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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Ori Katz   **Email/Phone:** ORIKATZ.OK@gmail.com

Speaker's Name: Yifeng Yu

Talk Title: Optimal rate of convergence in periodic homogenization of Hamilton-Jacobi equations

Date: 10/09/18   **Time:** 9:15 (am) / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Yu presents recent progress in obtaining the optimal rate of convergence $O(\varepsilon)$ in periodic homogenization of Hamilton-Jacobi equations, with a new method. The new method uses a natural connection between convergence rate & the underlying Hamiltonian system, allowing employment of powerful tools from the Aubry-Mather theory & the weak KAM theory.

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Optimal rate of convergence in periodic homogenization of Hamilton-Jacobi equations

Yifeng Yu

Department of Mathematics

University of California, Irvine

Joint work with Hiroyoshi Mitake and Hung V. Tran

Hamiltonian systems, from topology to applications through analysis I

MRSI, 2018

Homogenization Theory of Hamilton-Jacobi Equation

Assume $H(p, x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is uniformly coercive in the p variable and periodic in the x variable.
For each $\epsilon > 0$, let $u^\epsilon \in C(\mathbb{R}^n \times [0, \infty))$ be the viscosity solution to the following Hamilton-Jacobi equation

$$\begin{cases} u^\epsilon_t + H\left(Du^\epsilon, \frac{x}{\epsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u^\epsilon(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1}$$

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It was known (**Lions-Papanicolaou-Varadhan, 1987**), that u^ϵ, as $\epsilon \to 0$, converges locally uniformly to u, the solution of the effective equation,

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{2}$$

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$\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is called "effective Hamiltonian" or "α function", a nonlinear averaging of the original H.

Cell problem: for any $p \in \mathbb{R}^n$, there exists a **UNIQUE** number $\overline{H}(p)$ such that

$$H(p + Dv, y) = \overline{H}(p) \quad \text{in } \mathbb{T}^n.$$

has periodic viscosity solutions $v = v(y, p)$ ("corrector").

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$$u^\epsilon(x, t) = u(x, t) + \epsilon v\left(\frac{x}{\epsilon}, Du\right) + O(\epsilon^2).$$

Note: The corrector $v(x, p)$ for $p = Du(x, t)$ basically captures the oscillation of Du^ϵ at (x, t). $y = \frac{x}{\epsilon}$.

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According to the obove formal expansion, we "have" that

$$|u^\epsilon - u| = O(\epsilon).$$

However, there is NO way to justify this expansion rigorously!

Previous Results

Why does the expansion not hold generically?

(I) The solution of the effective equation $u(x,t)$ is in general not even C^1;
(2) There does not even exist a continuous selection of

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Strategy: (1) Using solutions to an auxiliary equation v_λ to replace v.

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• *Note: Armstrong, Cardaliaguet and Souganidis (2014) extended this to convex H in the i.i.d setting and obtained* $O(\epsilon^{1/8})$

Open Question

Whether the convergence rate $O(\epsilon^{1/3})$ can be improved?

In particular, when can we obtain the optimal one $O(\epsilon)$?

Note: It is basically impossible to modify or refine the **Capuzzo-Dolcetta–H. Ishii method** to achieve this goal. A completely new approach has to be introduced.

Main Result 1: General Convex Case ($p \to H(p, x)$)

Theorem (Mitake, Tran, Y. 2018)

Assume H is onvex in p and $g \in \mathrm{Lip}(\mathbb{R}^n)$.
(i)

$$u^\epsilon(x,t) \geq u(x,t) - C\epsilon \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [0,\infty).$$

The constant $C > 0$ in (i) and (ii) below depend only on H and $\|Dg\|_{L^\infty(\mathbb{R}^n)}$.

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The constant $C > 0$ in (i) and (ii) below depend only on H and $\|Dg\|_{L^\infty(\mathbb{R}^n)}$.

