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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Charles Godfrey **Email/Phone:** cgodfrey@uw.edu

Speaker's Name: Joe Harris

Talk Title: Compactifying moduli of line bundles on curves

Date: 5 / 6 / 2019 **Time:** 3 : 30 **am / pm (circle one)**

Please summarize the lecture in 5 or fewer sentences: Compactification of the moduli spaces of degree d line bundles on genus g curves is discussed, including Caporaso's construction of a compact moduli space and focussing on difficulties involved in building a universal line bundle. This leads to a consideration of normal vector conditions on families intersecting certain strata of the boundary of Caporaso's moduli space,

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [X] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [X] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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- [X] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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- [ ] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Compactifying moduli of line bundles on curves

Speaker: Joe Harris

Basic objects:

$$P_{d,g} = \left\{ (C, L) : \begin{matrix} C \in M_g \\ L \in \mathrm{Pic}^d(C) \end{matrix} \right\} / \mathrm{isom}$$

$$H^{\circ}_{d,g,r} = \left\{ \begin{matrix} \text{smooth, irreducible non-degen} \\ C \subset \mathbb{P}^r \text{ degree } d, \text{ genus } g \end{matrix} \right\}$$

$$\begin{matrix} H^{\circ}_{d,g,r} \\ \downarrow \\ P_{d,g} \\ \downarrow \\ M_g \end{matrix}$$

Goal: compactify $P_{d,g}$ in a nice way

Enumerative examples

Say $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ is a pencil of plane cubics with base points $p_1, \ldots, p_9$, let $a_1, \ldots, a_9 \in \mathbb{Z}$ with $\sum a_i = 0$.

For how many λ is

$$\mathcal{O}_{C_\lambda}\left(\sum a_i p_i\right) \simeq \mathcal{O}_{C_\lambda} \; ?$$

Sim: Say $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ is a pencil of plane quartics w/ b.p. $p_1, \ldots, p_{16}$ & say $a_1, \ldots, a_{16} \in \mathbb{Z}$, $\sum a_i = 2$, not all ≥ 0.

For how many λ is $h^0(\mathcal{O}_{C_\lambda}(\sum a_i p_i))$
> 0 ?

For fixed curve C, $\mathrm{Pic}^d(C)$
$\cong \mathrm{Pic}^e(C)$
for all d, e.

<u>But</u>: don't have $P_{d,g} \cong P_{e,g}$.

True if $e = -d$ or $e \equiv d \mod (2g-2)$

<u>Kouvidakis</u> $P_{d,g} \cong P_{e,g}$ ~~$\Leftrightarrow d \equiv \pm e (2g-2)$~~
iff $d \equiv \pm e \pmod{2g-2}$.

C very general: $NS(\mathrm{Pic}^d(C)) = \mathbb{Z}$.

- If $\mathcal{L}$ is any line bundle on $P_{d,g}$, its restriction to $\mathrm{Pic}^d(C)$ is alg. equiv to $\mathcal{O}_{\mathrm{Pic}^d(C)}(m\Theta)$ for some m.

<u>Q</u>: what is the smallest such m?

<u>Goal</u>: compactify $P_{d,g}$.

<u>Desiderata</u>: 1)

$$\begin{array}{ccc} \overline{P}_{d,g} & \hookleftarrow & P_{d,g} \\ \downarrow & & \downarrow \\ \overline{M}_g & \hookleftarrow & M_g \end{array}$$

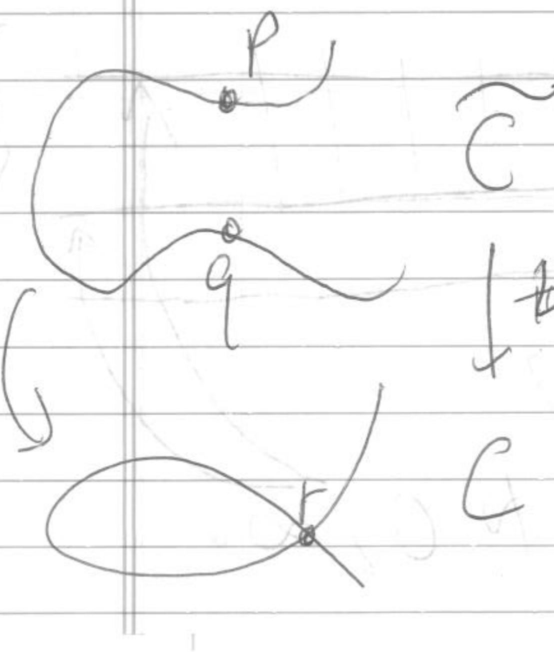
Also:

2) $\overline{P}_{d,g} \times_{\overline{M}_g} \overline{C}_g$ should have a universal line bundle.

