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Name: Charlie Godfrey **Email/Phone:** cgodfrey@uw.edu

Speaker's Name: Angelo Vistoli

Talk Title: The Chow ring of the stack of stable curves of genus 2

Date: 5 / 7 / 2019 **Time:** 3 : 30 **am / pm (circle one)**

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The Chow ring of the stack of stable curves of genus 2

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Joint work in progress with Andrea Di Lorenzo

Berkeley, May 7, 2019

Consider the moduli spaces $\mathrm{M}_{g,n}$ and $\overline{\mathrm{M}}_{g,n}$ of smooth, or stable, n-pointed curves of genus g. These are singular, but have quotient singularities, mild enough to have rational Chow rings $\mathrm{CH}^*(\mathrm{M}_{g,n})_{\mathbb{Q}}$ and $\mathrm{CH}^*(\overline{\mathrm{M}}_{g,n})_{\mathbb{Q}}$, first defined by David Mumford, which have been the subject of a lot of work by many authors.

There is an integral version of these rings. They are not defined for the spaces, but for the corresponding algebraic stacks $\mathscr{M}_{g,n}$ and $\overline{\mathscr{M}}_{g,n}$.

They are *quotient stacks*. Recall that if G is an affine algebraic group acting on a scheme X, the quotient stack is defined so that a map from a variety S to $[X/G]$ consists of a G-torsor (a.k.a. a principal G-bundle) $P \to S$, and a G-equivariant map $P \to X$. Then $X \to [X/G]$ is a G-torsor.

The *classifying stack* $\mathscr{B}G$ is the quotient stack $[\mathrm{pt}/G]$, parametrizing G-torsors.

If $\mathscr{X}$ is an algebraic stack representing a moduli problem, one way to show that $\mathscr{X}$ is a quotient stack is by adding to objects enough data to kill the non trivial automorphisms, resulting in fine moduli space X for objects with the additional data, so that there is an affine algebraic group G that permutes the additional data in a unique transitive way. In this case $\mathscr{X} \simeq [X/G]$.

For example, take $\mathscr{M}_g$ with $g \geq 2$. A map $S \to \mathscr{M}_g$ is a family $\pi\colon C \to S$ of smooth curves of genus g. We can rigidify it by adding a frame for the Hodge bundle, that is, a trivialization $\pi_*\omega_{C/S} \simeq \mathscr{O}_S^g$. Isomorphism classes of such objects give a functor that is represented by a smooth quasiprojective variety X_g of dimension $g^2 + 3g - 3$. There is an obvious action of GL_g on X_g, and $\mathscr{M}_g = [X_g/\mathrm{GL}_g]$.

Let $\mathscr{X} = [X/G]$ be a quotient stack. The geometry of $\mathscr{X}$ is the G-equivariant geometry of X. For example, vector bundles on $\mathscr{X}$ correspond to G-equivariant vector bundles on X. More precisely, they are of the form $[V/G] \to [X/G]$, where $V \to X$ is an equivariant vector bundle. In particular, vector bundles over $\mathscr{B}G$ correspond to representations of G.

The cohomology of $\mathscr{X}$ is the equivariant cohomology of X, which can be defined via the Borel construction. Totaro and Edidin–Graham have defined the Chow group of $\mathscr{X}$, via a variant of the Borel construction, as follows.

Let i be a non-negative integer, $G \to \mathrm{GL}(V)$ a representation of G, with an open subset $U \subseteq V$ on which G acts freely, and such that $V \smallsetminus U$ has codimension larger than i. Then G acts freely on $X \times U$, and

$$\mathrm{CH}^i \mathscr{X} \stackrel{\text{def}}{=} \mathrm{CH}^i\big((X \times U)/G\big)$$

and

$$\mathrm{CH}^* \mathscr{X} \stackrel{\text{def}}{=} \bigoplus_{i \geq 0} \mathrm{CH}^i \mathscr{X}\,.$$

This definition turns out to be independent of the presentation $[X/G]$.

If X is smooth, $\mathrm{CH}^* \mathscr{X}$ has a natural structure of commutative graded ring.

If the action has finite stabilizers, and has a moduli space $M = X/G$, then it was proved by Edidin and Graham that $(\mathrm{CH}^* \mathscr{X}) \otimes \mathbb{Q} = (\mathrm{CH}^* M) \otimes \mathbb{Q}\,.$

These Chow rings have almost all the properties of usual Chow groups. They are contravariant for equivariant morphisms of smooth stacks. Furthermore, vector bundles on $\mathscr{X}$ have Chern classes in $\mathrm{CH}^* \mathscr{X}$ with the usual properties. A representation $G \to \mathrm{GL}(V)$ gives a vector bundle on $\mathscr{B}G$, so there are Chern classes $\mathrm{c}_i(V) \in \mathrm{CH}^i \mathscr{B}G$.