(ii) For fixed $(x,t) \in \mathbb{R}^n \times (0,\infty)$, if u is differentiable at (x,t) and $\overline{H}$ is twice differentiable at $p = Du(x,t)$, then

$$u^\epsilon(x,t) \leq u(x,t) + \widetilde{C}_{x,t}\epsilon.$$

if the initial data $g \in C^2(\mathbb{R}^n)$ with $\|g\|_{C^2(\mathbb{R}^n)} < \infty$. If g is merely Lipschitz continuous, then

$$u^\epsilon(x,t) \leq u(x,t) + C_{x,t}\sqrt{\epsilon}.$$

Optimal Rate when $n = 2$

Theorem (Mitake, Tran, Y. 2018)

Assume $n = 2$ and $g \in \mathrm{Lip}(\mathbb{R}^2)$. Assume further that H is convex and positively homogeneous of degree k in p for some $k \geq 1$, that is, $H(\lambda p, x) = \lambda^k H(p, x)$ for all $(\lambda, x, p) \in [0, \infty) \times \mathbb{T}^2 \times \mathrm{R}^2$. Then,

$$|u^\epsilon(x,t) - u(x,t)| \leq C\epsilon \qquad \text{for all } (x,t) \in \mathrm{R}^2 \times [0,\infty).$$

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Here $C > 0$ is a constant depending only on H and $\|Dg\|_{L^\infty(\mathrm{R}^2)}$.

Note that $k = 1$ corresponds to Hamiltonians associated with the front propagation, which is probably one of the most physically relevant situations in the homogenization theory. For example,

$$u_t + a(x)|Du| = 0 \quad \text{in crystal growth, etc}$$

and the well known **G-equation** in turbulent combustion

$$u_t + |Du| + V(x) \cdot Du = 0.$$

Optimal Rate when $n = 1$

Theorem (Mitake, Tran, Y. 2018)

Assume that $n = 1$ and $H = H(p, x)$ is convex in p. Assume further that $g \in \mathrm{Lip}(\mathbb{R})$. Then, for each $T > 0$,

$$\|u^\epsilon - u\|_{L^\infty(\mathbb{R}\times[0,T])} \leq C\epsilon.$$

Here C is a constant depending only on H and $\|g'\|_{L^\infty(\mathbb{R})}$.

- Son N.T. Tu extended to $H(u_x, x/\epsilon, x)$ when $n = 1$ for some H (arxiv).

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• For the one dimension case, the remaining question is to find the optimal rate for general coercive H (i.e. **Nonconvex** H). Recall that the **Capuzzo-Dolcetta–Ishii** result says that

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$$\|u^\epsilon - u\|_{L^\infty(\mathrm{R}\times[0,T])} \leq C\epsilon^{\frac{1}{3}}.$$

• We conjecture that the optimal rate is $O\left(\sqrt{\epsilon}\right)$.

Sketch of Proof of the Lower Bound $u^\epsilon \geq u - C\epsilon$

$$u^\epsilon(0,1) = \inf_{\eta(0)=0} \left\{ g\left(\epsilon\eta\left(-\epsilon^{-1}\right)\right) + \epsilon \int_{-\epsilon^{-1}}^{0} L(\eta(t), \dot\eta(t))\, dt \right\}$$

Here $L(q,x) = \sup_{p\in\mathbb{R}^n}\{p\cdot q - H(p,x)\}$. Also,

$$u(0,1) = \inf_{y\in\mathbb{R}^n} \left\{ g(y) + \overline{L}(-y) \right\}.$$

For any $p \in \mathbb{R}^n$ and a "corrector" v_p:

$$H(p + Dv_p, y) = \overline{H}(p),$$

$$\int_{-\epsilon^{-1}}^{0} L(\eta(t), \dot\eta(t)) + \overline{H}(p)\, dt \geq p\cdot\eta(0) - p\cdot\eta\left(-\epsilon^{-1}\right) + v_p(\eta(0)) - v_p\left(\eta\left(-\epsilon^{-1}\right)\right)$$

