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Local systems on M_2 and the top weight cohomology of $\mathcal{M}_2$

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(notes by Charles Godfrey)

$M_{g,n}$: moduli space of smooth genus g curves with n marked points / $\mathbb{C}$

$H^\bullet(M_{g,n};\mathbb{Q})$: cohomology with coefficients in $\mathbb{Q}$

- carries mixed Hodge structures
- also a representation

of symmetric group Σ_n.

Small genus:

$M_{0,n}$ parametrizes configurations of points on a line ($\mathbb{P}^1$)
(see Arnold 1969)

- cohomology is well known
- Weight filtration is "boring":

$H^\bullet(M_{0,n})$ is minimally pure:

$H^k(M_{0,n})$ has weight $2k$.

$H^\bullet(M_{g,n})$ too complicated?

Special cases: part of $H^k(M_{g,n})$ with weight k

Further special case: algebraic part of $H^\bullet(M_{g,n})$

Further special case:
Subalgebra generated by tautological classes.
(tautological ring $RH^\bullet(M_{g,n})$)

Other special case: maximal weight

classes in $H^\bullet(M_{g,n})$ have weight bounded by $2(3g-3+n)$

They are Poincaré dual to weight 0 classes in $H^\bullet_c(M_{g,n})$.
(strongly combinatorial)

Interpretation in tropical geometry (Chan-Galatius-Payne)

Weight 0 part of $H^\bullet_c(\mathcal{M}_{2,n})$ via local systems on $\mathcal{M}_2$ (joint with Dan Petersen (Stockholm))

Strategy:

Symplectic local systems on $\mathcal{M}_2$.

Let $\pi : \mathcal{M}_{2,1} \to \mathcal{M}_2$ be the universal curve on $\mathcal{M}_2$.

Set $\mathbb{V}_{1,0} := R^1\pi_*\mathbb{Q}$ (loc syst on $\mathcal{M}_2$, rank 4)

for $[C] \in \mathcal{M}_2$, $(\mathbb{V}_{1,0})_{[C]} = H^1(C;\mathbb{Q}) \circlearrowleft Sp_4$

~~[illegible]~~ comes with a standard action of symplectic group Sp_4.

Irreducible symplectic local systems

$\leftrightarrow$ irreducible representations of $Sp(4)$

$\leftrightarrow$ partitions $\lambda := (\lambda_1, \lambda_2)$, $\lambda_1 \geq \lambda_2$

(weight $w(\lambda) = \lambda_1 + \lambda_2$)

$$\mathbb{V}_\lambda \subset \mathrm{Sym}^{\lambda_2 - \lambda_1}(\mathbb{V}_{1,0}) \otimes \mathrm{Sym}^{\lambda_2}(\Lambda^2 \mathbb{V}_{1,0})$$

is the highest weight part

e.g. $\mathbb{V}_{(k,0)} = \mathrm{Sym}^k \mathbb{V}_{1,0}$

$$\Lambda^2 V_{1,0} = V_{1,1} \oplus V_{0,0}(-1)$$

Here the Tate twist adjusts the weight.

To study $M_{2,n}$, consider

$$\pi_n : M_{2,n} \longrightarrow M_2 ;$$

Fiber over $[C] \in M_2$ is

$$F(C,n) = \text{configurations of } n \text{ points on } C$$

$$= C^n \setminus \cup\{\text{diagonals}\}.$$

Now use Leray spectral sequence:

$$E_2^{pq} = H^p(M_2; R^q\pi_{n*}\mathbb{Q}) \Rightarrow H^{p+q}(M_{2,n}; \mathbb{Q})$$

<u>idea</u>: decompose $H^q(F(C_a); \mathbb{Q})$ into symplectic representations

<u>Proposition</u>: the Leray spectral sequence degenerates on page 2.

<u>Consequence</u>: <u>multiplicative</u> decomp. of associative motive (Petersen - Tavakol - Yin 2017)

What are the other missing ingredients?

1) $H^q_c(M_2, V_\lambda)$ for all λ

Let $\mathcal{M}_2 \xrightarrow[\text{map}]{\text{Torelli}} \mathcal{A}_2$ moduli of abelian surfaces

be the Torelli map sending $C \mapsto J(C)$

Idea: local systems $\mathbb{V}_\lambda$ come from $\mathcal{A}_2$ (obtained on $\mathcal{M}_2$ by restriction)

- Cohomology of $\mathbb{V}_\lambda$ on $\mathcal{A}_2$ can be studied via arithmetic methods.