In particular, if $\mathscr{X} = [X/G]$, the morphism $X \to \mathrm{pt}$ gives a morphism $[X/G] \to \mathscr{B}G$, hence a ring homomorphism $\mathrm{CH}^* \mathscr{B}G \to \mathrm{CH}^* \mathscr{X}$.

If $\mathscr{Y} \subseteq \mathscr{X}$ is a closed substack of $\mathscr{X}$, then there is a an exact localization sequence

$$\mathrm{CH}^* \mathscr{Y} \longrightarrow \mathrm{CH}^* \mathscr{X} \longrightarrow \mathrm{CH}^*(\mathscr{X} \smallsetminus \mathscr{Y}) \longrightarrow 0\,.$$

If $G = \mathbb{G}_{\mathrm{m}}$, and we can let G act by multiplication of $\mathbb{A}^n$, and $(\mathbb{A}^n \smallsetminus \{0\})/\mathbb{G}_{\mathrm{m}} = \mathbb{P}^{n-1}$. If $\mathrm{id}\colon \mathbb{G}_{\mathrm{m}} = \mathrm{GL}_1$ is the tautological representation and $t \stackrel{\mathrm{def}}{=} \mathrm{c}_1(\mathrm{id})$, then id descends to $\mathscr{O}(-1)$ on $\mathbb{P}^{n-1}$, and $\mathrm{CH}^* \mathbb{P}^{n-1} = \mathbb{Z}[t]/(t^n)$; hence $\mathrm{CH}^* \mathscr{B}\mathbb{G}_{\mathrm{m}} = \mathbb{Z}[t]$.

Analogous arguments using Grassmannians show that if c_1, …, c_n are the Chern classes of the tautological representation of GL_n, then $\mathrm{CH}^* \mathscr{B}\,\mathrm{GL}_n = \mathbb{Z}[c_1, \dots, c_n]$.

Theorem (V., 1998). Consider the affine space $\mathbb{A}^7$ of forms of degree 6 in two variables, with the action of GL_2 defined by $(A \cdot f)(x) = (\det A)^2 f(A^{-1}x)$, and its stable open subvariety $X \subseteq \mathbb{A}^7$ consisting of smooth forms. Then $\mathscr{M}_2 = [X/\operatorname{GL}_2]$, and

$$\mathrm{CH}^* \mathscr{M}_2 = \mathbb{Z}[\lambda_1, \lambda_2]/(10\lambda_1, 2\lambda_1^2 - 24\lambda_2)\,.$$

Here the λ_i are the Chern classes of the Hodge bundle on $\mathscr{M}_2$ (Mumford's λ classes). The ring $\mathrm{CH}^* \mathscr{M}_2$ is not generated by κ classes (for example, Mumford's famous formula $\kappa_1 = 12\lambda_1$ holds integrally). It seems to be a general fact that integral Chow rings of stacks of smooth curves in low genus tend to be generated by λ-classes rather than by κ-classes.

Here is a sketch how to embed $\mathscr{M}_2$ into a representation of GL_2. Geometrically, such a representation corresponds to a vector bundle on $\mathscr{B}\,\mathrm{GL}_2$. Since every X in $\mathscr{M}_2$ has a unique degree 2 map $X \to C$, where C is a smooth curve of genus 0, we can use the standard description of double covers to give an equivalent description of $\mathscr{M}_2$ as the stack of triples (C, L, s), where C is a smooth curve of genus 0, L is a line bundle of degree 3 on C, and s is a section of $L^{\otimes 2}$ vanishing at six distinct points. We have a stack of pairs (C, L), where C and L are as above: since every such pair is non-canonically isomorphic to $(\mathbb{P}^1, \mathscr{O}(3))$, this is the classifying stack of torsors under the automorphism group of the pair $(\mathbb{P}^1, \mathscr{O}(3))$, which is GL_2. There is vector bundle $\mathscr{V}$ of rank 7 on this stack, whose fiber over (C, L) is $\mathrm{H}^0(C, L)$; since the condition on s that it has six distinct zeroes is open, this gives an open embedding $\mathscr{M}_2 \subseteq \mathscr{V}$.

