

Kloosterman crystals for
reductive groups

Speaker: Xinwen Zhu

Bessel's equation

$$(t \frac{\partial}{\partial t})^2 u - tu = 0$$

Kloosterman sum:

$$Kl_2(a) = \sum_{x \in \mathbb{F}_p^*} \exp\left(\frac{2\pi i}{p} \left(x + \frac{a}{x}\right)\right)$$

Integral
representation: \mathbb{G}_m^2 $\xrightarrow{\text{add}}$
 \mathbb{G}_m $\mathbb{G}_a = \mathbb{A}^1$

Bessel's D-mod:

$$B_2(\lambda) := d - \left(\lambda^2 t \right) \frac{dt}{t}$$

(RFP) Kloosterman loc syst:

$$Kb_2(\psi) = m_1 \text{ add}^* L_\psi$$

L_ψ Artin-Schreier loc system

on \mathbb{A}^1 associated to $\psi: \mathbb{F}_p \rightarrow \mathcal{O}^*$

$$\textcircled{B} B_2(\lambda) \simeq Kb_{2,de} := m_1 \text{ add}^* \left[e^{\lambda z} \right]$$

$e^{\lambda z}$ the exponential D-mod on \mathbb{A}^1
($\mathcal{O}, d - \lambda dz$)

Dwork: Let $\lambda^{p^2} = -p$ and regard

~~the~~ \mathbb{C}_2 as a D -mod on $\mathbb{C}_m/K = \mathbb{C}_p(\lambda^p)$

Frob pullback $t \mapsto t^p$

$$d - (\lambda^{2t} \mathbb{1}) \frac{dt}{t} \mapsto d - p (\lambda^{2t^p} \mathbb{1}) \frac{dt}{t}$$

Thm (Dwork) $p > 2$.

There exists a unique $F(t) \in \text{GL}_2(\mathcal{O}^{\dagger})$
such that

$$\begin{aligned} \text{I) } t \frac{dF}{dt} \cdot F^{-1} + F \begin{pmatrix} \lambda^{2t} & \\ & \mathbb{1} \end{pmatrix} F^{-1} \\ = P \begin{pmatrix} \lambda^{2t^p} & \\ & \mathbb{1} \end{pmatrix} \end{aligned}$$

Where

$\mathcal{O}^+ =$ ring of analytic functions
on a disc of radius > 1

$$= \bigcup_{r > 1} (K \langle \frac{t}{r} \rangle = \{ \sum_i a_i \binom{n}{i} \mid |a_i|_p \rightarrow 0 \})$$

(2) For every $a \in \mathbb{F}_p^x$ with $[\alpha] \in \mathbb{Z}_p^x$
Teichmüller lift

$$\text{to } F([\alpha]) = \text{Kl}_2(a, \psi) \quad \psi \leftrightarrow \lambda$$

(3) Let α, β be the 2 eigenvalues of
 $F([\alpha])$. Then $|\alpha|_p = 1, |\beta|_p = \frac{1}{p}$.

Rank: (Be_2, F) is an example of
an over-convergent \bar{F} -iso crystal.

$\simeq m_1 \text{ add}^* (\underbrace{e^{2z}, F}_{\text{Dwork's crystal}})$

Let G be a split-reductive group

($G = GL_n$, worked out by Deligne, Katz, Spitzer)

Frenkel Gross =

B_{reg} = trivial G -bundle on \mathbb{G}_m
with flat connection.

