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MSRI

Mathematical Sciences Research Institute

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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Ori Katz **Email/Phone:** ORIKATZ.OK@gmail.com

Speaker's Name: Daniel Scheeres

Talk Title: Minimum energy configurations in the N-body problem & the celestial mechanics of granular systems.

Date: 10/08/18 **Time:** 11:00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: The talk dealt with finding minimal energy configurations of N-body systems with a fixed angular momentum, for finite density distributions of the masses. For point masses, this question is ill-defined. This approach naturally leads to a "granular mechanics" extension of celestial mechanics with links to small solar system bodies.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - → **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

☆ Email notes to Scheeres

Minimum Energy Configurations in the N-body Problem and the Celestial Mechanics of Granular Systems

D.J. Scheeres

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University of Colorado Boulder

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D.J. Scheeres. 2012. “Minimum Energy Configurations in the N-Body Problem and the Celestial Mechanics of Granular Systems,” Celestial Mechanics and Dynamical Astronomy 113: 291-320.

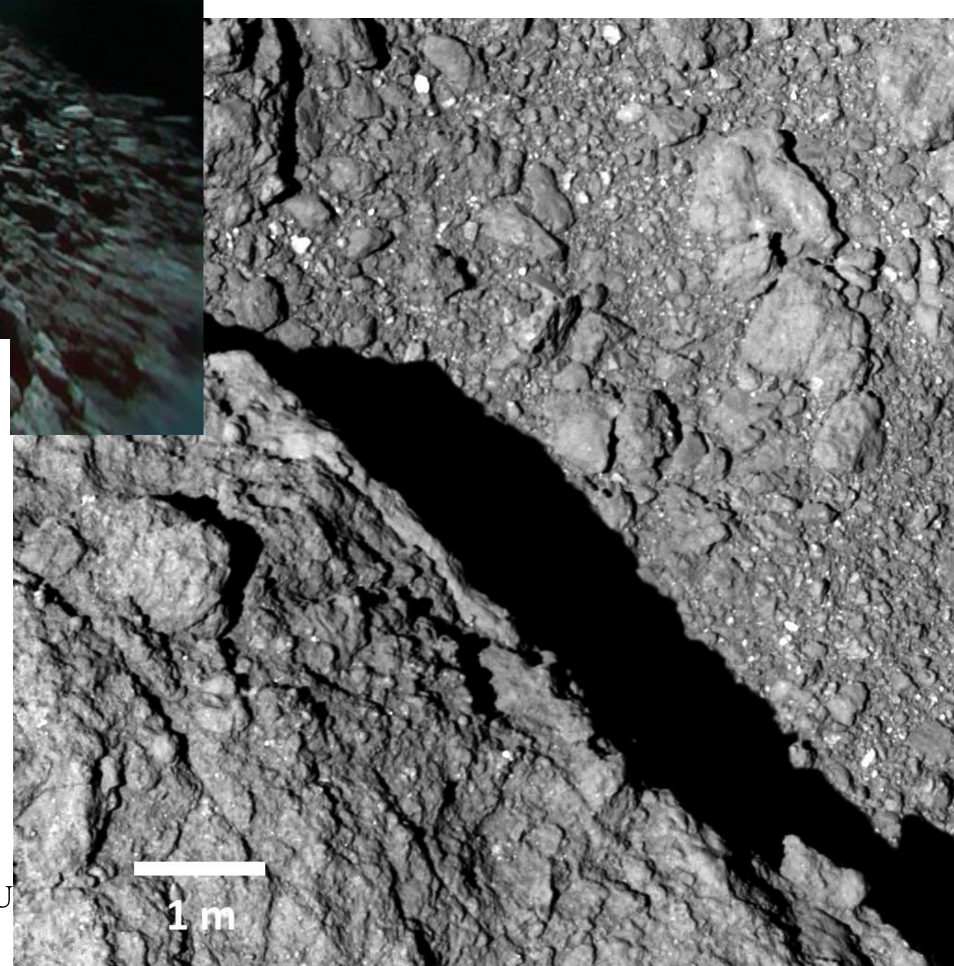
D.J. Scheeres. 2016. Relative Equilibria in the Full N-Body Problem with Applications to the Equal Mass Problem, in Recent Advances in Celestial and Space Mechanics (M. Chyba and B. Bonnard, eds.) Springer.