Theorem, (Arsie–V., 2004). Let $g \geq 2$, and call $\mathscr{H}_g \subseteq \mathscr{M}_g$ the smooth substack consisting of hyperelliptic curves. Consider the affine space $\mathbb{A}^{2g+3}$ of forms of degree $2g+2$ in two variable, with the open subvariety $X_g \subseteq \mathbb{A}^{2g+3}$ consisting of smooth forms.

If g is even, set $G = \mathrm{GL}_2$, with the action on X_g defined by $(A \cdot f)(x) = \det(A)^g f(A^{-1}x)$.

If g is odd, set $G = \mathbb{G}_{\mathrm{m}} \times \mathrm{PGL}_2$, with the action of X_g defined by $\big((\alpha, [A]) \cdot f\big)(x) = \alpha^{-2} \det(A)^{g+1} f(A^{-1}x)$.

Then $\mathscr{H} = [X_g/G]$. Furthermore

$$\operatorname{Pic} \mathscr{H}_g = \begin{cases} \mathbb{Z}/(4g+2) & \text{if } g \text{ is even, and} \\ \mathbb{Z}/(8g+4) & \text{if } g \text{ is odd}. \end{cases}$$

The case when g is even is much easier, because in this case G is *special*, that is, every G-torsor is locally free in the Zariski topology.

Theorem (Edidin–Fulghesu, 2008). If g is even, then

$$\mathrm{CH}^* \mathscr{H}_g = \mathbb{Z}[c_1, c_2]/\bigl(2(2g+1)c_1, g(g-1)c_1^2 - 4g(g+1)c_2\bigr).$$

Since $\mathscr{H}_2 = \mathscr{M}_2$, this recovers my previous result for $g = 2$.

Andrea Di Lorenzo has recently found a different presentation of $\mathscr{H}_g$ when g is odd, in which the group is $\mathbb{G}_{\mathrm{m}} \times \mathrm{GL}_3$, which is special.

Theorem (Fulghesu–Viviani, Di Lorenzo). If g is odd, then

$$\mathrm{CH}^* \mathscr{H}_g = \mathbb{Z}[c_1, c_2, c_3]/\bigl(4(2g+1)c_1, 8c_1^2 - 2(g^2-1)c_2, 2c_3\bigr).$$

The stack $\mathscr{M}_3 \smallsetminus \mathscr{H}_3$ of non-hyperelliptic curves of genus 3 can be shown to be the quotient $[X/\operatorname{GL}_3]$, were X is the space of smooth quartic forms in three variables, by the action of GL_3 defined by $(A \cdot f)(x) = \det(A) f(A^{-1}x)$.

Theorem (Di Lorenzo – Fulghesu – V.).

$$\operatorname{CH}^*(\mathscr{M}_3 \smallsetminus \mathscr{H}_3) = \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3]/(\text{some complicated relations})\,.$$

Once again, here we see λ classes as generators.

In all these cases the technique is the same: one expresses a stack $\mathscr{M}$ as a quotient $[X/G]$, where X is an open subscheme of a representation V of a linear algebraic group G (or something closely related). If $Y = V \smallsetminus X$, we have a localization sequence

$$\mathrm{CH}^*[Y/G] \longrightarrow \mathrm{CH}^*[V/G] \longrightarrow \mathrm{CH}^*\mathscr{M} \longrightarrow 0\,.$$

Thus $\mathrm{CH}^*\mathscr{M}$ is a quotient of $\mathrm{CH}^*[V/G] = \mathrm{CH}^*\mathscr{B}G$; the issue is computing the relations coming from $\mathrm{CH}^*[Y/G]$. This is usually achieved by projectivizing, stratifying $\mathbb{P}(Y)$, finding resolutions of singularities of the closures of the strata that are geometrically natural, and using localization techniques to compute pushforwards of cycles from the resolutions.

However, very few stacks are of this form. Some have stratifications in which the strata are of this type, for different representations of different groups. For example, consider $\mathscr{M}_3$; the open substack $\mathscr{M}_3 \smallsetminus \mathscr{H}_3$ has the description as a quotient stack for an action of GL_3 on the space of quartic form in three variables; of course this comes from the fact that every non-hyperelliptic smooth curve of degree 4 has a canonical embedding as a quartic in $\mathbb{P}^2$. But the canonical sheaf of a hyperelliptic curve is not ample.