(1) solved by Caporaso $\approx$ 20 yrs ago.

First: Compactify $Pic^d(C)$ for C singular.

1) Curves w/ 1 node 



Line bundle on C = line bundle $\mathcal{L}$ on $\tilde{C}$ + identification $\mathcal{L}_p \cong \mathcal{L}_q$.

~~Get a short exact seq~~

Get an <u>extension</u>

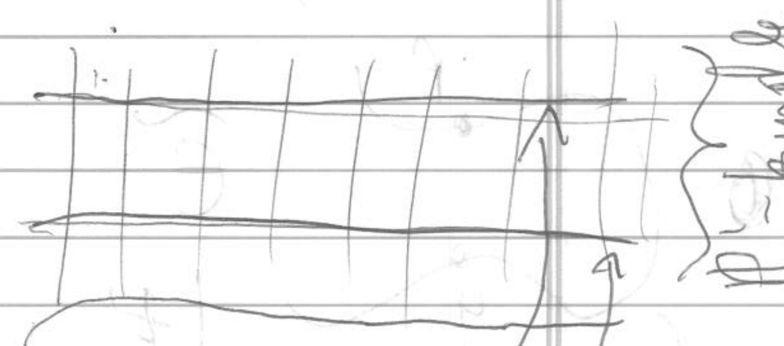
$\cancel{0 \to \mathrm{Pic}^d(C)}$

$$0 \to \mathbb{C}^* \longrightarrow \mathrm{Pic}^d(C) \to \mathrm{Pic}^d(\tilde{C}) \to 0$$

<u>To compactify</u>: allow <u>torsion free sheaves of rank 1.</u>

<u>Describe this</u>:

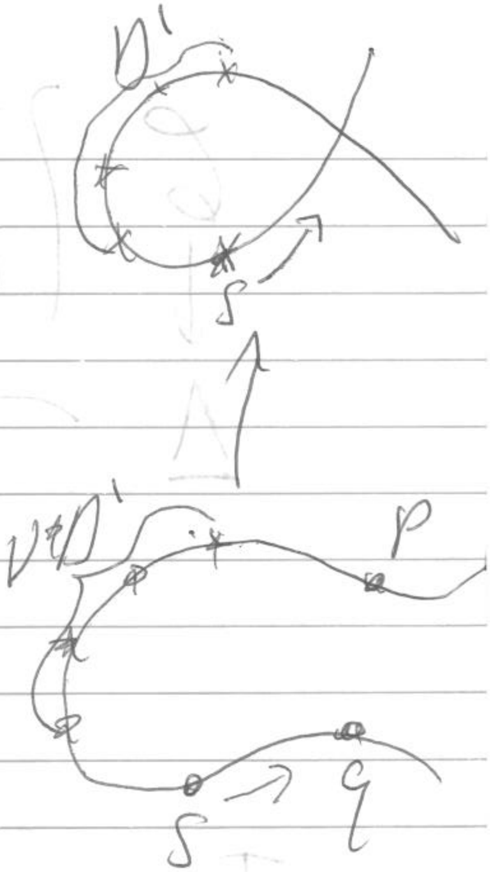
Picture:



$\mathrm{Pic}^d(C)$

sections corresponding to $0, \infty$.

$$\lim_{s\to r} \nu^* \mathcal{O}_C(D'+s)$$



$$= \begin{cases} \mathcal{O}_C(D'+p) \ \underline{\text{or}} \\ \mathcal{O}_C(D'+q) \end{cases}$$

<u>So</u>: need to <u>glue</u> $0, \infty$ sections along <u>translation</u> by $q-p$.

2) $C = C_1 \cup C_2$

In this case $\mathrm{Pic}^{0,0}(C) = \mathrm{Pic}^0(C_1) \times \mathrm{Pic}(C_2)$

<u>Problem</u>: limitting line bundle is not unique.

$$\begin{array}{c}\mathcal{C}\\ \downarrow\\ \Delta\end{array}$$

Say have $\mathcal{L}^*$ on $\mathcal{C}^* = \pi^{-1}(\Delta \setminus \{0\})$

$\mathcal{L}^*$ extends to a line bundle on $\mathcal{C}$, but <u>not</u> uniquely.

