Tropicalizing the moduli space of stable ~~space~~ spin curves

Speaker: Margarida Melo
(Joint work with L. Caporaso, M. Pacini)

$S_g^+ \cup S_g^- = S_g = \{(C,L) \mid \substack{C \text{ smooth curve} \\ \text{of genus } g,\ L^{\otimes 2} = \omega_C}\}$

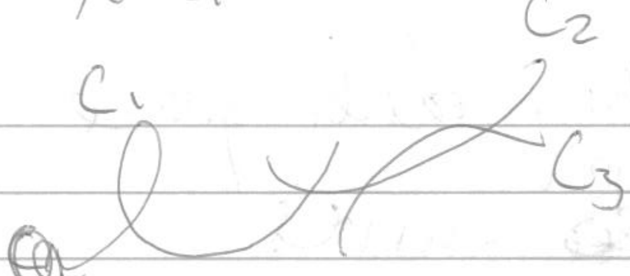
$\in$
(C,L) s.t.
$h^0(C,L) \equiv 0 \bmod 2$

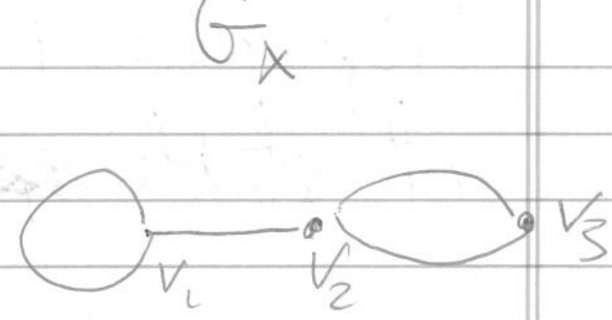
$$\begin{array}{ccc} S_g & \hookrightarrow & \overline{S_g} \\ {\scriptstyle 2^{2g}}\downarrow & & \downarrow \\ M_g & \hookrightarrow & \overline{M_g} \end{array}$$

$\overline{S_g}$ moduli of stable spin curves of genus g.

Remark: not all stable curves admit theta characteristics.