Accordingly, since $\overline{L}(q) = \sup_{p\in\mathbb{R}^n}\{p\cdot q - \overline{H}(p)\}$,

$$\epsilon \int_{-\epsilon^{-1}}^{0} (L(\eta(t), \dot\eta(t))\, dt \geq \overline{L}\left(-\epsilon\eta\left(-\epsilon^{-1}\right)\right) - C\epsilon.$$

The Upper Bound and the Hamiltonian System

For any $p \in \mathbb{R}^n$, if $\xi : \mathbb{R} \to \mathbb{R}^n$ is a **global charateristics** of a corrector v_p, i.e.,

$$p \cdot (\xi(t_2) - \xi(t_1)) + v_p(\xi(t_2)) - v_p(\xi(t_1)) = \int_{t_1}^{t_2} L(\dot{\xi}, \xi) + \overline{H}(p)\, ds.$$

for all $t_1 < t_2$. The collection of those ξ is the so called **"Mané set"** in weak KAM theory. Such ξ is an **absolute minimizer** of the action

$$A[\gamma] = \int L(\dot{\gamma}(t), \gamma(t)) + \overline{H}(p)\, dt.$$

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Question: Does the average slope

$$\frac{\xi(t)}{t}$$

converge as $t \to \infty$? More importantly, what is the **convergence rate**?

• It is known that in **weak KAM theory/Aubry-Mather theory** that if $\overline{H}$ is differentiable at p, then

$$\lim_{t\to\infty} \frac{\xi(t)}{t} = D\overline{H}(p). \tag{3}$$

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Connection with the convergence rate in homogenization: for $p = Du(x,t)$

$$(I).\quad \left|\frac{\xi(t)}{t}-D\overline{H}(p)\right|\leq\frac{C}{t}\Rightarrow|u^{\epsilon}-u|\leq O(\epsilon)\quad \text{for } g\in Lip(\mathbb{R}^n)$$

$$(II).\quad \left|\frac{\xi(t)}{t}-D\overline{H}(p)\right|\leq\frac{C}{\sqrt{t}}\Rightarrow\begin{cases}|u^{\epsilon}-u|\leq O(\sqrt{\epsilon}) & \text{for } g\in Lip(\mathbb{R}^n)\\ |u^{\epsilon}-u|\leq O(\epsilon) & \text{for } g\in C^2(\mathbb{R}^n).\end{cases}$$

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$$(II). \quad \left|\frac{\xi(t)}{t} - D\overline{H}(p)\right| \leq \frac{C}{\sqrt{t}} \Rightarrow \begin{cases} |u^{\epsilon} - u| \leq O(\sqrt{\epsilon}) & \text{for } g \in Lip(\mathbb{R}^n) \\ |u^{\epsilon} - u| \leq O(\epsilon) & \text{for } g \in C^2(\mathbb{R}^n). \end{cases}$$

By modifying the argument of (3), it is easy to show that if $\overline{H}$ is twice differentiable at p, then (Gomes 2002)

$$\left|\frac{\xi(t)}{t} - D\overline{H}(p)\right| \leq \frac{C}{\sqrt{t}}.$$

$n = 2$ and the Aubry-Mather Theory

Key ingredient: 2d topology + the fact that two absolute minimizers ξ cannot intersect twice lead to good description of the structure of absolute minimizers (**Aubry-Mather sets** basically consist of recurrent ones).

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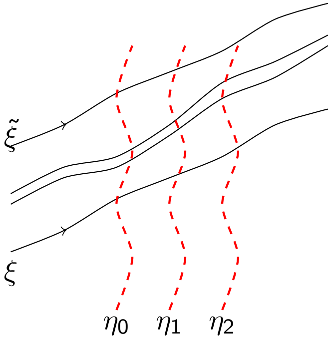
• In particular, each absolute minimizer can be identifed with a **circle map**: $f : \mathbb{R} \to \mathbb{R}$, continuous, increasing and $f(x+1) = f(x) + 1$.



There exists a rotation number α such that $|f^i(x) - x - \alpha i| \leq 1$ for all i.

