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Quadric rank loci on moduli spaces of curves and K3 surfaces

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(notes by Charlie Godfrey)

Thom-Porteous formula:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$ morphism of vector bundles on X

$e = \mathrm{rk}\, E$, $f = \mathrm{rk}\, F$

$X_k(\phi) = \{x \in X \mid \mathrm{rk}\, \phi(x) = k\}$

expected codim $= (e-k)(f-k)$

$$[X_k(\phi)] = \det \begin{vmatrix} c_{f-k} & c_{f-k+1} & \cdots \\ & \ddots & \\ & & c_{f-k} \end{vmatrix} \quad \begin{pmatrix} (e-k)\times(e-k) \\ \text{matrix} \end{pmatrix}$$

(Joint w/ R. Rimanyi)

E, F vector bundles on X as above

$e = \operatorname{rk} E$, $f = \operatorname{rk} F$

$$\begin{tikzcd} S^2 E \arrow[rr, "\phi"] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$

fix corank r.

$$\Sigma^r_{e,f}(\phi) = \left\{ x \in X \;\middle|\; \begin{array}{l} \exists\, q \neq 0,\ q \in \ker \phi(x) \\ \operatorname{rk} q \leq e - r \end{array} \right\}$$

Set $\Sigma^r = \{ Q \in S^2(\mathbb{C}^e) \mid \operatorname{rk} Q \leq e - r \}$

$\left(\operatorname{codim} = \binom{r+1}{2} \right)$

Expected codim of $\Sigma^r_{ef}(\phi)$:

$$\mathbb{P}(\ker \phi(x)) \cap \Sigma^r : \dim \binom{e+1}{2} - f - 1 - \binom{r+1}{2}$$

so expected codim: $f+1+\binom{r+1}{2}-\binom{e+1}{2}$

The universal situation:

$$\Sigma^r_{ef} = \left\{ \varphi : S^2\mathbb{C}^e \to \mathbb{C}^f \;\middle|\; \begin{array}{l} \exists q \neq 0,\ q \in \ker\varphi \\ \operatorname{rk} q \leq e-r \end{array} \right\}$$

Atiyah Bott localization gives residue formula for

$$[\Sigma^r_{ef}] \in H^*_G(\mathrm{Hom}(S^2\mathbb{C}^e, \mathbb{C}^f))$$

Diagonal case: $f = \binom{e+1}{2} - \binom{r+1}{2}$

Thm 1 (Farkas - Rimanyi):

$$[\Sigma^r_{e,f}(\phi)]^{virt} = A^r_e\left(c_1(F) - \frac{2f}{e}c_1(E)\right) \in H^2(X,\mathbb{Q})$$

where $A^r_e = \dfrac{\binom{e}{r}\binom{e+1}{r-1}\cdots\binom{e+r-1}{1}}{\binom{1}{0}\binom{3}{1}\cdots\binom{2r-1}{r-1}}$

Degenerate pencils: $f = \binom{e+1}{2} - 2$

$\ker\phi(x)$ = pencil of quadrics

pencil is degenerate if $\mathbb{P}(\ker\phi(x)) \cap \Sigma^1$ is non-reduced

Thm 2: $\mathfrak{D}_p = \{x \in X \mid \ker\phi(x) \text{ degenerate pencil}\}$

$$[\mathfrak{D}_p]^{virt} = e(e-1)\left(c_1(F) - \frac{2f}{e}c_1(E)\right)$$

Applications

Moduli of K3 surfaces

$$\mathcal{F}_g = \{ (X, \mathcal{L}) \text{ polarized K3 surface} \mid \mathcal{L}^2 = 2g-2 \}$$

19-dim'l irred. variety

$\pi : \mathcal{X} \to \mathcal{F}_g$ univ K3 surface

$\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ univ polarization

Marian-Oprea-Pandharipande:

$$\kappa_{a,b} = \pi_*\left(c_1(\mathcal{L})^a c_2(T_\pi)^b\right) \in CH^{a+2b-2}(\mathcal{F}_g)$$

In codimension 1

$K_{3,0} = \pi_*(c_1(\mathcal{L})^3)$, $K_{1,1} = \pi_*(c_1(\mathcal{L})c_2(T_\pi))$
$\in CH^1(F_g)$

Linear combo

$\gamma = K_{3,0} - \frac{g-1}{4} K_{1,1}$ is ~~___~~ intrinsic (doesn't depend on choice of $\mathcal{L}$)

NL divisor: fix d, h.

$$D^g_{d,h} = \left\{ [X, \mathcal{L}] \;\middle|\; \begin{array}{l} \exists\ \mathbb{Z}L \oplus \mathbb{Z}C \hookrightarrow \mathrm{Pic}\,X \\ C \cdot L = d,\ C^2 = 2h-2 \end{array} \right\}$$

Bergeron-Li-Millson-Moeglin

$\mathrm{Pic}(F_g)$ is spanned by NL loci

Borcherds Bruinier computes the ~~rk~~ rank of $Pic(F_g)$

<u>Question</u>: Find explicit NL representations for

$\lambda = \pi_*(\omega_\pi)$,

$\delta = K_{3,0} - \frac{g-1}{4} K_{1,1}$

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1) K3's & rk 4 quadrics.