$$d - (N + \lambda^k t^k) \frac{dt}{t}$$

$$w/ \quad N = \sum_{\alpha \in \Delta} e_{\alpha}$$

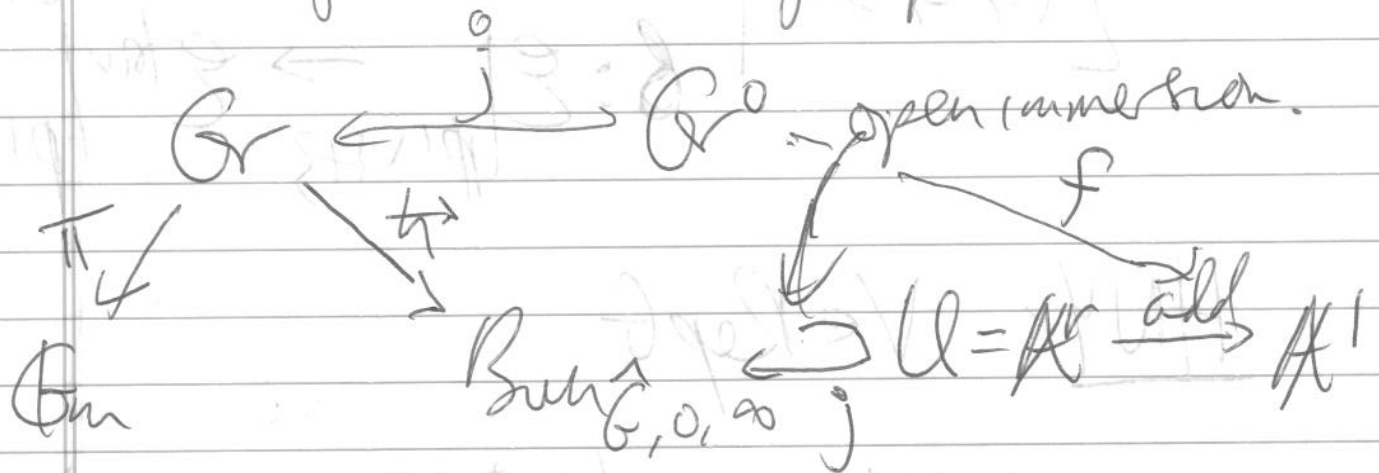
$$E = E_{-1}$$

Δ the set of
simple roots of G

θ highest root
 e_{α} root vector
assigned to α .

Heinloth-Ngô-Yun

\check{G} Langlands dual group of G



$U / \text{Bun}_{G,0,\infty} =$ moduli of \check{G} bundles
 on \mathbb{P}^1 with certain level
 structure at $0, \infty$.

- at 0 , parabolic level (e.g. $\check{G} = GL_n \quad \mathcal{E}_0 \supset F^1 \mathcal{E}_0 \supset \dots \supset F^r \mathcal{E}_0$)
- at ∞ , beyond parabolic level.

$G_r = \text{Affine Grassmannian}$

$$\approx \left\{ (t, \varepsilon, \zeta) \mid t \in \mathbb{G}_m, \varepsilon \in \text{Ker} \hat{\sigma}_{1,0,\infty} \right\}$$
$$\left. \begin{array}{c} \mathcal{B}: \varepsilon \mid_{\mathbb{P}^1 \setminus \{3\}} \rightarrow \varepsilon \mid_{\mathbb{P}^1 \setminus \{3\}} \end{array} \right\}$$

HNY: $V \in \text{Rep } G$

$$Kl_G^V = \pi_1 \left(\mathcal{H}^{\text{odd}} \left(\begin{array}{c} \mathbb{C} \\ \mathbb{C} \end{array} \right) \otimes \text{Sat}(V) \right)$$

$$\text{Sat}: \text{Rep}(G) \longrightarrow \text{Per } V(G)$$

Geometric Satake.

$\text{Sat}(V)$ supported on some finite dimensional $G_V \subset G$.

Also have

$$Kl_V = \pi_! \left(\int_! (F^* \alpha_\psi) \otimes \text{Sat}(V) \right)$$

Thm (Ku-Z) $\text{Fix } G.$

$\exists F(t) \in G(\mathcal{O}^+)$ so that

$$\left(\frac{\partial F}{\partial t} F^{-1} + \text{Ad}_F(N + \lambda^a F) \right)$$

$$= p(N + \lambda^a F)$$

and so that for every $a \in \mathbb{F}_p^\times, V \in \text{Rep}(G)$

$$\text{tr}(F(\sigma a), V) = \text{tr}(\text{Frob}_G(Kl_G^V))$$

Corollary: "Generic Newton Slope"

$\varphi \neq \check{p}$

Conjecture: Newton slope $\varphi \neq \check{p}$ everywhere

Corollary: One can identify Kl_G from different groups G (Conj KM)

Ex: $Kl_{SO(2n)} = Kl_{Sp(2n)}$

$$Kl_{SO(2n+1)} = Kl_{SO(2n+2)} \stackrel{+3}{=} Kl_{G_2}$$

$$Kl_{SO(2n+1)}^{std} = Hyp(4, \underbrace{1, \dots, 1}_{2n}, n) \quad \text{Legendre}$$

$$\stackrel{n \neq 1}{\sim} q \left(\sum_{a \neq 0} \psi \left(n - \frac{a}{k} \right) \sum_{v \neq 0} \left(\frac{a}{v} - v \right) - 1 \right)$$

$$= \sum \psi \left(x_1 + x_2 + x_3 - y \right) \left(\frac{y}{p} \right)$$

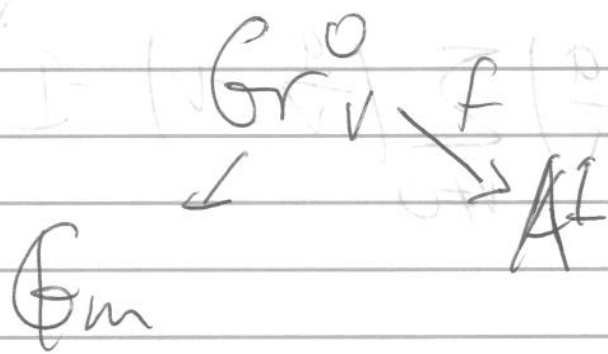
$$x_1 x_2 x_3 = ay$$

Pf of the thm

1) Thm(Z) $B_{\mathbb{C}} = K_{\mathbb{C}, dR}$

2) Geometric analogue for arithmetic D -modules.

3) Compare rigid & algebraic de Rham.



some specific V
 $Be_G^V \cong Kl_G^V$

\rightsquigarrow Mirror conjecture
 Lan Templier by Ritsch

Questions from audience

Q