Finiteness properties of hyperbolic varieties

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§ Brody hyperbolic

A variety X over $\mathbb{C}$ is <u>Brody Hyperbolic</u> iff every holomorphic map $\mathbb{C} \longrightarrow X^{an}$ is <u>constant</u>.

<u>Thm</u>: X smooth projective curve.
X Brody hyperbolic $\Leftrightarrow g(X) \geq 2$.

Remark: A abelian variety.

$\mathbb{C}^g \xrightarrow[\exp]{} A$ shows A is not Brody hyperbolic.

Theorem (Bloch - Ochiai - Kawamata)
'26 '83

$X \subset A$ closed subvariety,
A an abelian variety over $\mathbb{C}$.

X Brody hyperbolic
$\iff$ X does not contain the translate of a positive dim'l abelian subvariety.

Starting point:

Thm [Brody - Kobayashi]

If X is a Brody Hyperbolic projective variety $/\mathbb{C}$, then $\mathrm{Aut}(X)$ is finite.

Arithmetic hyperbolicity: k alg closed, char $k = 0$

A projective variety X/k is arithmetically hyperbolic $/k$ if $\forall K \subset k$ finitely generated subfield & $\forall \mathcal{X}/K$ model the set $\mathcal{X}(K)$ is finite

Remark X proj/$\mathbb{Q}$. $X \subseteq \mathbb{P}^n_{\mathbb{Q}}$.

$X_{\bar{\mathbb{Q}}}$ arithmet. hyp $\iff$ $\forall K$ Number field $X(K)$ finite

$X_{\mathbb{C}}$ arithmet hyp. $\iff$ $\forall K \subset \mathbb{C}$ finitely generated $X(K)$ finite

Ex: $\mathbb{P}^1$ is not arithmet. hyperbolic.
[arithmet. hyp. vars can't contain rational curves]

Thm [Faltings '85] X smooth projective curve.

X arithmet. hyp $\iff g(X) \geq 2$.

Thm: Hassett - Tschinkel:

A an abel. var./k.

$\exists\, K \subset k$ fin. gen. subfield +

$\mathcal{A}/_K$ model with $\mathcal{A}(K) \subset A$ dense.

Thm: X closed subvariety of abelian variety $A_{\bar{k}}$.

X arithmet. hyp $\Longleftrightarrow$ X does not contain the translate of a non-0 abel. subvariety.

Conjecture (Lang-Vojta)
X projective variety over $k \subset \mathbb{C}$

X arithmet. hyp $\iff$ $X_{\mathbb{C}}$ Brody hyp.

[J] Thm: If X is an arithmetically hyperbolic projective variety$/k$, then $\mathrm{Aut}(X)$ is finite.

$k \subseteq \mathbb{C}$, X projective variety over k

Lang-Vojta predicts: $X/_k$ arithmet. hyp.

$\implies X_{\mathbb{C}}$ arithmet. hyp.

Thm [JT]: True if $X_{\mathbb{C}}$ is Brody

Thm [Van Bommel - J - Kamenova]

X proj surface$/k$ $\quad k \subseteq \mathbb{C}$

Assume $Alb(X) \neq 0$, equiv

X admits a non-constant map to some abelian variety.

~~Then~~ Then $X/_k$ arithmet. hyp

$\Rightarrow X_{\mathbb{C}}$ arithmet. hyp.

Uses ~~Lang's specialization theo~~ Silverman's specialization theorem.

§ Varieties of general type

C curve genus $g \geq 2$, arithmet. hyperbolic

$C \times C$ surface general type: arithmet hyp.

$Bl_p\, C \times C$ still general type, <u>not</u> arithmet. hyp.

<u>Conjecture</u> (Lang) X proj. $/k$.

X general type $\Longleftrightarrow$ X pseudo arithmetically hyperbolic!

Theorem [Iitaka, Kobayashi-Ochiai]

X projective variety of general type

Then 1) $\mathrm{Bir}_k(X)$ is finite

2) for all Y proj $/k$, the set

$\mathrm{Surj}_k(Y, X)$ is finite.

(still true if "Surj" is replaced w/ dominant rational maps)

Thm [J-Xie] if X is projective ~~&~~ & pseudo-arithmet. hyp variety $/k$. Then

$\mathrm{Bir}_k(X)$ is finite.

§ Boundedness

A projective variety $X/_k$ is pseudo-bounded if $\exists\, \Delta \subset X$ closed such that $\forall\, Y$ proj curve

$$\underline{\mathrm{Hom}}(Y, X) \setminus \underline{\mathrm{Hom}}(Y, \Delta) \text{ is of finite type.}$$

Conj [Lang] $X/\mathbb{C}$

X general type $\Longleftrightarrow$ X is pseudo-bounded

Thm 5 [J-Xie] X pseudo-bounded proj. var $/k$. Then

1) $\mathrm{Bir}_k(X)$ is finite & 2) $\mathrm{Sur}(Y, X)$ is finite for all proj var Y.

Proof that X arithmet. hyp $\Rightarrow \mathrm{Aut}_k(X)$ is finite.

Step I: $\mathrm{Aut}_k(X)$ is torsion:

$\sigma \in \mathrm{Aut}_k(X)$, $p \in X(k)$

Choose $K \subset k$ ~~fine~~ fin. gen,

$\mathcal{X}/K$ model with $\tilde{p} \in \mathcal{X}(K)$
and $\tilde{\sigma} : \mathcal{X} \longrightarrow \mathcal{X}$

Then $\{\tilde{\sigma}^i \tilde{p}\}_{i=1}^{\infty} \subseteq \mathcal{X}(K)$ hence finite.

Thm [Amerik 2010] if $f : X \to X$ is an endomorphism s.t. $\forall p \in X(k)$, $\{f^i p\}_{i=1}^{\infty}$ is finite, then f has finite order.

Pf (when k is uncountable)

For $i, j \geq 1$, set $X^{i,j} = \{p \in X \mid f^i p = f^j p\}$

Assumption: $X(k) = \bigcup_{\substack{i,j \geq 1 \\ i \neq j}} X^{i,j}$

k uncountable $\Rightarrow X(k) = X^{i,j}$ for some i,j. $\square$

Thm $k = \bar{k} \supset \mathbb{Q}$, X proj $/k$.

$\mathrm{Aut}(X)$ torsion $\Rightarrow \mathrm{Aut}(X)$ finite.

Pf: $\underbrace{\mathrm{Aut}(X)}_{\text{torsion}} \longrightarrow \underbrace{\mathrm{Aut}(NS(X))}_{\text{virtually torsion free}}$

hence image is finite.

Questions from audience

Q: What about dominant rational maps $Y \dashrightarrow X$?

A: Difficulty is understanding a moduli space for these.