$[X, L] \in F_g$, $X \xrightarrow{|L|} \mathbb{P}^g$

Look @ ex. seq

$$0 \to q_2(L) \to S^2 H^0(X,L) \xrightarrow{\mu_L} H^0(X, L^2) \to 0$$

Hs: $\binom{g-2}{2}$ $\binom{g+2}{2}$ $4g-2$

$$D_g^{rk4} = \{ [X, L] \mid \exists\, q \neq 0,\ q \in \ker \mu_L,\ \operatorname{rk} q \leq 4 \}$$

<u>D_g^{rk4} is a NL divisor.</u>

Have $Q \subset \mathbb{P}^g$

$$\begin{tikzcd} Q \arrow[dr, dashed] \arrow[r, phantom, "\subset"] & \mathbb{P}^g \arrow[d, dashed, "i"] \\ & \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, hook] & Q \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \\ \mathbb{P}^1 & \mathbb{P}^1 \end{tikzcd}$$

$$L = \underbrace{\pi_1^* \mathcal{O}_{\mathbb{P}^1}(1)}_{A_1} \otimes \underbrace{\pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)}_{A_2}$$

Fact: $h^0(X, A_i) \geq 2$ for $i = 1, 2$

$$[D_g^{rk4}] = A_{g+1}^{g-3} \left((2g+1)\lambda + \frac{2}{g+1}\delta \right) \text{ for } g = 2i$$

~~g odd~~

Lazarsfeld-Mukai bundle

$$\boxed{E^2 \longrightarrow X,\ \det E = L \quad \mathcal{M}_X(2, L, i) = \{E\} \quad h^0(X, E) = i+2}$$

Let $\mathcal{E} \longrightarrow \mathfrak{X}$ be the universal LM bundle

$$\begin{array}{c} \mathfrak{X} \\ \downarrow \pi \\ \mathcal{F}_g \end{array}$$

(we'll have $\mathcal{L} = \det \mathcal{E}$)

Set $\mathcal{V} = \pi_*(c_1(\mathcal{E})\, c_2(\mathcal{E}))$

Consider

$$0 \to I_2(E) \to S^2 H^0(X, E) \xrightarrow{\mu_E} H^0(X, S^2 E) \to 0$$

rks $\quad \binom{i-3}{2} \qquad \binom{i+1}{3} \qquad 6i-3$

set $D_g^{\mathrm{rk}\,6} = \{[X,Z] \in T_g \mid \exists q \neq 0,\ q \in \ker \mu_E,\ \mathrm{rk}\, q \le 6\}$

We have a map $X \xrightarrow{\phi_E} G(2, H^0(E)^\vee)$

$= G(2, i+2)$

and $\mathrm{Im}\, \phi_E \subseteq G(2,q) =: G_q$

$= \{\text{lines on } q \subset \mathbb{P}H^0(E)^\vee\}$

When $\mathrm{rk}\, q \le 6$, $\mathcal{O}_{G_q}(1)$ splits as a tensor product of 2 line bundles with ≥ 2 sections

$Q = G(2,4) = G(2,U) \subset \mathbb{P}^5$

$\mathbb{P}(U) = \mathbb{P}^3$

Any line on Q has the form

$\{p_0 \in l \subset H\}$, w/ $p_0 \in \mathbb{P}^3$, $H \in \mathbb{P}^{3\vee}$

so $G_Q = \{(p_0, H) \in \mathbb{P}^3 \times (\mathbb{P}^3)^\vee \mid p_0 \in H\}$

$$\begin{tikzcd} & G_Q \arrow[dl] \arrow[dr] & \\ \mathbb{P}^3 & & (\mathbb{P}^3)^\vee \end{tikzcd}$$

(incidence correspondence)

<u>3rd relation</u>: ~~consid~~

consider $\det : \Lambda^2 H^0(X,E) \to H^0(X,L)$

(recall $\det E = L$)

$$D_{res} = \left\{ [X,L] \in F_{2i} \;\middle|\; \frac{S^i H^0(X,E) \otimes H^0(X,E)}{S^{i+1} H^0(X,E)} \to S^{i-1} H^0(X,E) \otimes H^0(X,L) \text{ is degenerate} \right\}$$

Another application: Hurwitz spaces.

$$\mathcal{H}_{g,k} = \left\{ C \xrightarrow{f} \mathbb{P}^1 \;\middle|\; \begin{array}{l} \deg f = k \\ g(C) = g \end{array} \right\}$$

Question: birational geometry of $\mathcal{H}_{g,k}$?

Case $g = 2k-1$:

$$\mathcal{H}_{2k-1,k} \longrightarrow \text{divisor on } \mathcal{M}_g.$$

$$A = f^*\mathcal{O}_{\mathbb{P}^1}(1) \; ; \; L = \omega_C \otimes A^\vee$$

$$C \xhookrightarrow{|L|} \mathbb{P}^{k-1}$$

$$\text{Consider } 0 \to I_2(L) \longrightarrow S^2 H^0(X,L) \xrightarrow{\mu_L} H^0(X,L^2) \to 0$$

ranks $\binom{k-3}{2}$ $\quad \binom{k+1}{2}$ $\quad 2(3k-4)+1 - 2k+1$

$$\mathcal{D}^{rk 4}_{Hur} = \{[C, A] \mid \exists q \neq 0,\ q \in \ker \mu_L,\ rk\, q \leq 4\}$$

Theorem (Farkas-Rimanyi)

$$\sigma : \overline{H}_{2k-1,k} \longrightarrow \overline{M}_{2k-1}$$

$$K_{\overline{H}_{2k-1,k}} = (k-12)\, \mathcal{D}^{rk 4}_{Hur} + \sigma^*(? \lambda - \delta_0)$$

Corollary: for $k \geq 12$ $\overline{H}_{2k-1,k}$ "is" of "general type"

(Modulo questions about singularities of $\overline{H}_{2k-1,k}$).

also: $k \leq 8$: $\overline{H}_{2k-1,k}$ is unirational.

Also: if $g = 16$

$$\underbrace{\overline{H}_{16,g}}_{\text{Kodaira dim} \geq 0} \xrightarrow{\text{finite}} \overline{M}_{16} \text{ uniruled}$$