Another example is $\overline{\mathscr{M}}_2$. We have seen that $\mathscr{M}_2 = [X/GL_2]$, where X is the space of smooth forms of degree 6 in two variables. This has to do with the fact that every smooth curve of genus 2 is a double cover of $\mathbb{P}^1$. Recall that $\overline{\mathscr{M}}_2$ contains a smooth divisor Δ_1 consisting of curves with a separating node, that is, union of two smooth or nodal irreducible curves of arithmetic genus 1 attached at smooth point. Every curve in $\overline{\mathscr{M}}_2 \smallsetminus \Delta_1$ is a double cover of $\mathbb{P}^1$ with a ramification divisor of degree 6. From this one can deduce that $\overline{\mathscr{M}}_2 \smallsetminus \Delta_1 = [X'/GL_2]$, where X' is the space of forms of degree 6 in two variables with at most a double zero, and the action is given, as before, by $(A \cdot f)(x) = (\det A)^2 f(A^{-1}x)$. The stack Δ_1 also a description as a quotient stack, but it is completely different.

In cases like these it can be very hard to patch the descriptions of different strata in a way that is suitable for calculations.

Let $\mathscr{Y}$ be a smooth closed substack of a smooth quotient stack $\mathscr{X}$; call $i\colon \mathscr{Y} \to \mathscr{X}$ and $j\colon \mathscr{X} \smallsetminus \mathscr{Y} \to \mathscr{X}$ the embeddings. There is a localization sequence

$$\mathrm{CH}^* \mathscr{Y} \xrightarrow{i_*} \mathrm{CH}^* \mathscr{X} \xrightarrow{j^*} \mathrm{CH}^*(\mathscr{X} \smallsetminus \mathscr{Y}) \longrightarrow 0\,.$$

We would like to compute $\mathrm{CH}^* \mathscr{X}$ from $\mathrm{CH}^* \mathscr{Y}$ and $\mathrm{CH}^*(\mathscr{X} \smallsetminus \mathscr{Y})$; we can use the sequence above to get generators for $\mathrm{CH}^* \mathscr{X}$, but one can not get the relations without some control over the kernel of i_*.

A possible approach is to use higher intersection theory; this present serious problems, because while one can often get a reasonable description of $\mathrm{CH}^*(\mathscr{X} \smallsetminus \mathscr{Y}; 1)$, getting information on the boundary map $\mathrm{CH}^*(\mathscr{X} \smallsetminus \mathscr{Y}; 1) \to \mathrm{CH}^* \mathscr{Y}$ is harder.

For $\overline{\mathscr{M}}_2$, this strategy was carried out very recently by Eric Larson, who proved the following result.

Theorem (E. Larson). We have

$$\mathrm{CH}^* \overline{\mathscr{M}}_2 = \mathbb{Z}[\lambda_1, \lambda_2, \delta_1]/(24\lambda_1^2 - 48\lambda_2, 20\lambda_1\lambda_2 - 4\delta_1\lambda_2, \delta_1^3 + \delta_1^2\lambda_1, 2\delta_1^2 + 2\delta_1\lambda_1)$$

where $\delta_1 = [\Delta_1]$.

I would like to present an approach to the study of $\mathrm{CH}^* \overline{\mathscr{M}}_2$ due to Andrea Di Lorenzo and myself, which has some independent interest. There is a case in which we can can get a lot of information from the localization sequence, using ideas that go back to Borel, Atiyah and Segal, and have been exploited by many authors in the contexts of equivariant cohomology, equivariant Chow rings and equivariant cohomology.

Let $\mathscr{Y}$ be a smooth closed substack of a smooth quotient stack $\mathscr{X}$ as before, and take the localization sequence

$$\mathrm{CH}^* \mathscr{Y} \xrightarrow{i_*} \mathrm{CH}^* \mathscr{X} \xrightarrow{j^*} \mathrm{CH}^*(\mathscr{X} \smallsetminus \mathscr{Y}) \longrightarrow 0\,.$$

If N is the normal bundle of $\mathscr{Y}$ in $\mathscr{X}$, then the composite $\mathrm{CH}^* \mathscr{Y} \xrightarrow{i_*} \mathrm{CH}^* \mathscr{X} \xrightarrow{i^*} \mathrm{CH}^* \mathscr{Y}$ is multiplication by the top Chern class $\mathrm{c}_{\mathrm{top}}(N)$. If $\mathrm{c}_{\mathrm{top}}(N)$ is not a zero-divisor in $\mathrm{CH}^* \mathscr{Y}$, then i_* is injective. Furthermore, the ring homomorphism $(i^*, j^*)\colon\ \mathrm{CH}^* \mathscr{X} \times \mathrm{CH}^*(\mathscr{X} \smallsetminus \mathscr{Y})$ is injective (one can also characterize its image).