Connection with the Convergence Rate

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Conjecture: For a general convex and coercive $H(p, x)$ when $n = 2$, we are working on to show that

$$|u^{\epsilon}(x,t) - u(x,t)| \leq C_{x,t}\,\epsilon \quad \text{for a.e. } (x,t) \in \mathbb{R}^2 \times (0, +\infty).$$

Some Remarks about the Higher Dimension Case $n \geq 3$

Consider a simple metric Hamiltonian with smooth, positive and periodic $a(x)$

$$H(p, x) = a(x)|p|.$$

and the associated effective Hamiltonian $\overline{H}(p)$:

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Does there exist a non-constant smooth $a(x)$ such that $\overline{H} \equiv |p|$?

When $n = 2$, the answer is "No" (**Bangert, 1994**) based on Aubry-Mather

Lack of Examples with Fractional Convergence Rate

For $n \geq 3$, consider

$$H(p, x) = a(x)|p|.$$

Although it is very reasonable to believe that the optimal convergence rate $O(\epsilon)$ is not achievable in general, we haven't been able to construct an example with **fractional convergence rate** since this involves handling chaotic behaviors.

Lack of Examples with Fractional Convergence Rate

For $n \geq 3$, consider

$$H(p,x) = a(x)|p|.$$

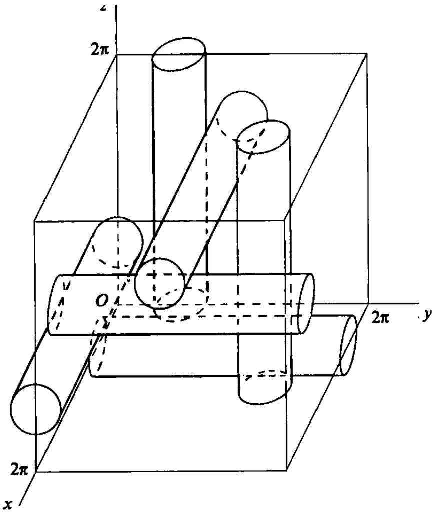
Although it is very reasonable to believe that the optimal convergence rate $O(\epsilon)$ is not achievable in general, we haven't been able to construct an example with **fractional convergence rate** since this involves handling chaotic behaviors.

When $n \geq 3$, the only well-understood interesting example is the classical **Hedlund example**: The metric function $a(x)$ is a smooth periodic singular pertubation of 1 such that any minimizing geodesics is basically confined in a small neighbourhood of one of three disjoint parallel lines.

So the **Aubry-Mather set** is very small and

$$\overline{H}(p) = C\max\{|p_1|,\ |p_2|,\ |p_3|\}.$$

• The level surface is a cube, in particular not C^1, which is **different** from $n = 2$.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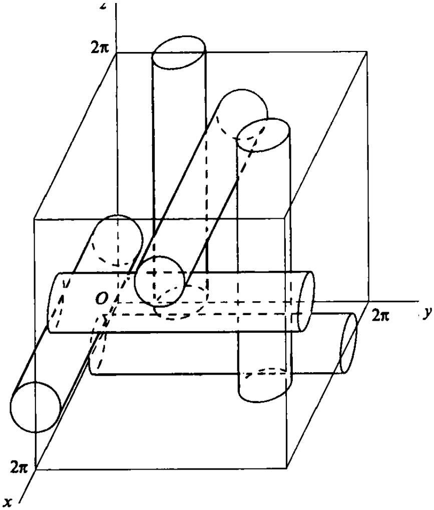


However, for this sort of “bad” example, the convergence rate is $O(\epsilon)$.

Summary: Optimal convergence rate ⇒ Dynamical system.

(1) For general dimension n and C^2 initial data g,

$$|u^{\epsilon}(x,t) - u(x,t)| = O(\epsilon) \quad \text{for "typical" } (x,t);$$

• **Key ingrient:** Use $\overline{H}(p)$ to control the behavior of orbits on the Mané set.

