Motive of $\mathrm{Quot}^n(E)$ for E a bundle on a smooth projective curve.

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(Joint with Bagnarol, Perroni)

Origin: Getzler - Pandharipande ('95, '05) ~~Betti~~ Betti #s of

$$\overline{M}_{0,n}(\mathbb{P}^r, d) \supset M_{0,n}(\mathbb{P}^r, d)$$

and (I) class of $M_{0,n}(\mathbb{P}^r, d)$ in the Grot. group of var. w/ S_n-action

Key idea: $M_{0,n}(\mathbb{P}^r, d) = \mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^r) \times_{\mathrm{Aut}(\mathbb{P}^1)} F(\mathbb{P}^1, n)$

$+ \mathrm{Mor}^d(\mathbb{P}^1, \mathbb{P}^r) \subset \mathbb{P}(\mathbb{C}[s,t]_d^{\oplus r+1})$

<u>Idea</u>: <u>Stratify</u> by degree of <u>common</u> <u>factor</u>.

Stratum comming from "common factor" has degree e":

$$\mathbb{P}(\mathbb{C}[s,t]_e) \times \mathrm{Mor}_{d-e}(\mathbb{P}^1, \mathbb{P}^r)$$

2) Write explicitly relationship

$$\mathcal{U} = \sum_{n \geq 0} \sum_{d \geq 0} q^d \, \overline{M}_{0,n}(\mathbb{P}^r, d) \in K(\mathrm{Var}, \mathbb{S})[[q]]$$

$$K(\mathrm{Var}, \mathbb{S}) = \prod_{n \geq 0} K(\mathrm{Var}, S_n).$$

Equivalently it's the Groth. group of the category $[P, Var]$,
where $P = \coprod_{n \geq 0} [pt/S_n]$

$\overline{\mathcal{U}}$ = same with $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$.

Main Theorem of [GP]: explicit relation between $\mathcal{U}, \overline{\mathcal{U}}$.

<u>Ingredient</u>: give $K(Var, \mathbb{S})$ + [Gq] structure of <u>composition</u> <u>algebra</u>.

<u>Ex</u>: $\mathbb{Q}[x]$ is filtered w/
$F_i \mathbb{Q}[x] = (x^i)$

$D: F_i \mathbb{Q}[x] \to F_{i-1}\mathbb{Q}[x]$ (<u>derivative</u>)

obtain $S_0, S_1, \ldots$ where $S_i \in F_i$,
$S_0 = 1$ & $D S_i = S_{i-1}$, eg. $S_n = \frac{x^n}{n!}$.

<u>Composition</u>:

$$\mathbb{Q}[x] \otimes_{F_1} \mathbb{Q}[x] \longrightarrow \mathbb{Q}[x]$$

$$f, g \longmapsto f \circ g$$

~~comp~~ note: $F_a \otimes F_b \to F_{ab}$.

<u>Key result</u>:

$$K(\mathrm{Var}, \$), \ K(MHS_{\mathbb{Q}}, \$)$$

& same w/ $[[q]]$ are <u>composition algebras</u>.

Fact: if A is a composition algebra & $\hat{A}$ is its completion, then there's an exponential

$\mathrm{Exp}: F_1\hat{A} \to F_1\hat{A}$ with an inverse Log.

Bagnarol's thesis: replace $\mathbb{P}^r$ with Grassmanian $G(r,V)$.

(2) is very similar (action is happening on the domain, i.e. the stable curve.)

(1): need to understand $\mathrm{Mor}_d(\mathbb{P}^1, G(r,V))$

$\mathrm{Mor}_d(\mathbb{P}^1, \mathrm{Gr}(r,V))$

$= \{V \otimes \mathcal{O}_{\mathbb{P}^1} \twoheadrightarrow F\}$, F a rk r vector bdl of deg d

$\subset \mathrm{Quot}_{\mathbb{P}^1}^{r(t+1)+d}(V \otimes \mathcal{O}_{\mathbb{P}^1})$

Stratifying: take $V \otimes \mathcal{O}_{\mathbb{P}^1} \twoheadrightarrow F$ in Quot.