D.J. Scheeres. 2017. “Relative Equilibria in the Spherical, Finite Density 3-Body Problem,” Journal of Nonlinear Science 26: 1445-1482.

D.J. Scheeres. 2018. “Celestial Mechanics of Rubble Pile Bodies,” chapter in a forthcoming book based on the 2017 CELMEC meeting and school.

Granular Mechanics and Asteroids

- A recent focus of space missions and astrophysical science are the mechanics of primitive asteroids
  - A key question is how does gravitational attraction, i.e., the *N*-body problem, affect these micro-gravity bodies.



Images of the asteroid Ryugu surface from the Hayabusa2 S/C and Minerva2 rover

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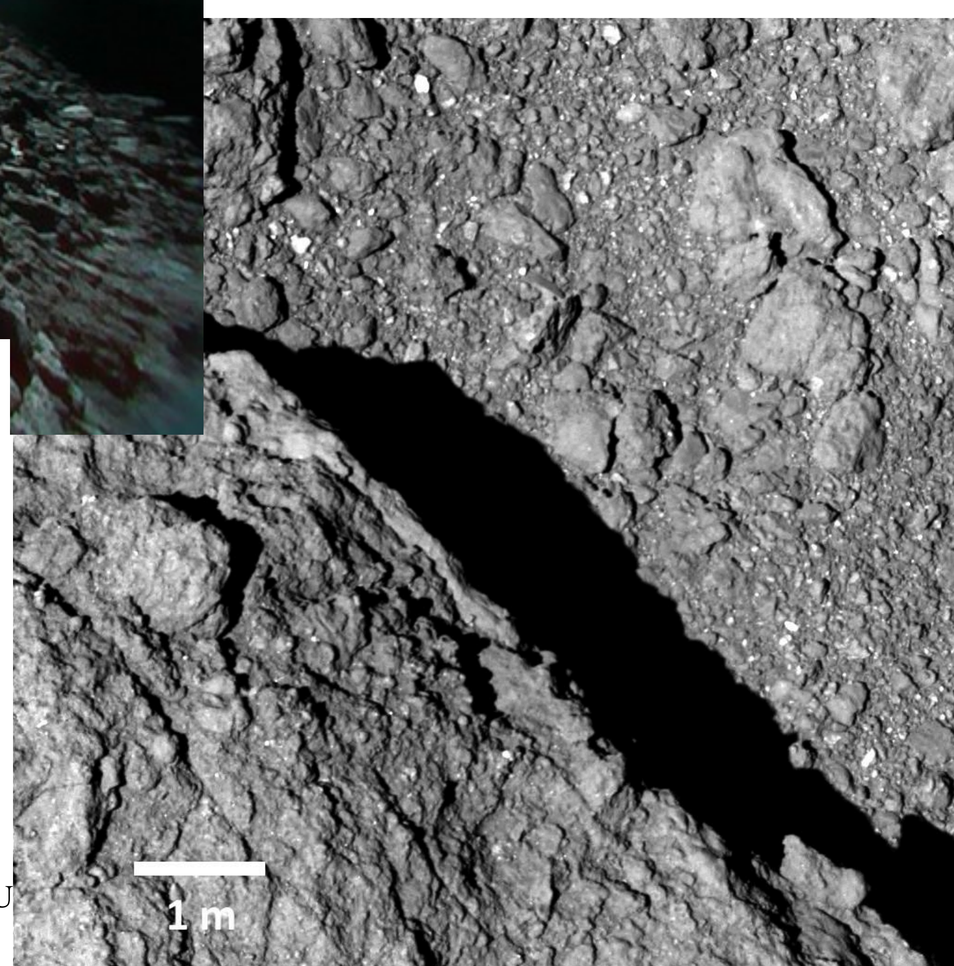
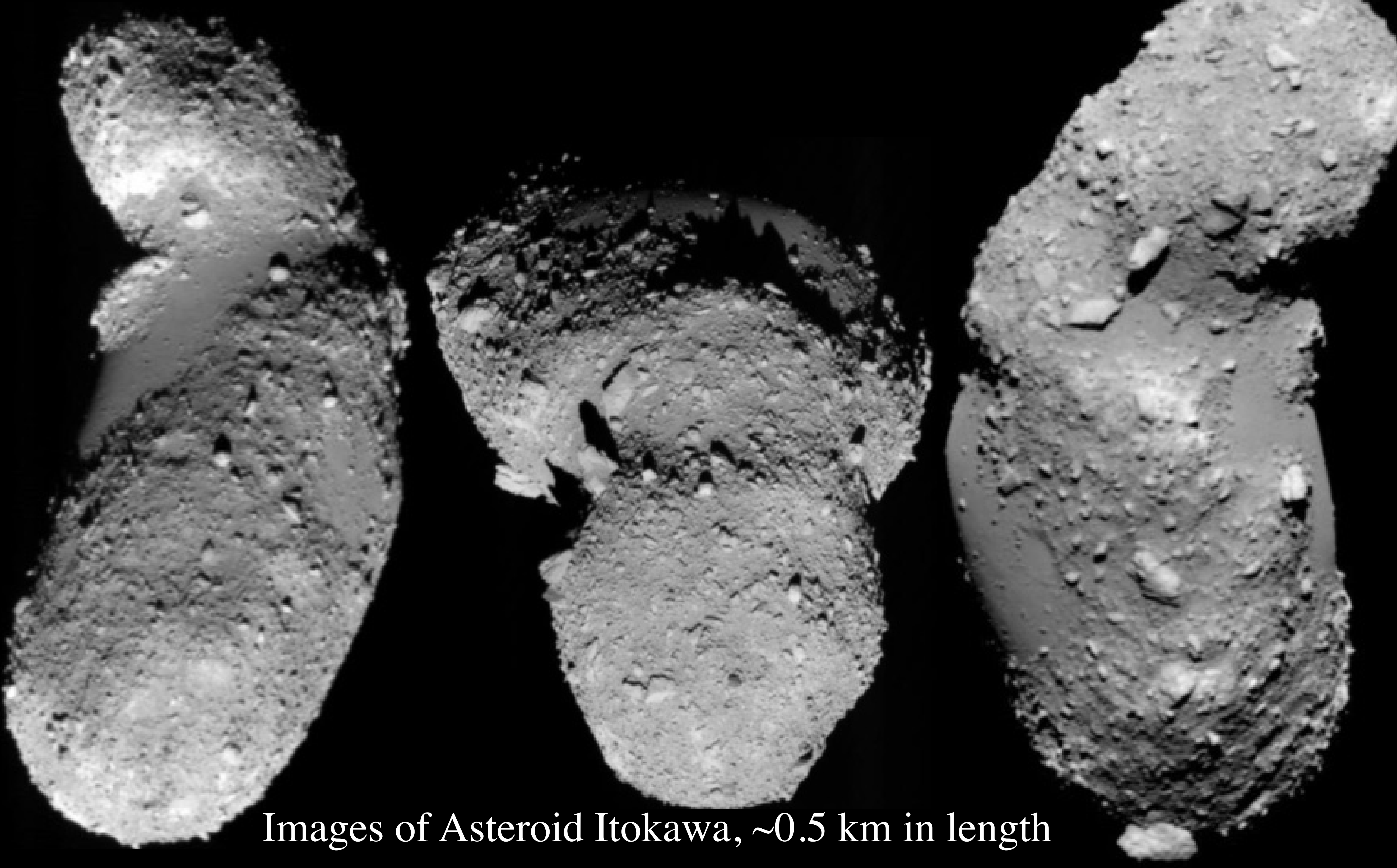


Images of the asteroid Ryugu surface from the Hayabusa2 S/C and Minerva2 rover

Fundamental and Simple Questions:

