Integral points on elliptic curves

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(Joint work with L. Alpoge)

$E = E_{A,B}$: $y^2 = x^3 + Ax + B$

$A, B \in \mathbb{Z}$,

$\Delta = -16(4A^3 + 27B^2)$

$\neq 0$

Q: how many integral points on E?

$E_{A,B}(\mathbb{Z}) = \{(x,y) \in \mathbb{Z}^2 \mid y^2 = x^3 + Ax + B\}$

Mordell/Siegel: finitely many integral points.

Helfgott-Venkatesh '06

$$\# E(\mathbb{Z}) \ll O(1)^{\omega(\Delta)} (\log|\Delta|)^2 \, 1.33^{\mathrm{rk} E}$$

where $\omega(n) = \#$ distinct prime divisors

Minimal models

Silverman: $\# E(\mathbb{Z}) \ll O(1)^{\mathrm{rk} E + \omega_{ss}(\Delta)}$

Hindry-Silverman: $O(1)^{\mathrm{rk} E + \sigma_{A,B}}$

$\cancel{\# E(\mathbb{Z}) \ll O(1)^{\mathrm{rk} E + \omega_{ss}(\Delta)}}$

Note: ABC conjecture would imply $\sigma_{A,B}$ is uniformly bounded.

idea: to get some result about average # of integral points wanted to get a bound like

$$\# E(\mathbb{Z}) \ll 2^{rk E} \cdots ??$$

Theorem 1 | $\cancel{\# E(\mathbb{Z}) \le \left(\frac{\text{fixed}}{\text{big}\,\#}\right) \cdot 2^{rk E}}$

$$\# E(\mathbb{Z}) \le \left(\frac{\text{fixed}}{\text{big}\,\#}\right) \cdot 2^{rk E} \prod_{p^2 | \Delta} \left(4\left\lfloor \frac{v_p(\Delta)}{2} \right\rfloor + 1\right)$$

Thm 2: $\cancel{F_{univ} = \{E_{A}}$

$F_{univ} = \{E_{A,B}\}$ ordered by height $H(E_{A,B}) = \max\{4|A|^3, 27|B|^2\}$

For any $0 < s < \log_2 5 = 2.3219\ldots$

$\mathrm{Avg}\left(|E_{A,B}(\mathbb{Z})|^s\right)$ is bounded

$E_{A,B} \in F_{univ}$

Moduli spaces for arithmetic statistics |

Q: How many rational points on genus g curves$/\mathbb{Q}$? On average.

~~Take~~ $\{$genus g curve $C/\mathbb{Q}$ and a point $p \in C(\mathbb{Q})$. $\}$

$$\downarrow$$

$\{$ genus g curve $C/\mathbb{Q} \}$

In other words look at

$\mathcal{C}_g$ want to count points

$\pi \downarrow$ in a compatible way.

$\mathcal{M}_g$

Example: $\mathcal{M}_{1,2} \longrightarrow \mathcal{M}_{1,1}$

$\left\{ \begin{matrix} \text{elliptic curve } E \\ + \text{non-}\mathcal{O} \text{ point } E \end{matrix} \right\} \to \left\{ \begin{matrix} \text{elliptic curve} \\ E \end{matrix} \right\}$

Here $\mathcal{M}_{1,1}$ will be modelled as $\mathbb{P}(4,6)$ (weighted proj. space)

$\mathcal{M}_{1,2}$ will be modelled as $\mathbb{P}(2,3,4)$

$$(y^2 + d_3 y = x^3 + d_2 x^2 + d_4 x)$$

$$\hookrightarrow y^2 = x^3 + \left(d_4 - \frac{d_2^3}{3}\right)x + \left(\frac{-d_2 d_4}{3} + \frac{2}{27}d_2^3 + \frac{d_3^2}{4}\right)$$

See video.

Difficulty is ordering points of $M_{1,2}$ & $M_{1,1}$ compatibly.