$X = C_1 \cup C_2 \cup C_2$

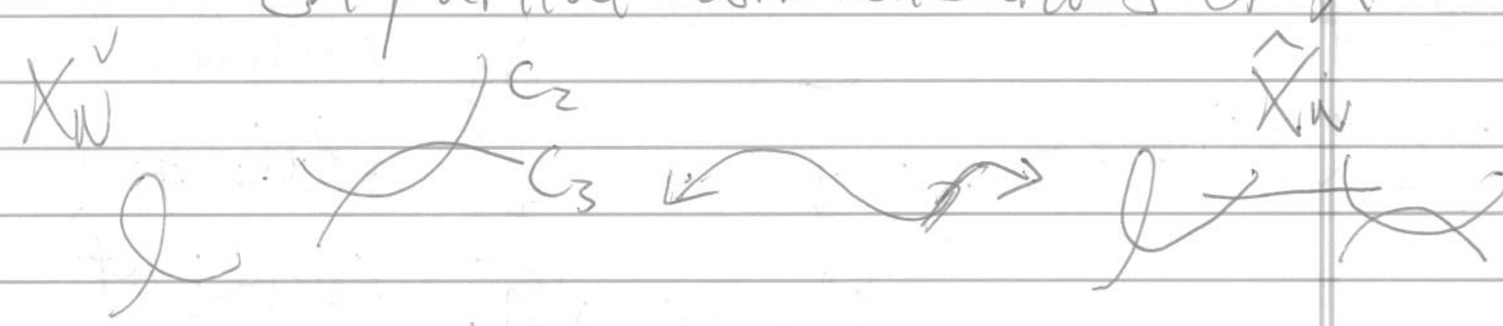


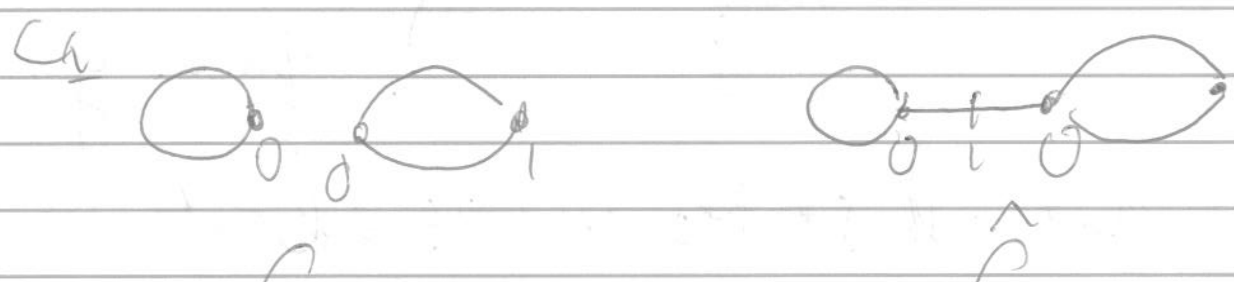


$\deg|_{C_i} \omega_X = 2g_i - 2 + \deg v_i$

$\underline{\deg}\, \omega_X = (1, 1, 2)$

S_g paramaterizes theta characteristics on partial normalizations of X





$R \subset X_{sing}$

$$\begin{tikzcd} X_R^\vee \arrow[r, hook] \arrow[dr, "\text{partial norm.}"'] & \hat{X}_R \arrow[d, "\text{``blow up'' of } X \text{ at } R"] \\ & X \end{tikzcd}$$

<u>Def</u>: a <u>stable spin curve</u> over X is a pair $(\hat{X}_R, L)$ where

a) $\hat{X}_R$ is the blow-up of X at R

b) $L \in \mathrm{Pic}^{g-1}(\hat{X}_R)$ and

1) $L^{\otimes 2}\big|_{X_R^\vee} \simeq \omega_{X_R^\vee}$

2) $L|_E \simeq \mathcal{O}_E(1)$, $\forall E \simeq \mathbb{P}^1$ exc. ~~des.~~ component

Thm (Cornalba '89)

$\overline{S}_g$ is a proper moduli space with a 2^{2g} ramified map onto $\overline{M}_g$, $\overline{S}_g = \overline{S}_g^{+} \cup \overline{S}_g^{-}$

Proposition: $S_g \subset \overline{S}_g$ is toroidal

$\Rightarrow$ Can apply Thuillier: $p: S_g^{an} \to \Sigma(\overline{S}_g^{an})$
or [ACP]

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later

~~Def: A stable spin graph of genus g is a triple (G, P, s) where~~

~~a) G is a stable weighted graph of genus g;~~

~~b) $P \subset G$ is a cyclic subgraph~~

<u>Aim</u> | 1) Describe boundary $\overline{S}_g \setminus S_g$ via stratification of combinatorial type

2) Give $\Sigma(\overline{S}_g^{an})$ a modular interpretation via moduli of tropical spin curves.