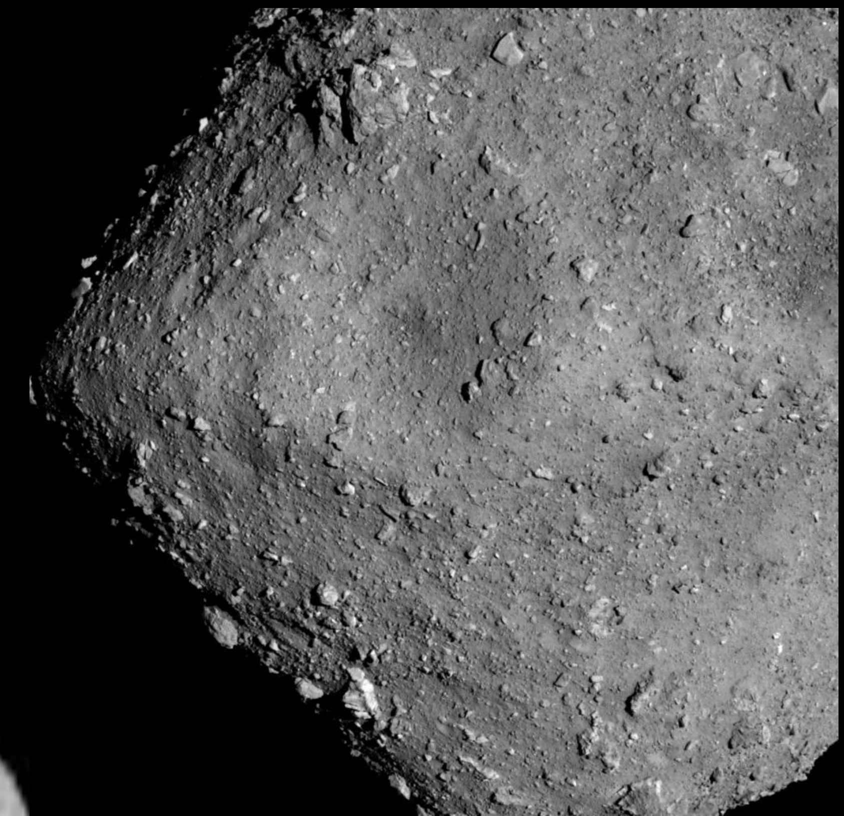
What is the expected configuration of a collection of self-gravitating grains?

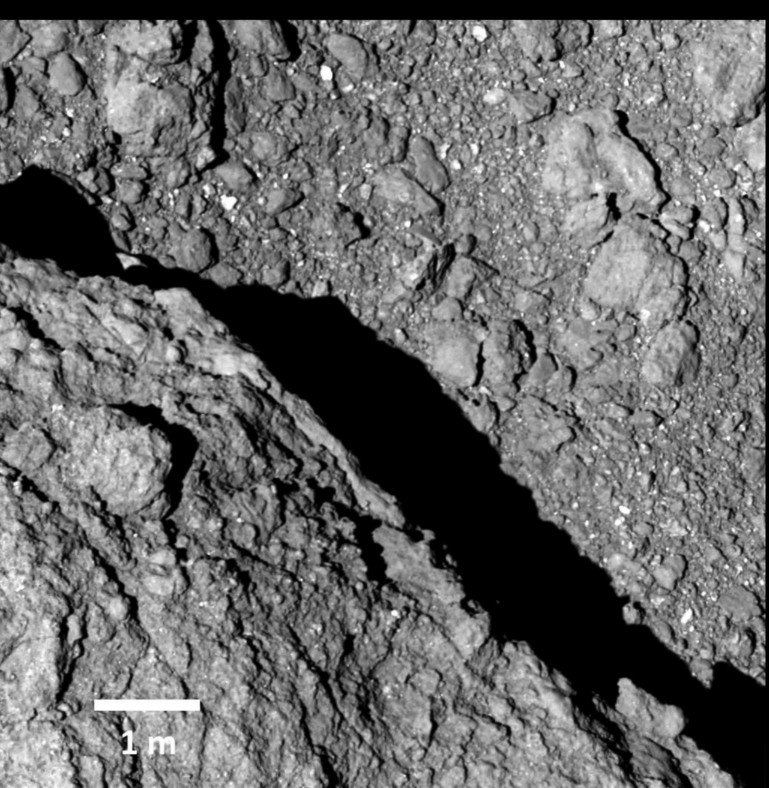


Images of Asteroid Itokawa, ~0.5 km in length

Fundamental and Simple Questions:

How will they reconfigure themselves under changing angular momentum?

