(1-1)

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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: ORI KATZ Email/Phone: ORIKATZ.OK@gmail.com

Speaker's Name: Antonio Siconolfi

Talk Title: The vanishing discount problem in a noncompact setting

Date: 10/12/18 Time: 9:30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Siconolfi presents a study of the asymptotic behavior of the solutions to a family of discounted Hamilton-Jacobi eqs when the discount factor goes to zero, in a non-compact setting. They prove that a distinguished critical solution of the equation with vanishing discount is selected @ the limit. The approach is based on some duality between suitable cones of Lagrangian functions & families of probability measures.

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(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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MSRI–Berkeley, October 2018

The vanishing discount problem in a noncompact setting

A. Siconolfi, Università di Roma "La Sapienza".

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The result of DFIZ is given for equations defined on a **compact manifold**, say the flat torus $\mathbb{T}^N$, while we study the same problem in the **whole Euclidean space** $\mathbb{R}^N$.

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They rely on some **functional analysis** and appropriate **duality principles** between spaces of Lagrangians and spaces of measures.

We do not employ **representation formulae** for solutions of the discounted problems or **property of curves** in the space of state variable.

We think that this alternative approach is interesting per se and can be handled to generalize the asymptotic result to **more general setting**, for instance in the case of **fully nonlinear second order equations**.

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Ziliotto 2018 has recently found a **counterexample** in the nonconvex case

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Under the above assumptions the definition is **well posed**. The constant c is a minimum thanks to basic stability properties of viscosity subsolutions.

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$$\lambda u + H(x, Du) = c \qquad \text{in } \mathbb{R}^N. \tag{E$_\lambda$}$$

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The proof by Crandall–Lions (1982) that the value function of the related control problem is the unique solution of the Hamilton–Jacobi–Bellman equation has been the **starting point of viscosity solution theory**.

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The equation (Ec) not only possess multiple solutions even in the compact setting , but the notion of **maximal solution cannot be given** since the pointwise supremum of all subsolution is apparently infinite.

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We name after Aubry and denote by $\mathcal{A}$ the set of points y such that

- $S(y, \cdot)$ is a solution in the whole space.

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- it is an **uniqueness set** for (Ec), namely any admissible trace on $\mathcal{A}$ can be uniquely extended to be a solution in the whole space
- given any $y \in \mathcal{A}$, there are no critical subsolutions w which are **strict** locally at y, namely satisfying

$$H(x, Dw(x)) \leq c - \epsilon \qquad \text{in a neighborhood of } y, \text{ for some } \epsilon > 0$$

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Roughly speaking, the **obstruction** to find subsolution below the critical value is **concentrated** on $\mathcal{A}$.

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- $\mathcal{A}$ is compact;
- there is **no Aubry set at infinity**. No obstruction to find subsolution below c at infinity.

It is the unbounded setting more close to the compact case. It is not clear if the asymptotic result we are looking for can be obtained by **relaxing** this set of geometric conditions.

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- There exists a locally Lipschitz continuous function ψ with

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Compare with Ishii 2008

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It is **relatively easy** to show that the u_λ are locally equibounded and locally equiLipschitz continuous, which implies local uniform convergence up to subsequences.

The **difficult point** is to show uniqueness of the limit for the whole family. In other term to prove that the asymptotic procedure is capable to **select** a special critical solution at the limit as $\lambda \to 0$.

Role of probability measures

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where

$$\langle \mu, \Phi \rangle = \int \Phi(x, q) \, d\mu(x, q)$$

for any $\Phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, μ probability measure on $\mathbb{R}^N \times \mathbb{R}^N$.

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We say that a locally Lipschitz continuous function u is a **subsolution for** Φ if

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We further say that u is a **strict subsolution** if

$$Du(x) \cdot q \leq \Phi(x, q) - \epsilon \qquad \text{for a.e. } x, \text{ any } q, \text{ some } \epsilon > 0.$$

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Imagine that there is a subsolution for Φ then there is a strict subsolution for $\Phi + \epsilon$, say u, and u can be **regularized** still remaining a strict subsolution for $\Phi + \epsilon$. So we can assume that u is smooth.

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Imagine that there is a subsolution for Φ then there is a strict subsolution for $\Phi + \epsilon$, say u, and u can be **regularized** still remaining a strict subsolution for $\Phi + \epsilon$. So we can assume that u is smooth.