<u>Def</u>: a <u>stable spin graph of genus g</u> is a triple (G, P, s) where

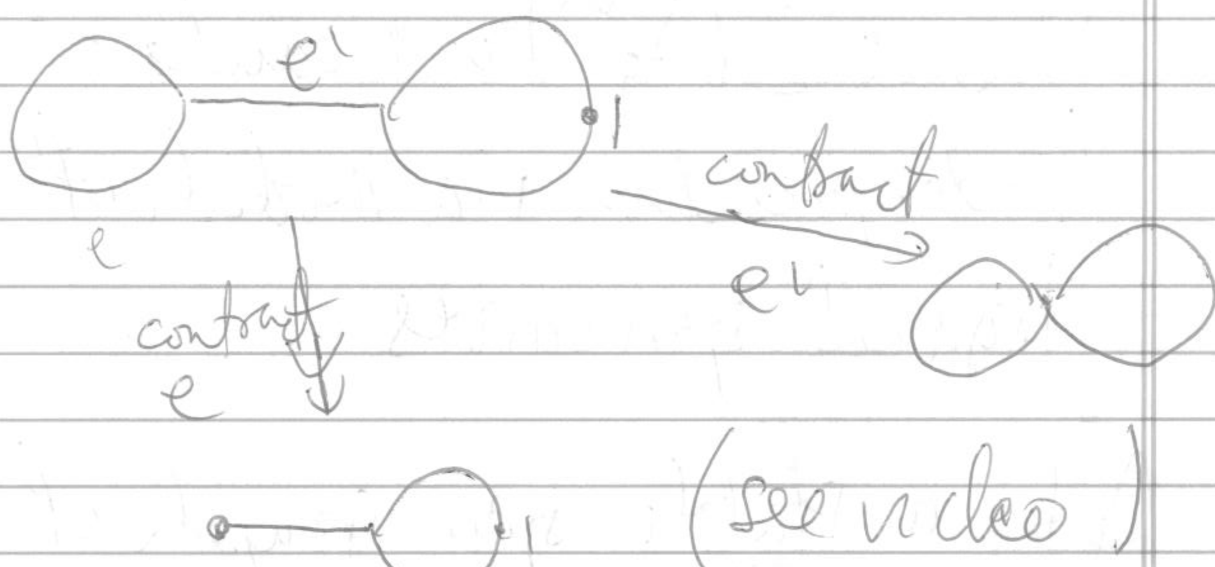
1) G is a stable weighted graph of genus g

2) $P \subset G$ is a cyclic subgraph

3) $s : V(G/P) \to \mathbb{Z}/2\mathbb{Z}$,

$w(v)=0 \Rightarrow s(v)=0$

<u>Rmk</u>: Stable spin graphs behave well under contraction.



(see video)

Let $S\Gamma_g$ be the set of stable spin graphs of genus g.

<u>Poset structure on $S\Gamma_g$</u>:

$(G,P,s) \geq (G',P',s')$ if there is a contraction

$(G,P,s) \rightsquigarrow (G',P',s')$

Def: $(\hat{X}_R, L)$ stable spin curve

$\longmapsto (G, P, s)$ stable spin graph

- $G = G_X$
- $P = G \setminus R$ $\left(P = G_{X_R^\nu}\right)$
- $s : V(G/P) \longrightarrow \mathbb{Z}/2\mathbb{Z}$

$$v \longmapsto h^0(Z_v, L|Z_v)$$

where $X_R^\nu = \bigcup_{v \in V(G/P)} Z_v$

Def: given $(G, P, s) \in S\Gamma_g$

$\rightsquigarrow S_{(G,P,s)} = \{(\hat{X}_R, L)$ stable spin curves / comb. type $(G, P, s)\}$

Theorem [Caporaso-Melo-Pacini]

~~$\overline{S}_g \setminus S_g$ has a stratification governed by $S\Gamma_g$:~~

~~a)~~ $\cancel{\overline{S}_g \setminus S_g = \bigcup_{(G,P,s)\in S\Gamma_g} S_{(G,P,s)}}$

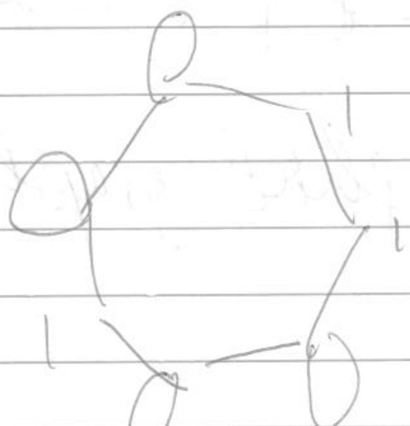
$\overline{S}_g \setminus S_g$ has a stratification governed by $S\Gamma_g$:

a) $\overline{S}_g \setminus S_g = \bigcup_{(G,P,s)\in S\Gamma_g} S_{(G,P,s)}$

b) each $S_{(G,P,s)}$ is irreducible

c) $\overline{S}_{(G,P,s)} = \bigcup_{(G',P',s')\geq (G,P,s)} S_{(G',P',s')}$

Note: For b), uses degeneration argument to reduce to "basic" stable spin curves



& give explicit description for theta characteristics on these.