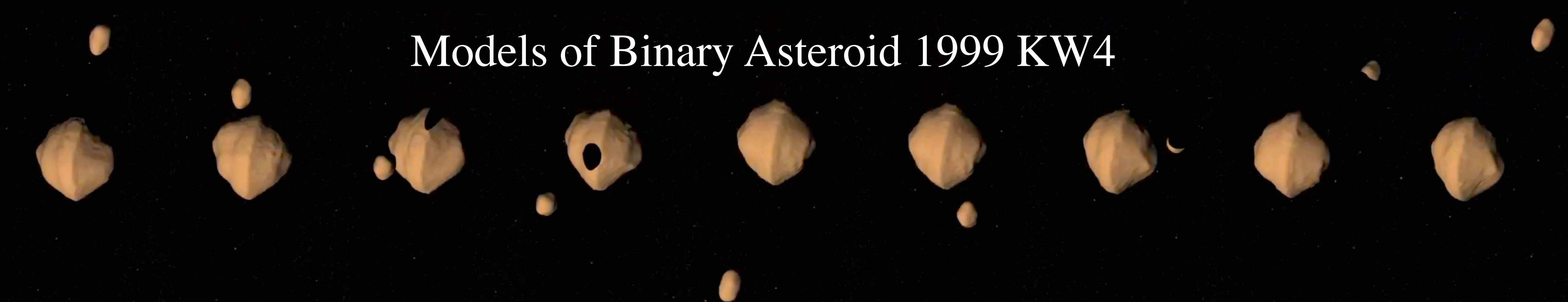




Images of Asteroid Ryugu, ~1.0 km in diameter

Fundamental and Simple Questions:

How can they form into multiple component systems?



Models of Binary Asteroid 1999 KW4

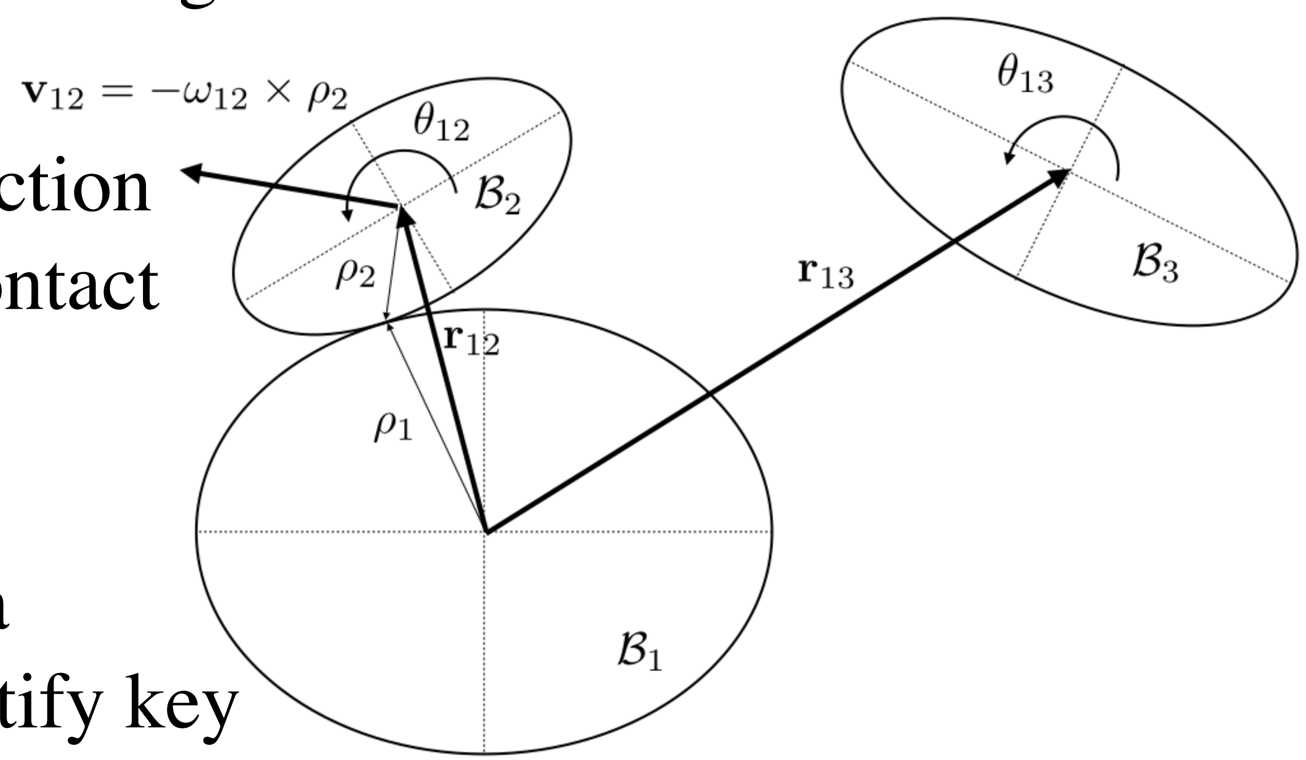
Fundamental and Simple Questions:

