Kloosterman crystals for reductive groups

Speaker: Xinwen Zhu

<u>Bessel's equation</u>

$$\left(t\frac{\partial}{\partial t}\right)^2 u - tu = 0$$

<u>Kloosterman sum</u>:

$$Kl_2(a) = \sum_{x \in \mathbb{F}_p^\times} \exp\left(\frac{2\pi i}{p}\left(x + \frac{a}{x}\right)\right)$$

Integral representation: $\mathbb{G}_m^2 \xrightarrow{\text{add}} \mathbb{G}_a = \mathbb{A}^1$, $\mathbb{G}_m^2 \xrightarrow{\text{m}} \mathbb{G}_m$

Bessel's D-mod:

$$Be_2(\lambda) := d - \begin{pmatrix} & 1 \\ \lambda^2 t & \end{pmatrix} \frac{dt}{t}$$

($\ell \neq p$) Kloosterman loc syst:

$$Kl_2(\psi) = m_! \, add^* \mathcal{L}_\psi$$

$\mathcal{L}_\psi$ Artin Schreier loc system on $\mathbb{A}^1$ associated to $\psi : \mathbb{F}_p \to \mathcal{O}^\times$

$$Be_2(\lambda) \simeq Kl_{2,dR} := m_! \, add^* \text{“}e^{\lambda z}\text{”} [1]$$

$e^{\lambda z}$ the exponential D-mod on $\mathbb{A}^1$

$(\mathcal{O}, d - \lambda dz)$

__Dwork__: Let $\lambda^{p-1} = -p$ and regard

Be_2 as a D-mod on $\mathbb{G}_m / K = \mathbb{Q}_p(\mu_p) \ni \lambda$

Frob pullback $t \mapsto t^p$

$$d - \begin{pmatrix} & 1 \\ \lambda^2 t & \end{pmatrix}\frac{dt}{t} \longmapsto d - p\begin{pmatrix} & 1 \\ \lambda^2 t^p & \end{pmatrix}\frac{dt}{t}$$

__Thm__ (Dwork) $p > 2$.

There exists a unique $F(t) \in GL_2(\mathcal{O}^\dagger)$ such that

1) $$t\frac{\partial F}{\partial t} \cdot F^{-1} + F\begin{pmatrix} & 1 \\ \lambda^2 t & \end{pmatrix}F^{-1} = p\begin{pmatrix} & 1 \\ \lambda^2 t^p & \end{pmatrix}$$

__Where__

$\mathcal{O}^{+}$ = ring of analytic functions on a disc of radius > 1

$$= \bigcup_{r>1} \left(K\langle \tfrac{t}{r} \rangle = \left\{ \sum_i a_i \left(\tfrac{t}{r}\right)^i \,\middle|\, |a_i|_p \to 0 \right\} \right)$$

(2) For every $a \in \mathbb{F}_p^\times$ with $[a] \in \mathbb{Z}_p^\times$ Teichmuller lift

$$\operatorname{tr} F([a]) = Kl_2(a, \psi) \qquad \psi \longleftrightarrow \lambda$$

(3) Let α, β be the 2 eigenvalues of $F([a])$. Then $|\alpha|_p = 1$, $|\beta|_p = \frac{1}{p}$.

<u>Rmk</u>: $(\mathcal{B}e_2, F)$ is an example of an over-convergent F-iso crystal.

$\simeq m_! \, add^* \underbrace{(e^{\lambda z}, F)}$ Dwork's crystal

Let G be a split-reductive group
($G = GL_n$, worked out by Deligne, Katz, Spector)

Frenkel Gross:

$\mathcal{B}e_G$: trivial G-bundle on $\mathbb{G}_m$
with flat connection.

$$d - (N + \lambda^k t E)\frac{dt}{t}$$

w/ $N = \sum_{\alpha \in \Delta} e_\alpha$, Δ the set of simple roots of G

$E = E_{-\theta}$, θ highest root

e_α root vector assigned to α.

Given a <u>representation</u> $G \to GL(V)$

<u>get</u> $\mathcal{B}e_G^V : d - d\rho(N + \lambda^k f E)\frac{dt}{t}$.