Corollary: X general stable curve, $(\hat{X}_R, L)$ stable spin curve over X

a) $V(G/p) = 1$, $h^0(\hat{X}_R, L) = \begin{cases} 1 & \text{if } s=1 \\ 0 & \text{else} \end{cases}$

b) In general, $h^0(\hat{X}_R, L) = \sum_{v \in V(G/p)} h^0(Z_v, L_{Z_v})$

Spin tropical curves

Def'n: A *tropical curve of genus g* is a pair $\Gamma = (G, \ell)$ where

a) G is a stable weighted graph and

b) $\ell : E(G) \to \mathbb{R}_{>0}$

A *divisor* $D \in \mathrm{Div}(\Gamma)$ is a formal sum

$$D = \sum_{p \in \Gamma} a_p p \quad \& \quad \deg D = \sum_{p \in \Gamma} a_p .$$

$$K_\Gamma \in \mathrm{Div}^{2g-2}(\Gamma)$$

$$\sum_{p \in \Gamma} (2w(p) - 2 + \deg p)\, p$$

A <u>theta characteristic</u> on Γ is a $D \in \mathrm{Div}^{g-1}(\Gamma)$ so that $[2D] = [K_\Gamma]$; there are $2^{b_1(\Gamma)}$ classes of such D.

<u>Def</u>: A <u>spin structure</u> on a tropical curve $\Gamma = (G, e)$ is a pair (P, s) so that (G, P, s) is a stable spin graph.

$\psi = (\Gamma, P, s)$ ⇝ tropical spin curve

$\hookrightarrow D_\psi \in \mathrm{Div}^{g-1}(\Gamma)$

$$D_\psi(p) = \begin{cases} w(v) - 1 + \frac{\deg_P(v)}{2} & \text{if } v = p \in V(G) \\ 1 & \text{if } p \text{ is the} \\ 0 & \text{otherwise} \end{cases}$$

Thm ([ICMP], Zharkov for $w \equiv 0$)

- D_ψ is a theta characteristic of Γ and all tropical theta characteristics on Γ are of the form D_ψ for $\psi = (\Gamma, P, s)$ a tropical spin curve.
- If $w \equiv 0$, $\exists!$ non-effective theta char. on Γ
- If $w \not\equiv 0$, all tropical theta char. are effective.

Moduli space of tropical spin curves

Given $(G,P,s) \in S\Gamma_g \rightsquigarrow \sigma_{(G,P,s)} = \mathbb{R}_{>0}^{|E(G)|}$

$\sigma_{(G,P,s)} / \mathrm{Aut}(G,P,s) \longleftrightarrow$ tropical spin curves of ~~combinatorial~~ comb. type (G,P,s)

$$S_g^{trop} = \varinjlim_{S\Gamma_g} \sigma_{(G,P,s)}$$

$$\left(S_g^{trop} = S_g^{trop,+} \cup S_g^{trop,-} \right)$$

$$\begin{tikzcd}
\overline{S}_g^{an} \arrow[r] \arrow[d] \arrow[rr, bend left] & \overline{\Sigma}(\overline{S}_g^{an}) \arrow[r, phantom, "\simeq"] \arrow[d] & \overline{S}_g^{trop} \arrow[d] \\
\overline{M}_g^{an} \arrow[r] \arrow[rr, bend right, "trop"'] & \overline{\Sigma}(\overline{M}_g^{an}) \arrow[r, phantom, "\simeq"] & \overline{M}_g^{trop}
\end{tikzcd}$$

Questions from audience

Q] What ~~are~~ role do the lengths play in def'n of theta characteristics?

A] No role, really.

Q] Higher r-spin curves?

A] Good question.