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Let μ be a closed measure and Φ an element of X admitting subsolution then

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We endow X with the **compact open topology** making it a **locally convex Hausdorff space**.

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Let E be a closed convex subset of X with nonempty interior, then $N_E(\Phi_0)$ contains nonzero elements for any $\Phi_0 \in \partial E$.

This is actually a simple consequence of the Hyperplane Separation theorem in locally convex Hausdorff spaces.

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The cone $\mathcal{G}_0$ is open in X endowed with the compact open topology.

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We denote by $\mathcal{M}_0$ the above set of probability measures. Taking into account that $\mathcal{G}_0$ is a cone

$$\langle \mu, L \rangle = 0 \qquad \text{and} \qquad \langle \mu, \Phi \rangle = 0 \quad \text{for any } \Phi \in \mathcal{G}_0$$

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- $\mathcal{M}$ is convex and compact with respect to the narrow topology.

It is convenient to define the notion of **locally closed** measure

$\langle \mu, Df(x) \cdot q \rangle$ for any **compactly supported** C^1 function f

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Given a function $F : A \times B \to \mathbb{R}$ with A compact convex subset of a topological vector space and B convex subset of another topological vector space, $F(x, y)$ satisfying suitable semicontinuity and convexity/concavity properties in x, y we can conclude

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We aim at constructing a class of measures enjoying suitable properties with respect to the discounted equation

We consider the **compact perturbation** of L of the form

$$\Phi(x,q) = t\,L(x,q) + f(x,q) \qquad t \geq 0,\ f \text{ compactly supported}$$

We denote by u_λ the maximal solution to the discounted equation

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We show

Theorem

Given $\lambda > 0$, $z \in \mathbb{R}^N$, there exist probability measures μ with

$$\langle \mu, L \rangle = \lambda\, u_\lambda(z), \quad \langle \mu, \Phi \rangle \geq \lambda\, u_{\lambda,\Phi}(z)$$

for any compact perturbation Φ of L.

We assume that $u_\lambda(z) = 0$ and consider the convex cone with vertex at the origin

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The idea of the proof is to consider, given a Φ_0 all the probability measures such that the inequality to be proved holds for L and Φ_0 namely the set

$$\mathcal{P}(\Phi_0) = \{\mu \mid \langle \mu, L \rangle \leq \langle \mu, \Phi_0 \rangle\}$$

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The assertion is equivalent to

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in other terms

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for any probability measure.

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This ends the proof.

Convergence of measures

We exploit that as consequence of the last assumption L possess a **compactly supported subsolution** plus the characterization of Mather measures as locally closed measures μ with $\langle \mu, L \rangle = 0$ to get

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Theorem

Given λ_j infinitesimal and $z \in \mathbb{R}^N$, we consider a sequence of measures μ_j with

$$\langle \mu_j, L \rangle = \lambda_j \, u_{\lambda_j}(z).$$

Then μ_j narrowly converges, up to subsequences, to a probability measure $\mu \in \mathcal{M}$.

Main result

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the second lemma is

Fact

We have that

$$\langle \mu, u_\lambda \rangle \leq 0 \qquad \textit{for any } \lambda > 0, \textit{ any Mather measure } \mu.$$

The limit function

Regarding the limit function, denoted by w_0 , we recover the formula **already discovered in DFIZ paper**.

$$w_0(x) = \max\{v(x) \mid v \text{ sol. to } (\mathrm{E}_\lambda) \text{ with } \langle \mu, v \rangle \leq 0 \, \forall \, \mu \in \mathcal{M}\}$$

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Given x, y in $\mathbb{R}^N$, $S(x, y)$, namely the value at x of the maximal subsolution vanishing at y, can be interpreted as an **intrinsic (semi)distance** related to the critical equation (E_λ).

- $S(\cdot, \cdot)$ satisfies a **triangular inequality**

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- $S(\cdot,\cdot)$ satisfies a **triangular inequality**
- it can be defined an **intrinsic length** on the curves in $\mathbb{R}^N$ in such a way that $S(x,y)$ is the infimum of the intrinsic lengths of curves linking x to y

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$$\overline{S}(\mu, \nu) = \inf \left\{ \int S(x, y) \, d\gamma(x, y) \right\}$$

where γ varies among the probability measures in $\mathbb{R}^N \times \mathbb{R}^N$ with first marginal μ and second marginal ν.

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