(2) When $n = 2$ and H is homogeneous $H(\lambda p, x) = \lambda^k H(p, x)$:

$$|u^{\epsilon}(x,t) - u(x,t)| = O(\epsilon) \quad \text{for all } (x,t);$$

• **Key ingredient:** the original Aubry-Mather theory—identification of an orbit with a circle map.

(3) When $n = 1$,

$$|u^{\epsilon}(x,t) - u(x,t)| = O(\epsilon) \quad \text{for all } (x,t);$$

• **Key ingredient:** integrability of 1d Hamiltonian system.

Optimal rate of convergence in periodic homogenization of Hamilton-Jacobi equations - Yifeng Yu

Lecture Notes (Ori S. Katz)

October 12, 2018

Abstract

In this talk, I will present some recent progress in obtaining the optimal rate of convergence $O(\epsilon)$ in periodic homogenization of Hamilton-Jacobi equations. Our method is completely different from previous pure PDE approaches which only provides $O(\epsilon^{1/3})$. We have discovered a natural connection between the convergence rate and the underlying Hamiltonian system. This allows us to employ powerful tools from the Aubry-Mather theory and the weak KAM theory. It is a joint work with Hiroyashi Mitake and Hung V. Tran.

1 Blackboard + Lecture notes

No x dependence in the effective Hamiltonian.

The two major PDE's in the talk:

$$u_t^\epsilon + H\left(Du^\epsilon, \frac{x}{\epsilon}\right) = 0$$

$$u^\epsilon(x,u) = g(x)$$

$$\xrightarrow[\epsilon\to 0]{} \star \begin{cases} u_t + \bar{H}(Du) = 0 \\ u(x,v) = g(x) \end{cases}$$

What is $\bar{H}$?

The "corrector" may not be unique.

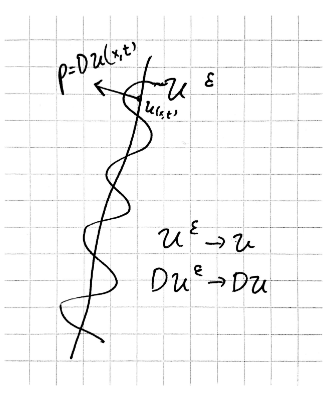
Why is it called the corrector?

u^ϵ has fluctuations, $u^\epsilon \to u$, $Du^\epsilon \to Du$.

Fix a point $u(x,t)$ and look at the direction $p = Du(x,t)$.

so the oscillation of the gradient is captured by

$$u^\epsilon \approx u(x,t) + \epsilon u(p, x/\epsilon).$$



So this is how we determine the effective Hamiltonian:

$$H(p + Dv, y) = \bar{H}(p)$$

For regularity:

1. Find a Lipshitz-continuous in p selection

$$p \to u(p, x)$$

2. Solution to the effective equation $u(x,t)$ to $\star$:

$$u_t + \bar{H}(Du) = 0$$

$$u(x, v) = g(x)$$

that is C^1.

For n=1, this is easy, but in higher dimensions, for example $n = 3$ there is no continuous selection. So the first condition is the most serious obstacle due to lack of general regularity.

Up to now, the best result was due to Capuzzo-Dolcetta and Ishii (2001).

By combining 2 strategies they got the result

$$|u^\epsilon - u| \leq \mathcal{O}\left(\epsilon^{1/3}\right).$$

The question of the talk - can we improve convergence

$$u^\epsilon \to u\left(\mathcal{O}\left(\epsilon^{1/3}\right)\right)$$

to $\mathcal{O}(\epsilon)$?

Note that in Main Result 1 we assume convex H.

Note that the constant in (ii) depends on x, t.

Note that if H is convex, then so is $\bar{H}$, where

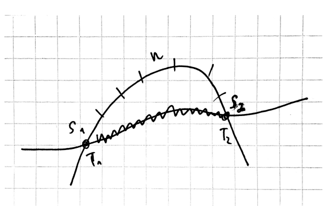
$$H(p + Du, y) = \bar{H}(p),$$

therefore it is twice differentiable.