How can they form into multiple component systems?

Models of Binary Asteroid 1999 KW4

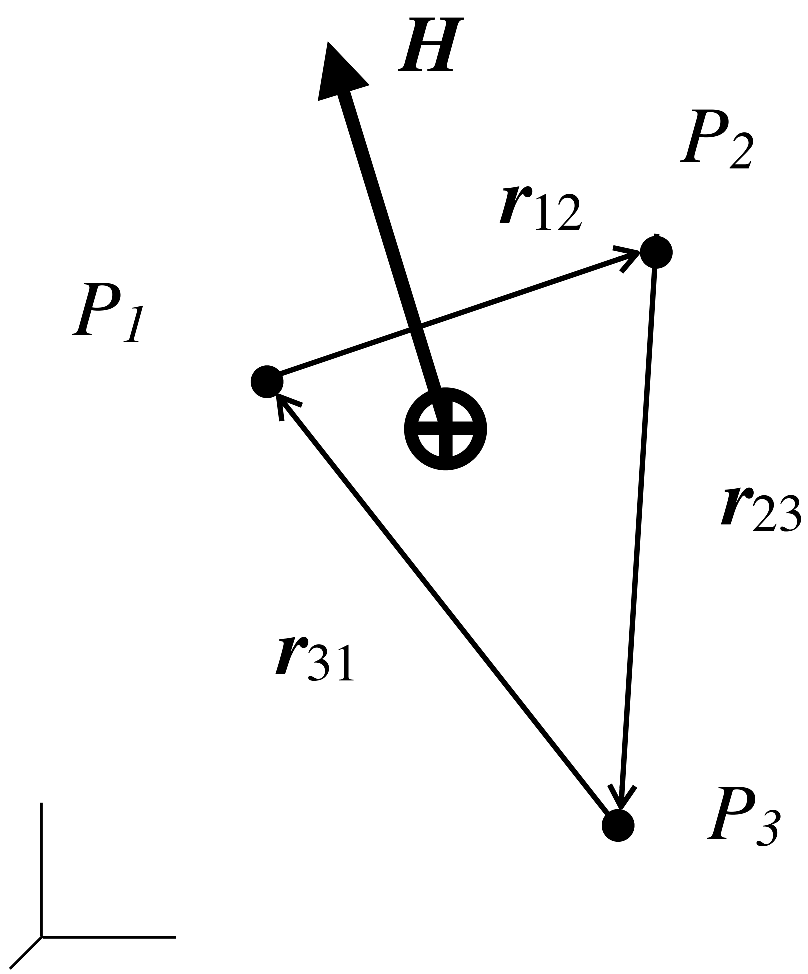
Mathematical Challenge

- Develop a realistic model for such systems that can be analyzed in a mathematically tractable way
- Approach:
  - *N*-body problem of gravitationally interacting rigid bodies.
  - Account for finite densities, surface friction and energy dissipation, allowing for contact mechanics and minimum energy configurations.
  - Track stable system configurations as a function of angular momentum to identify key transitions that can be associated with astrophysical observations.
  - Place rigorous constraints on the possible final outcomes of these systems.