<u>Ex</u>: $G = GL_n \quad V = Std.$

$$\mathcal{B}e_{GL_n}^{Std} = d - \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ \lambda^n t & & & 0 \end{pmatrix}\frac{dt}{t}.$$

<u>Ex</u>: $G = SO_{2n+1}, \; V = (Std, \langle\ \rangle = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$

$$\mathcal{B}e_{SO_{2n+1}}^V = d - \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ \lambda^{2n} t & & & 0 \\ & \lambda^{2n} t & & \end{pmatrix}$$

Heinloth - Ngô - Yun

$\check{G}$ Langlands dual group of G

$$\begin{tikzcd}
Gr \arrow[d, "\pi"'] \arrow[dr, "\hat{\pi}"] & & Gr^0 \arrow[ll, hook', "\mathring{j}"'] \arrow[d] \arrow[dr, "f"] & \\
\mathbb{G}_m & Bun_{\hat{G},0,\infty} & U = \mathbb{A}^r \arrow[l, hook', "j"'] \arrow[r, "\mathrm{add}"] & \mathbb{A}^1
\end{tikzcd}$$

open immersion.

w/ $Bun_{\hat{G},0,\infty}$ = moduli of $\hat{G}$ bundles on $\mathbb{P}^1$ with certain level structure at $0, \infty$.

- at 0, parabolic level (e.g. $\hat{G} = GL_n$ $\mathcal{E}_0 > F^1\mathcal{E}_0 > \cdots > F^n\mathcal{E}_0$)
- at ∞, beyond parabolic level.

Gr = Affine Grassmannian

$$\simeq \left\{ (t, \mathcal{E}, \beta) \;\middle|\; \begin{array}{l} t \in \mathbb{G}_m, \ \mathcal{E} \in \mathrm{Bun}_{\hat{G}(0,\infty)} \\ \beta : \mathcal{E}|_{\mathbb{P}^1 \setminus \{t\}} \longrightarrow \mathcal{E}^{triv}|_{\mathbb{P}^1 \setminus \{t\}} \end{array} \right\}$$

<u>HNY</u>: $V \in \mathrm{Rep}\, G$

$$Kl_G^V = \pi_! \left(h^* \left(j_! \, add^* \mathcal{L}_\psi \right) \otimes \mathrm{Sat}(V) \right)$$

$$\mathrm{Sat} : \mathrm{Rep}(G) \longrightarrow \mathrm{Perv}(\mathrm{Gr})$$

Geometric Satake.

Sat(V) supported on some finite dimensional $\mathrm{Gr}_V \subset \mathrm{Gr}$.

Also have

$$\mathrm{Kl}_V = \pi_! \left(\mathring{j}_! (f^* \mathcal{L}_\psi) \otimes \mathrm{Sat}(V) \right)$$

Thm (Kn-Z) Fix G.

$\exists\, F(t) \in G(\mathcal{O}^+)$ so that

$$t \frac{\partial F}{\partial t} F^{-1} + \mathrm{Ad}_F (N + \lambda^n t E)$$

$$= p(N + \lambda^n t^p E)$$

and so that for every $a \in \mathbb{F}_p^\times$, $V \in \mathrm{Rep}(\hat{G})$,

$$\mathrm{tr}(F(\{a\}), V) = \mathrm{tr}(\mathrm{Frob}_a \mid \mathrm{Kl}_G^V)$$

Corollary: "Generic Newton slope" is ρ

Conjecture: Newton slope is ρ everywhere

Corollary: One can identify Kl_G from different groups G (Conj KNY)

Eg: $Kl_{GL(2n)} = Kl_{Sp(2n)}$

$$Kl_{SO(2n+1)} = Kl_{SO(2n+2)} \overset{n=3}{=} Kl_{G_2}$$

$$Kl^{std}_{SO(2n+1)} = Hyp(\psi, \underbrace{1, \ldots, 1}_{2n}, \eta)$$

Legendre

$$\overset{n=1}{\simeq} q\left(\sum_{a\neq 0} \psi\left(n-\frac{a}{n}\right)\sum_{v\neq 0}\left(\frac{a}{v}-v\right)-1\right)$$

$$= \sum_{x_1x_2x_3=ay} \psi(x_1+x_2+x_3-y)\left(\frac{y}{p}\right)$$

Pf of the thm

1) Thm(Z) $Be_G = Kl_{\mathcal{E},dR}$

2) Geometric Satake for arithmetic D-modules.

3) Compare Rigid & algebraic de Rham.

$$\begin{tikzcd} & \mathrm{Gr}^{\circ}_{V} \arrow[dl] \arrow[dr, "f"] & \\ \mathbb{G}_m & & \mathbb{A}^1 \end{tikzcd}$$

some specific V

$Be^{V}_{G} \cong Kl^{V}_{G}$

$\rightsquigarrow$ Mirror conjecture by Rietsch

Lam Templier

Questions from audience

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