<u>Issue</u>: if $\mathcal{L}$ is one extension, so is

$$\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(C_1) = \mathcal{L}'$$

$$\mathcal{L}'|_{C_1} = \mathcal{L}|_{C_1}(-p), \quad \mathcal{L}'|_{C_2} = \mathcal{L}|_{C_2}(p)$$

Need to restrict "bidegree" of line bundles on $C = C_1 \cup C_2$.

Caporaso's thesis:

Can choose a degree distribution consistently over moduli.

Method:

Mumford-Knudson: embed smooth C of genus g w/ $m K_C$.

$$C \hookrightarrow \mathbb{P}^{m(2g-2)-g}$$

Look at $\mathcal{H}_{m(2g-2),\, g,\, m(2g-2)-g}$.

Locus H of ~~cam~~ m-canon. embedded C of genus G is loc closed.

Thm (Mumford-Knudsen):

$$H /\!/ PGL = \overline{M}_g .$$

To build $P_{g,d}$, drop the condition that C is canonically embedded.

What's the problem?

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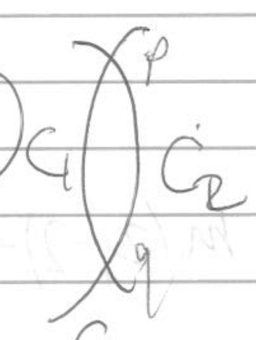
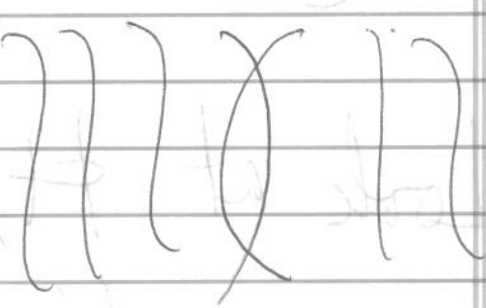




key: L on C is not determined by $L|_{C_1}$ and $L|_{C_2}$.

Get an extension

$$0 \to \mathbb{C}^{\times} \to \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C_1) \times \mathrm{Pic}^0(C_2) \to 0$$

Say $\mathcal{L}$ is a line bundle on C.

$\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}_C(C_1)$,

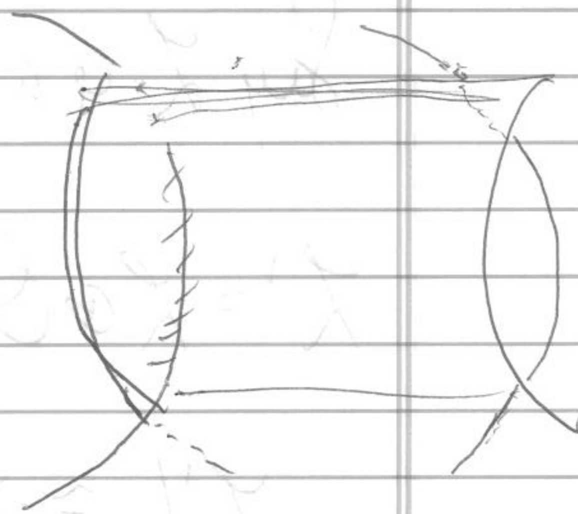
$\mathcal{L}'|_{C_1} \cong \mathcal{L}|_{C_1}(-p-q)$, $\mathcal{L}'|_{C_2} \cong \mathcal{L}|_{C_2}(p+q)$

So if $\mathcal{L}$ is a line bundle on C w/ deg 0, then $(\mathcal{L}|_{C_1}, \mathcal{L}|_{C_2})$ have degrees either (even, even) or (odd, odd).

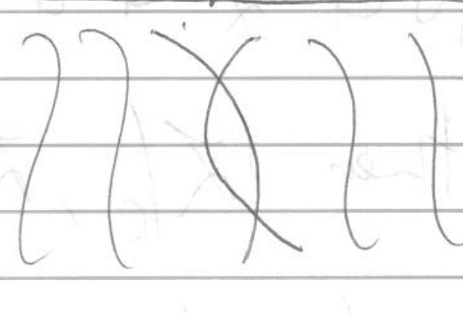
Get a priori two components in the limit:

$\mathrm{Pic}^{0,0}(C)$ $\quad$ $\mathrm{Pic}^{1,-1}(C)$ or $\mathrm{Pic}^{-1,1}(C)$

identify these by twisting w/ C_1



Issue: identification depends on the family.



Proposed solution:

identification $Pic^{1,-1}(C) = Pic^{-1,1}(C)$ depends only on the normal vector of the family as it intersects the locus of curves like 

Thm (Mano) there's an isomorphism

$$\mathbb{C}^* \longleftrightarrow \mathbb{C}^*$$

normal direction identification.

⑥ Audience questions