Of course, if $\mathscr{Y}$ is Deligne–Mumford (that is, it represents a moduli problem for object with finite automorphism groups) and Y is its moduli space, then $\mathrm{CH}^i \mathscr{Y} = \mathrm{CH}^i\, Y = 0$ for $i > \dim Y$; hence all elements of positive degree in $\mathrm{CH}^* \mathscr{Y}$ are zero-divisors. Hence, this can only work in the presence of object with positive-dimensional automorphism groups.

We embed $\overline{\mathscr{M}}_2$ into a vector bundle $\mathscr{V}$ over a stack $\mathscr{L}$ that is not of the form $\mathscr{B}G$, but has a smooth divisor $\mathscr{L}_1 \subseteq \mathscr{L}$ such that both $\mathscr{L}_1$ and $\mathscr{L} \smallsetminus \mathscr{L}_1$ are of the form $\mathscr{B}G$. If N is the normal bundle of $\mathscr{L}_1$ in $\mathscr{L}$, then $\mathrm{c}_1 N \in \mathrm{CH}^* \mathscr{L}_1$ is not a zero-divisor, so the previous idea applies. Our construction is based on an extension of my construction of $\overline{\mathscr{M}}_2$ as a quotient stack.

Let X be a curve in $\overline{\mathscr{M}}_2$, and call σ the hyperelliptic involution. If X is not in Δ_1 the quotient $X/\langle\sigma\rangle$ a smooth curve of genus 0. If X is in Δ_1 then $X/\langle\sigma\rangle$ is a nodal conic, and the projection $X \to X/\langle\sigma\rangle$ is not flat at the separating node.

This can be fixed by giving a stack structure around the node to the quotient $X/\langle\sigma\rangle$, in which the node itself becomes a copy of $\mathscr{B}\boldsymbol{\mu}_2$; call C the resulting orbispace curve. In other word, C acquires the structure of a twisted curve. A *twisted conic* will be either a smooth curves of genus 0, or one of the nodal curves that we have just described. Twisted conics form a smooth connected quotient algebraic stack.

Thus, a curve in $\overline{\mathscr{M}}_2$ can be described as a double cover $X \to C$, where C is a twisted curve. With the standard description of double covers we see that $\overline{\mathscr{M}}_2$ is equivalent to the stack of triples (C, L, s), where C is a twisted conic, L is a line bundle of degree 3 on C, and s is a section of $L^{\otimes 2}$ that vanishes on six distinct points. If C is reducible we assume that the zero locus of s consists of three points on each component of C; in other words, L must have degree $3/2$ on each component. This can only happen thanks to the stacky node.

We have a stack $\mathscr{L}$ whose objects are pairs (C, L) as above. Call $\mathscr{L}_1 \subseteq \mathscr{L}$ the divisor of pairs (C, L) in which C is singular: we already saw that $\mathscr{L} \smallsetminus \mathscr{L}_1 = \mathscr{B}\,\mathsf{GL}_2$. One can describe $\mathscr{L}_1$ explicitly as a classifying stack $\mathscr{B}G$ for a group G of the form $(\mathbb{G}_{\mathrm{m}}^3 \rtimes \mathrm{C}_2) \ltimes \mathbb{G}_{\mathrm{a}}^2$. It is possible to compute both $\mathsf{CH}^* \mathscr{L}_1$ and $\mathsf{CH}^*(\mathscr{L} \smallsetminus \mathscr{L}_1)$.

If N is the normal bundle of $\mathscr{L}_1$ in $\mathscr{L}$, then $\mathsf{c}_1(N)$ is not a zero-divisor in $\mathsf{CH}^* \mathscr{L}_1$; this allows one to compute $\mathsf{CH}^* \mathscr{L}$. There is a vector bundle $\mathscr{V}$ on $\mathscr{L}$, whose fiber over (C, L) is $\mathsf{H}^0(C, L^{\otimes 2})$; we have an embedding $\overline{\mathscr{M}}_2 \subseteq \mathscr{V}$.

The chow ring of the stack of stable curves of genus 2

Speaker: Angelo Vistoli

<u>Questions</u>:

Q: How common are stacks such as $\mathcal{L}$?

<u>A</u>: For instance, if $G \curvearrowright X$ is a group acting on a variety with finitely many orbits, $[X/G]$ is stratified by finitely many BG_i's.

Q: what about rigidifying $\overline{M}_2$ to eliminate the generic μ_2-stabilizer?

A: In that case the Chow group should be simpler.