{J \otimes k[s]/s^2} \in \ker\left(\mathrm{Br}(J \otimes k[s]/s^2) \to \mathrm{Br}(J)\right)$ is a generator for $H^2(J, \mathcal{O}_J)$.

$$\text{Given } \begin{array}{c} X \\ \downarrow \\ \mathbb{P}^1 \end{array} \Longleftrightarrow B \in Br(J(X)) \quad \begin{array}{c} J(X) \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

There is a family

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathbb{P}^1 \times \mathbb{A}^1 \\ \cup & & \cup \\ J & \longrightarrow & \mathbb{P}^1 \times \{0\} \end{array} \quad ; \quad \mathcal{X} \Longleftrightarrow \alpha \in Br(J \times \mathbb{A}^1)$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

These are known as ~~tw~~ twistor families/Artin-Tate families.

They cover the moduli space of lattice polarized K3s.

## Criterion for perfect curves

Picture:

$$\begin{tikzcd} J^{(e)} \arrow[r, "\varphi_e"] \arrow[d] & J \arrow[d] \\ \mathbb{P}^1 \arrow[r, "F^e"] & \mathbb{P}^1 \end{tikzcd}$$

here $J^{(e)}$ is a resolution of $\mathbb{P}^1 \times_{F^e, \mathbb{P}^1} J$.

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Given $X \longrightarrow \mathbb{P}^1$ w/ $J(X) \to \mathbb{P}^1$
$\updownarrow$
$\alpha \in Br(J)$

$$J(X) \to \mathbb{P}^1 \;=\; J \longrightarrow \mathbb{P}^1$$

there is a perfect curve on $X$ of degree $p^e$ over $\mathbb{P}^1$ $\Longleftrightarrow$ $\varphi_e^* \alpha = 0 \in Br(J^{(e)})$

Nonexistence:

$$J \underset{F}{\rightleftarrows} \mathbb{P}^1 \qquad \mathcal{X} \longrightarrow \mathbb{P}^1 \times \mathbb{A}^1$$

univ family

$\alpha \in Br(J \times \mathbb{A}^1)$

Let $\infty \to \mathbb{A}^1$ be a geometric generic point.

Q: when does $\mathcal{X}_\infty \xrightarrow{f_\infty} \mathbb{P}^1_\infty$

$\downarrow$

$\alpha_\infty \in Br(J \otimes k(\infty))$

admit a perfect curve?

Criterion: $f_\infty$ admits a perfect curve

$\iff$ $\varphi_e^*(\alpha_\infty) = 0 \in \mathrm{Br}(J^{(e)} \otimes k(\infty))$
for some $e$

$\iff$ $\alpha|_{J^{(e)} \times \mathbb{A}^1} = 0$

$\Rightarrow$ $\alpha|_{J^{(e)} \otimes k[s]/s^2} = 0$

$\Rightarrow$ $\varphi_e^*: H^2(J, \mathcal{O}) \to H^2(J^{(e)}, \mathcal{O})$
is 0.

Upshot: If $\varphi_e^*: H^2(J, \mathcal{O}) \to H^2(J^{(e)}, \mathcal{O})$

is injective for all $e$, then
$X_\infty \to \mathbb{P}^1_\infty$ has no perfect curves.

<u>Question</u>: how to compute $\varphi_e^*$?

$$\begin{tikzcd}
H^2(J,\mathcal{O}) \arrow[r] \arrow[d, equal] & H^2(J^{(e)},\mathcal{O}) \arrow[d, equal] \\
H^1(\mathbb{P}^1, R^1f_*\mathcal{O}) \arrow[r] & H^1(\mathbb{P}^1, R^1f^{(e)}_*\mathcal{O})
\end{tikzcd}$$

associated to $g_e : R^1f_*\mathcal{O} \to F^e_* R^1 f^{(e)}_*\mathcal{O}$

<u>Key</u>: $R^1f^{(2e)}_*\mathcal{O}_{J^{(2e)}} = \left((F^{2e})^* R^1 f_*\mathcal{O}\right)\left(\frac{p^{2e}-1}{12}\Delta(f)\right)$

Follow Tate's algorithm.

<u>Cor</u>: $g_{2e}$ is identified w/

$$R^1f_*\mathcal{O} \otimes \left(\mathcal{O} \to F^{2e}_*\mathcal{O}\left(\frac{p^{2e}-1}{12}\Delta(f)\right)\right)$$

$\mathcal{O} \to F^{2e}_* \mathcal{O}\left(\frac{p^{2e}-1}{12}\Delta(f)\right)$ is adjoint to

$\mathcal{O} \longrightarrow \mathcal{O}\left(\frac{p^{2e}-1}{12}\Delta(f)\right)$

associated to the divisor $\frac{p^{2e}-1}{12}\Delta(f)$

Sully: $\varphi^*_{2e}$ inj $\Leftarrow$ $\mathcal{O} \to F^{2e}_* \mathcal{O}\left(\frac{p^{2e}-1}{12}\Delta(f)\right)$ splits.

Fun math problem: $\pi: \mathbb{P}^1 \longrightarrow \mathbb{P}^1$
$x \longrightarrow x^n$

$D \in |\mathcal{O}_{\mathbb{P}^1}(d)| \Longleftrightarrow f(s,t) \in \Gamma(\mathbb{P}^1, \mathcal{O}(d))$

get $\mathcal{O} \longrightarrow \pi_* \mathcal{O}(d)$. When does it split?

Answer: ford<2n

Answer: it splits iff there is a term $a\, s^i t^j$ in $f(s,t)$ with $i < n,\ j < n$.

Ex 8II: $\frac{p^{2e}-1}{12} \Delta \mapsto \prod_{i=1}^{8} (t - \alpha_i s)^{\frac{p^{2e}-1}{6}}$

Lemma: space cut out by vanishing of coefficients has codim $\geq 5$