Also have

$\mathcal{A}_{1,1} = \mathrm{Sym}^2 \mathcal{A}_1 \longrightarrow \mathcal{A}_2$

(products of elliptic curves)

$H^\bullet_c(\mathcal{A}_2, \mathbb{V}_\lambda)$ is known

(Faber–VdGeer 2004, Petersen 2016)

For weight 0 part:

- no issues with inclusion $A_{1,1} \hookrightarrow A_2$
- weight 0 classes belong to $H^3_c(M_2)$

(2) Need to compute $H^\bullet(F(C,n))$ together with symplectic representation structure.

One method: study Leray spectral sequence of inclusion

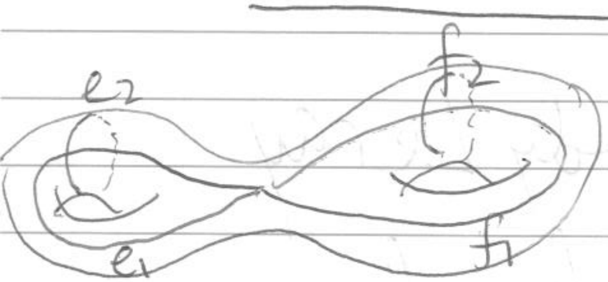
$$F(C,n) \hookrightarrow C^n$$

expresses $H^\bullet(F(C,n))$ ~~as~~ explicity as coho of a differential graded algebra.

Another method: Getzler spectral sequence for $H_c^i(F(C,n))$ obtained from stratification of C^n by looking at which p_i of $(p_1, \ldots, p_n)$ coincide

Idea: only local systems V_λ with no twist contribute to weight 0 part.

How do these arise?



C a curve of genus 2

$$H^0(C;\mathbb{Q}) = \mathbb{Q}$$

$$H^1(C;\mathbb{Q}) = V_{1,0}$$

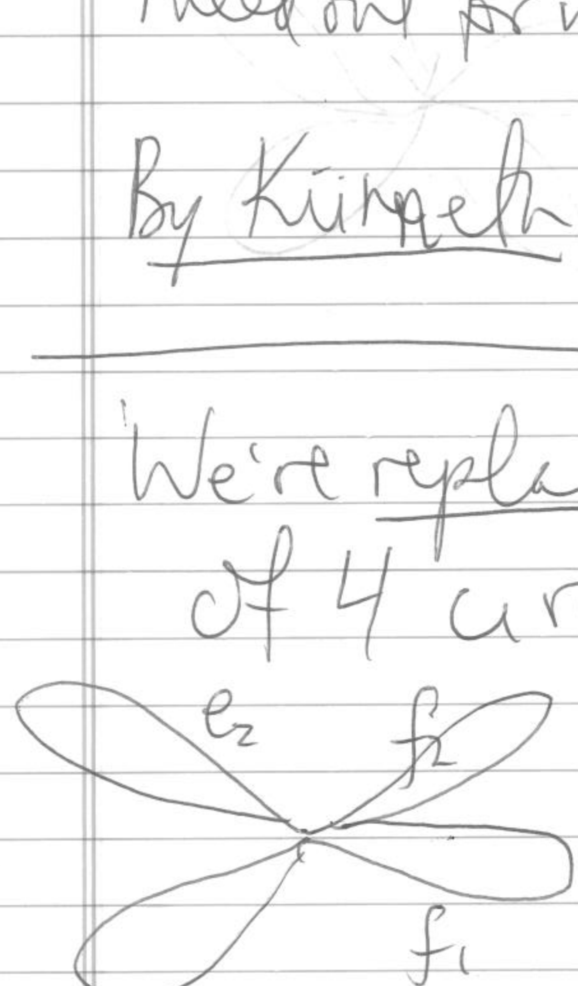
note: $H^1(C;\mathbb{Q}) = \mathrm{Hom}(\pi_1(C),\mathbb{Q})$; basis given by e_1, e_2, f_1, f_2

~~$H^2(C;\mathbb{Q}) = \mathbb{Q}(-1)$ gen by fundamental class~~

ruled out for weight reasons

<u>By Künneth</u>: $H^\bullet(C^\ell) = (H^\bullet(C))^{\otimes \ell}$

We're <u>replacing</u> C w/ a wedge of 4 circles, say Σ:

Σ: 

can replace $F(C,n)$ with $F(\Sigma,n)$

get $$F(\Sigma,n) = F(\overset{e_1}{\mathbb{R}} \sqcup \overset{e_2}{\mathbb{R}} \sqcup \overset{f_1}{\mathbb{R}} \sqcup \overset{f_2}{\mathbb{R}}, n)$$

$$\cup \bigcup_n F(\underset{e_1}{\mathbb{R}} \sqcup \underset{e_2}{\mathbb{R}} \sqcup \underset{f_1}{\mathbb{R}} \sqcup \underset{f_2}{\mathbb{R}}\ n-1)$$

where the copies of $F(\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}\sqcup\mathbb{R}, n-1)$ arise when 'L point is at the center of the wedge: