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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Charles Godfrey **Email/Phone:** cgodfrey@uw.edu

Speaker's Name: Ana-Maria Castravet

Talk Title: Exceptional collections on moduli spaces of stable rational curves

Date: 5 / 7 / 2019 **Time:** 2 : 00 **am / pm (circle one)**

Please summarize the lecture in 5 or fewer sentences: Investigates Orlov's question about S_n-equivariant exceptional collections on moduli spaces of rational curves with n-points, incorporating Kapronov's blow-up construction, Hassett's work on weighted curves, and the Losev-Manin spaces.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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EXCEPTIONAL COLLECTIONS ON MODULI SPACES OF STABLE RATIONAL CURVES

ANA-MARIA CASTRAVET

Links

- Derived category of moduli of pointed curves – I, https://arxiv.org/abs/1708.06340 [CT17].

References

[CT17] Ana-Maria Castravet and Jenia Tevelev, *Derived category of moduli of pointed curves-i*, arXiv preprint arXiv:1708.06340 (2017).

Date: May 7, 2019.

5/7/2019

Exceptional collections on moduli spaces of stable rational curves

Speaker: Ana-Maria Castravet.

Q: (Orlov): does $\overline{M}_{0,n}$ have a full, S_n-invariant exceptional collection?

Def'n: Let X be a smooth projective variety / $\mathbb{C}$.

$E \in D^b(X)$ is exceptional iff

$$\mathrm{Ext}^\alpha(E,E) = \begin{cases} 0 & \forall \alpha \neq 0 \\ \mathbb{C} & \forall \alpha = 0 \end{cases}$$

- $E_1, \ldots, E_r$ is an <u>exceptional collection</u> iff

  i) ~~Θ~~ each E_i is exceptional

  ii) $\mathrm{Ext}^\alpha(E_i, E_j) = 0$ for $i > j$ for all α

Implies $E_1, \ldots, E_r$ are linearly independent in $K(X)$

- $E_1, \ldots, E_r$ is <u>full</u> iff

$$\langle E_1, \ldots, E_r \rangle = D^b(X)$$

Implies $E_1, \ldots, E_r$ <u>span</u> $K(X)$.

<u>Ex</u>: $X = \mathbb{P}^n$. $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ is a full exc. collection.

$$\tilde{X} =$$

<u>Orlov</u>: $Y \subset X$ $\underset{\text{codim } r+1}{}$. form
$$\begin{tikzcd} E \arrow[r] \arrow[d, "q"'] & Bl_Y X \arrow[d, "p"] \\ Y \arrow[r, hook] & X \end{tikzcd}$$

Suppose $\{F\}$ is a full exc. coll on Y

$\{G\}$ is a full exc. coll. on X.

Then $\underbrace{\{q^*F \otimes \mathcal{O}_E(-r)\}, \to \{q^*F \otimes \mathcal{O}(-1)\}}$,

$Rp_*(-) = 0$ $\quad \{p^*G\}$ is a full exc. coll. on $\tilde{X}$.

<u>Rmk</u>: $\tilde{X} \xrightarrow{p} X$ a prop birational map w/ $Rp_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$.

<u>Set</u>

$D^b_{cusp}(\tilde{X}) \overset{def}{=} \{E \mid Rp_*E = 0\}$,

then $D^b(\tilde{X}) = \langle D^b_{cusp}(\tilde{X}), p^*D^b(X) \rangle$

$\overline{M}_{0,n}$ = moduli space of stable, rational curves w/ n-markings

Kapranov:

$\overline{M}_{0,n} \xrightarrow[|\psi_i|]{} \mathbb{P}^{n-3}$ blow up $p_1, \dots, p_{n-1}$, lines, planes, ...

$$\begin{tikzcd} \overline{M}_{0,n+1} \arrow[d, "\pi = \pi_{n+1}"'] \\ \overline{M}_{0,n} \arrow[u, bend right, "\sigma_i"'] \end{tikzcd} \qquad \psi_i = \sigma_i^*(\omega_\pi)$$

$$\begin{tikzcd} & \overline{M}_{0,n} \arrow[dl, "\psi_i"'] \arrow[dr, "\psi_j"] & \\ \mathbb{P}^{n-3} \arrow[rr, dashed, "\text{Cremona}"'] & & \mathbb{P}^{n-3} \end{tikzcd}$$

S_n-invariant collection?

<u>Ex</u>: $\overline{M}_{0,4} = \mathbb{P}^1$, $\mathcal{O}(-1)$, $\mathcal{O}$ works

- $\overline{M}_{0,5}$: $\Omega^1(\log \partial \overline{M}_{0,5})^\vee$,

  $\{\pi_i^* \mathcal{O}(-1)\}_{i=1}^5$, $\mathcal{O}$.

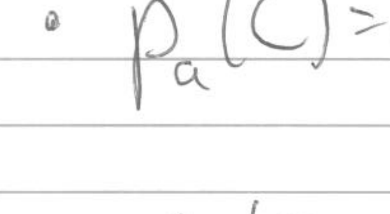
- $\mathbb{P}^5$

Hassett's work on weighted curves

$\mathcal{A} = (a_1, \dots, a_n)$, $0 < a_i \le 1$, $\sum a_i > 2$

$\exists$ moduli space $\overline{M}_{\mathcal{A}}$ of $\mathcal{A}$-stable curves:

- $p_a(C) = 0$, $p_1, \dots, p_n$ smooth points on C
- $\omega_C(\sum a_i p_i)$ ample
- If $\{p_i\}_{i \in I}$ coincide, $\sum_{i \in I} a_i \le 1$.



$$\boxed{M_{0,n} \subseteq \overline{M}_{\mathcal{A}}}$$

If $\mathcal{B} = (b_1, \dots, b_n)$ s.t.

$a_i \geq b_i$ for all i,

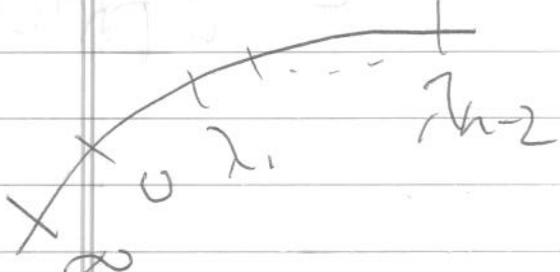
$$\overline{M}_{\mathcal{A}} \longrightarrow \overline{M}_{\mathcal{B}}$$

<u>Ex</u>: 0) $\mathcal{A} = (\underbrace{1, 1, \dots, 1}_{n})$: $\overline{M}_{0,n}$.

1) $\mathcal{A} = (1, \varepsilon, \dots, \varepsilon)$, w/

$$\frac{1}{n-1} < \varepsilon \leq \frac{1}{n-2}$$

$\mathcal{A}$-stable $\Rightarrow$ <u>irreducible</u>.



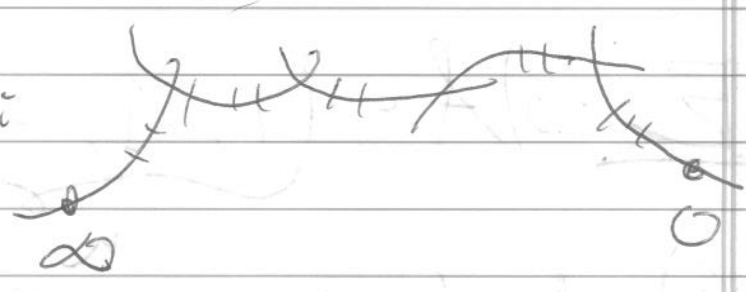
$$\overline{M}_{\mathcal{A}} = \mathbb{C}^{n-2} \setminus \{0\} / \mathbb{C}^* = \mathbb{P}^{n-3}$$

Realizes Kapranov map:

$$\overline{M}_{0,n} = \overline{M}_{(1,\dots,1)} \longrightarrow \overline{M}_{(1,\varepsilon,\dots,\varepsilon)} = \mathbb{P}^{n-3}$$

3) $A = (1, 1, \overbrace{\varepsilon, \dots, \varepsilon}^{n})$ Losev-Manin $\overline{LM}_n$

A-stable:



$p_1, \dots, p_n$ smooth points (may coincide)

each $\mathbb{P}^1$ has ≥ 3 points

$(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^\times)^n$,

get $(\mathbb{C}^\times)^n / \mathbb{C}^\times \simeq (\mathbb{C}^\times)^{n-1}$,

a torus

$$\begin{tikzcd} & \overline{LM}_n \arrow[dl, "\psi_0"'] \arrow[dr, "\psi_\infty"] & \\ \mathbb{P}^{n-1} \arrow[rr, dashed, "\text{cremona}"'] & & \mathbb{P}^{n-1} \end{tikzcd}$$

- $\overline{LM}_n$ is a toric variety of dim'n $n-1$
- Comes with an action

$$S_2 \times S_n \curvearrowright \overline{LM}_n; \quad S_2 \text{ permutes } 0, \infty.$$

3) $\sum a_i = 2$. $\overline{M}_A = \overline{M}_{(a_1+\varepsilon, \dots, a_n+\varepsilon)}$

$$\overline{M}_A = (\mathbb{P}^1)^n /\!/ PGL_2$$

<u>Particular case</u>: $A = (\underbrace{a, \dots, a}_{p}, \underbrace{b, \dots, b}_{q})$
with $pa + qb = 2$

set $\overline{M}_{p,q} := \overline{M}_A$. Eg. if $p = 2$ ...

Thm 4 (Castravet-Tevelev)

$\overline{M}_{0,n}$ has a full S_n-equivariant exc. coll provided the following do:

1) $\mathbb{P}^{n-3}$ (S_{n-1})

2) $\overline{LM}_n$ $(S_2 \times S_n)$

3) $\overline{M}_{p,q}$ $(S_p \times S_q)$

Cases where 1, 2, 3 hold:

(1) always holds $(\mathcal{O}, \mathcal{O}(1), \ldots \mathcal{O}(n-3))$

(2) always holds; this is $\boxed{\text{Thm 2}}$

Theorem 3:

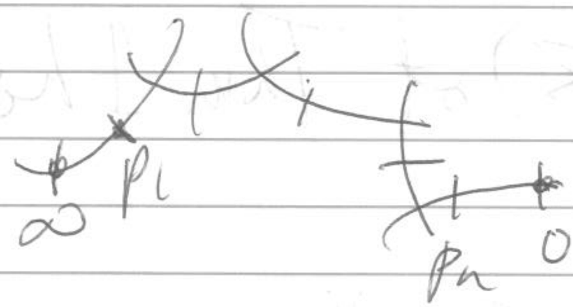
Condition (3) of Thm 1 holds when:

- p is odd, for all q
- p is even, q is odd or $q = 0, 2$

Part of a full exc. coll. is obtained when both p, q are even.

Corollary: Orlov's question is answered affirmatively for $n \leq 15$.

Consider again $\overline{\mathcal{M}}_n$:



For $\overline{\mathcal{M}}_n$ $\operatorname{rk} K = \operatorname{rk} CH^* = \operatorname{rk} H^*$

$= \chi_{top} = \#$ torus invar points

$= n!$

$\overline{\mathcal{M}}_n \xrightarrow{\psi_0} \mathbb{P}^{n-1}$ blow up $q_1, \ldots, q_n$ + lin subspaces

The ψ_i correspond to line bundles

$G_1, \ldots, G_{n-1}$

$\psi_0 \quad \psi_\infty$

Divisor associated to G_a
is a degree a hypersurface, with
mlt $(a-1)$ at points
mlt $(a-2)$ at lines, etc.

There's a S_2-action permuting
$G_a \longleftrightarrow G_{n-a}$. Recall

$D^b_{cusp}(\overline{LM}_n) = \{E \mid R\pi_{i*}E = 0 \text{ for } i=1,\ldots,n\}$

<u>Theorem 2</u> (II): $D^b_{cusp}(\overline{LM}_n)$ has a full, $S_2 \times S_n$-invariant exc. coll. $\{G_{a_1}^\vee \boxtimes \cdots \boxtimes G_{a_t}^\vee\}$

Thm 2 (Z)

$$D^b([\overline{M}_n]) = \langle D^b_{cusp}([\overline{M}_n]), \{\pi_k^* D^b_{cusp}([\overline{M}])\}, \mathcal{O} \rangle$$

Thm 3 (1) p odd, for all q. Say $p=2r+1$

$\{F_{l,E}\}$ $e_p = |E \cap p|$

$l + \min(e_p, p-e_p) \leq r+1$

is a full, $S_p \times S_q$ invariant exc. collection

(2) ~~p odd q even~~ p even, q odd

say $q = 2s+1$, $p = 2r$

The following is a $S_p \times S_q$-invariant exc. coll. of expected length

$\{\tilde{F}_{l,E}\}$ $l + \min(e_p, p+1-e_p) \leq r+1$

$\{\tilde{G}_{l,E}\}$ $\underbrace{|E \cap P|}_{E_p} = r$, $l + \min(e_q, q-e_q) \leq s+1$

($l+e$ even)

$$T_{l,E} = j_*\left(-\frac{r+l}{2}\psi_x - \sum_{j\in E_q}\psi_j\right)$$

$$x = E_p \xrightarrow{j} \overline{M}_{p,q}$$

3) $p = 2r+1,\ q = 0$

$$(\mathbb{P}^1)^p_{ss} \xrightarrow{\varphi} \overline{M}_p = \text{GIT quot.}$$

$$\varphi^* T_{l,E} = \mathcal{O}(L_1, \dots, L_n) \otimes V_l$$

$$V_l = H^0(\mathbb{P}^1, \mathcal{O}(l))$$

$$\cancel{\operatorname{Ext}^i_{\overline{M}_p}(T, T') = \operatorname{Ext}_{(\mathbb{P}^1)^p_{ss}}(F, F')^{PGL_2}}$$

now

$$\mathrm{Ext}^{\bullet}_{\overline{\mathcal{M}}_p}(F, F') = \mathrm{Ext}^{\bullet}_{(\mathbb{P}^1)^p_{ss}}(F, F')^{PGL_2}$$

$$= \mathrm{Ext}^{\bullet}_{(\mathbb{P}^1)^p}(F, F')$$

by Teleman, Halpern-Leistner