Have ex. seq

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & F_{tors} \arrow[r] & F \arrow[r, two heads] \arrow[u] & F^{\vee\vee} \arrow[r] \arrow[u] & 0 \\
 & & V \otimes \mathcal{O}_{\mathbb{P}^1} \arrow[r, equal] \arrow[u, two heads] & V \otimes \mathcal{O}_{\mathbb{P}^1} \arrow[u, two heads] & \\
 & & \tilde{S} \arrow[r] \arrow[u] & S \arrow[r] \arrow[u] & F_{tors}
\end{tikzcd}$$

Note: $F^{\vee\vee}$ is loc free of rk r & deg $e < d$.

Moreover, $F^{\vee}$ is loc free of deg $e<d$

so $F^{\vee} = \bigoplus_{i=1} \mathcal{O}_{\mathbb{P}^1}(a_i)$, $a_i \geq 0$,

$\sum a_i = e < d$.

So: stratify by $F^{\vee}$ + each stratum $\mathrm{Quot}^{\cancel{d-e}}_{\mathbb{P}^1}(S)$

Note: S is loc free of rk $\dim V - r$ & degree $-e$.

Question: Fix C a smooth proj curve, E rk r vector bdl on C

$a \geq 0$.

$[\mathrm{Quot}^a_C(E)] \in K(\mathrm{Var})$?

Fact: $\mathrm{Quot}^1_C = \mathbb{P}_C(E) = \mathbb{P}^{r-1}_k[C] \in K(\mathrm{Var})$.

~~Fact~~ Idea: use

$\mathrm{Quot}^a(E) \longrightarrow \mathrm{Sym}^a(C)$

& $\exists\, U \subset C$ open s.t. $E|_U \cong \mathcal{O}_U^{\oplus r}$

& $Z \setminus U$ is a finite set.

Prop: [Bagnarol-Fantechi-Perroni]

$$[\mathrm{Quot}^a_C(E)] = [\mathrm{Quot}^a_C(\mathcal{O}_C^{\oplus \mathrm{rk} E})]$$

Then Bifet '89 gives

$$[\mathrm{Quot}^a(\mathcal{O}_C^{\oplus b})] = \sum_{\substack{v \in \mathbb{N}^{\oplus b} \\ |v| = a}} [\mathrm{Sym}^{v_1} C] \times \cdots \times [\mathrm{Sym}^{v_b} C] \times \mathbb{A}^{d(v)}$$

where $d(v) = \sum_{i=1}^{b} (i-1) v_i$

~~Op~~ Operads

(Kelly 2005) $\mathcal{V}$ a <u>cosmos</u>

(symmetric monoidal category, complete + cocomplete)

There is a (non symmetric) tensor product on $[\mathbb{P}, \mathcal{V}]$

If you take $\mathcal{V} = \mathrm{Var} \rightsquigarrow$ comp algebra

<u>But</u>: Var isn't a cosmos (not complete / cocomplete)

So: use a fully-faithful embedding of Var in a cosmos.

E.g. $\mathrm{Var} \underset{ff}{\subset} [\mathrm{Aff}^{op}, \mathrm{Sets}]$

But isn't compatible w/ finite quotients

So: use $\mathrm{Var} \underset{ff}{\subset} \mathrm{Sh}_{fppf}(\mathrm{Aff})$

(sheaves in the fppf topology on Aff.)

Thm (Bagnarol) $K(\mathcal{C}, P)[[M]]$ is a ~~comp~~ composition algebra, where M = monoid of effective curve classes on a smooth

projective variety, $\mathcal{C}$ is either
Varieties or $MHS_{\mathbb{Q}}$

<u>Questions from Audience</u>

Q] Relation to "twisted algebras"?

A] Maybe?

Q]