Idea: use "Selmer groups" in number theory

$$M_{1,2} \longrightarrow \underbrace{\left\{ \begin{array}{l} \text{nice genus 1 curves} \\ + \text{degree } d \text{ line bundles } L \end{array} \right\}}_{/\sim} \quad d$$

$$(E, P) \longmapsto (C = E, L = \mathcal{O}((d-1)O + P))$$

Key idea: Coarse space is open $\mathbb{P}(4,6)$

Example: for $d = 3$,

$\mathcal{N}_d = \{\text{plane cubics}\}$

$$\left(\{\text{ternary cubic forms}\} \big/ \mathbb{G}_m \times GL_3 \right)$$

invariants: deg 4 S, deg 6 T

Picture:

$$\begin{tikzcd}
M_{1,2} \arrow[r] \arrow[dr, "\text{forget}"'] & \mathcal{N}_d \arrow[d, "\text{Jac.}"] \\
& M_{1,1}
\end{tikzcd}$$

Count w.r.t. height of Jacobian

Count w.r.t. height of $\overline{E}$

Above a specific E, over $\mathbb{Q}$:

$$E(\mathbb{Q}) \longrightarrow \left\{ \begin{array}{c} \text{genus } 1\ C \\ \text{w/ } \mathrm{Jac}(C) \simeq E \\ \deg d\ L \end{array} \right\}$$

$$= H^1(\mathbb{Q}, \Theta_{E,d}) \subset H^1(\mathbb{Q}, E[d])$$

$$E(\mathbb{Q})/dE(\mathbb{Q}) \to \mathrm{Sel}_d(E)$$

Today:

$$\begin{tikzcd} E(\mathbb{Z}) \arrow[r] \arrow[drr, "\Psi"'] & E(\mathbb{Q}) \arrow[r] & E(\mathbb{Q})/2E(\mathbb{Q}) \arrow[d, hook] \\ & & H^1(\mathbb{Q}, \Theta_{E,2}) \end{tikzcd}$$

Q: what is $\operatorname{im} \Psi$?

$$|\text{image of } \Psi| \le |E(\mathbb{Q})/2E(\mathbb{Q})| \le 4 \cdot 2^{\mathrm{rk} E}$$

$$(= 2^{\mathrm{rk} E + 2})$$

Fibers of Φ?

$\mathcal{M}_2 = \{\text{binary quartic forms}\}/\sim$

Binary quartics:

$f(x,y) = \cancel{ax^4+bx^3y+cx^2y^2}$

$ax^4+bx^3y+cx^2y^2+dxy^3+ey^4$

$SL_2 \curvearrowright Sym^4(2)$

poly invariants:

$I = 12ae-3bd+c^2$

$J = 72ace-27ad^2 \cdots$

$\Delta(f) = \frac{1}{27}(4I^3-J^2)$

$$\left\{\Delta \neq 0 \text{ binary quartics}\right\} \Big/ \mathbb{G}_m \times GL_2 \simeq \mathcal{M}_2$$

$$f \longmapsto C_f = z^2 = f(x,y)$$

L = pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$.

$$\mathrm{Jac}(C_f): y^2 = x^3 - \frac{I}{3}x - \frac{J}{27}$$

<u>Def</u>: f <u>integral</u> $a,b,c,d,e \in \mathbb{Z}$

<u>integer matrix</u> $4 \mid b, d,\ 6 \mid c$

<u>flattened</u>: $a = 1,\ b = 0$, integral

$$E_{A,B}(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, \Theta_{E,2})$$ binary quartics

$$P \longmapsto f(x,y) = x^4 - 6x_0 x^2 y^2 + 8y_0 x y^3 + (-4A - 3x_0^2) y^4$$

(x_0, y_0)

Thm (Mordell):

$$\left\{ (E_{A,B}, \text{int. pt. } (x_0,y_0)) \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{l} \text{BQs} \\ x^4 + 6cx^2y^2 + 8dxy^3 + ey^4,\ e \equiv c^2 (4), \\ c,d,e \in \mathbb{Z} \end{array} \right\}$$

Prop: f flattened integral binary quartic. Then

$\#\, \gamma \in PGL_2\mathbb{Q}$ s.t. $\gamma \cdot f$ flattened

$$\# \ll \prod_{p^2 \mid \Delta} \left(4 \left\lfloor \frac{v_p(\Delta)}{2} \right\rfloor + 1\right)$$

<u>Idea</u>: Thue equations

BQ $\quad f(x,y) = \delta^2 \quad \delta^2 \mid \Delta$

how many sol'ns

<u>Thm 1 $\Rightarrow$ Thm 2</u>: Average both sides!

Use Thm (Bhargava-Shankar)

Avg $\mathrm{Sel}_5(E) = 6$

$\Rightarrow$ avg 5^{th} bounded.

Audience questions

Q| What about function fields?

A| Methods don't work in the function field case + there are counterexamples.