Thus, we get

$$|u^\epsilon - u| \leq \mathcal{O}(\epsilon)$$

for "typical" (x,t). That's the best we can say in the general high dimensional case.



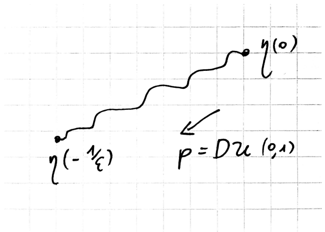
If we reduce the dimension we get the optimal convergence rate, with a constant independent on (x,t).

In 1D everything is perfect. Initially, we didn't intend to obtain $\mathcal{O}(\epsilon)$. In the beginning, we wanted to find a specific example for $\mathcal{O}(\epsilon)$.

What is the optimal rate for a general coercive H?

Even for 1D, non-convex H becomes very complicated.

Sketch of proof - lower bound is not hard. How to get the other direction? The difficulty is that the integral is over $1/\epsilon$. Start at a point $\eta(0)$ and integrate until $\eta(1/\epsilon)$.



The general direction of integration is $p = Du\,(0,1)$. Minimizing this, how do you identify the final fate of the trajectory? So we have to find a trajectory that is integrable in the sense that you can identify the final point of the trajectory. So we look at the global characteristics of the PDE.

Absolute minimizer: Choose any two points on a curve t_1, t_2, then another curve that goes through the same two points will have the same energy difference.

Problem is reduced to - if I have a trajectory, how can I find the final point?

1. KAM for $n \geq 3$ - Given a near integrable Hamiltonian

$$H^{\delta} = H\,(p) + \delta H\,(p,x)\,,$$

there is the KAM torus Q and near it $\left|\frac{\zeta(t)}{t} - Q\right| \leq C/t$. But away from the KAM tori this is not true.

2. $n = 2$ - Aubry-Mather

3. $n = 1$ - integrable.

What about the case $n \geq 3$, non-perturbative? Then KAM doesn't hold. But there is one thing to do - use an effective Hamiltonian $\bar{H}\,(p)$ to control the trajectory. So if we know $\bar{H}$ is twice differentiable we can find the limit Gomes 2002.

$n = 2$ and the Aubry-Mather theory:

We want

$$\left|\frac{\zeta\,(t)}{t} - D\bar{H}\,(p)\right| \leq C/t,$$

we have

$$\left|\frac{f^{i}\,(x)}{i} - 2\right| \leq 2/i,$$

but to apply the second to the first we need a special type of Hamiltonian.

This is a topological statement.

Conjecture - work in progress - we think the constant should be 2.

The missing step: Look at a graph of $\bar{H}(p)$. Think of the mechanical Hamiltonian

$$\frac{1}{2}|p|^2 + V(x).$$

If $D\bar{H}(p)$ is rational, everything works. But if it is irrational, we need to find a non-tangential direction such that $h(t) = \bar{H}(\tilde{p} + tq)$ is twice differentiable, $h''(u)$ exists, then we are done. But we haven't figured out, if it's irrational, how to figure out if it's twice differentiable.

Some remarks about the higher dimensional case $n \geq 3$: to show how difficult this is.

Bangert 1994 even proved a stronger case.

Question - don't you need $|p|^2$?

Answer:

$$\frac{1}{2}|p|^2 + V(x) = \bar{H}(p) > maxV$$

is equivalent to

$$\frac{1}{\sqrt{\bar{H} - V(x)}}|p| = 1.$$

Question: Are we able to find just one example, one point (x_0, t_0), such that

$$|u^\epsilon(x_0, t_0) - u(x_0, t_0)| \geq C\epsilon^2, \alpha \in (0, 1)$$

Level set is a cube. For this example we get $\mathcal{O}(\epsilon)$. The trouble you may obtain with the cube are the edges that may be non-differentiable. So you are not able to find a trajectory on the edges that points to $q \in Du(x,t)$. But this can be obtained by jumping from one trajectory to another. The combines trajectory will not be a minimizer but that's OK.