Celestial Mechanics of Point Mass Bodies

- A collection of point mass bodies will conserve Angular Momentum and Energy. If it is "forced" to dissipate energy it will eventually violate some aspect of its configuration space.



Point Mass *N*-Body Problem

Configuration Space:

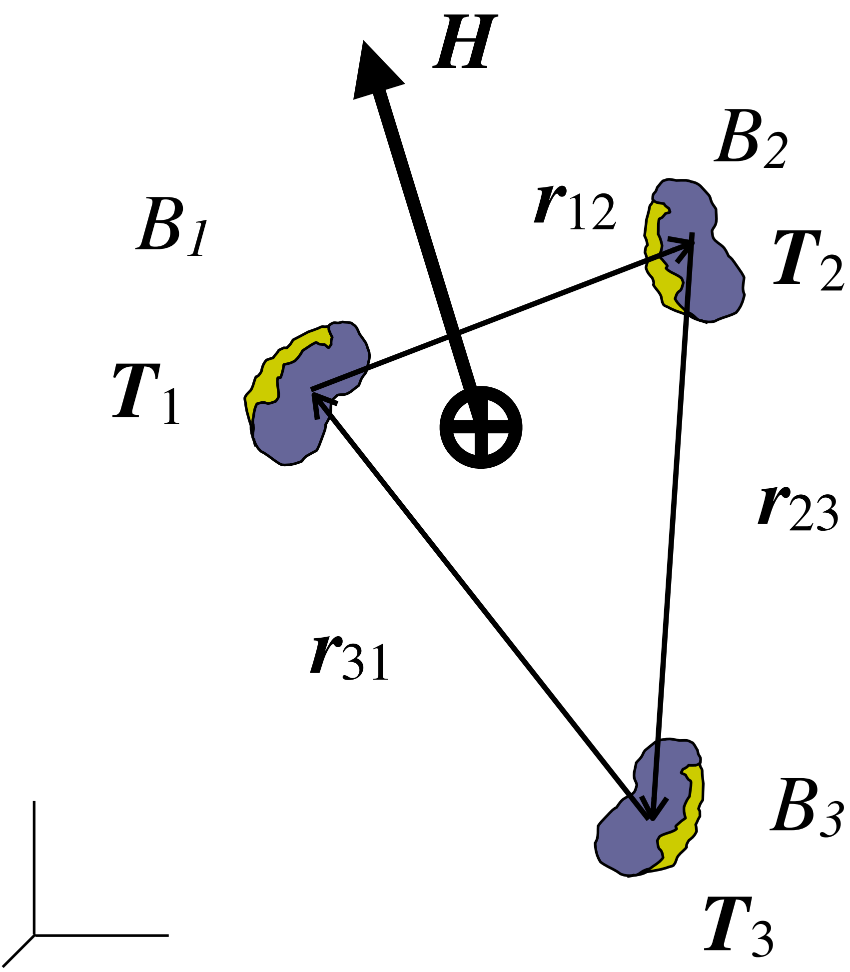
$$\mathbf{Q} = \left\{ \mathbf{r}_{ij} \; ; \; i, j = 1, 2, \ldots, N \mid \sum_j r_{ij} > 0 \right\}$$

Masses:

$$m_i ; i = 1, 2, \ldots, N$$

Celestial Mechanics of Finite Density Bodies

- A collection of quasi-rigid bodies. Inherit all of the classical *N*-body conservation principles, but are also subject to surface forces and tidal distortion
  - Enables "resting equilibria" to exist
  - Adds "moments of inertia" and attitude



Finite Density *N*-Body Problem

Configuration Space:

$$\mathbf{Q} = \{\mathbf{r}_{ij}, \mathbf{T}_{ij}; i, j = 1, 2, \ldots, N \mid r_{ij} \geq D_{ij}(\mathbf{T}_{ij})\}$$

$$\mathbf{T}_{ij} = \mathbf{T}_j^T \cdot \mathbf{T}_i$$

Mass and Inertias:

$$m_i, \mathbf{I}_i \ ; \ i = 1, 2, \ldots, N$$

General System Formulation

- The Lagrangian of N interacting rigid bodies can be expressed conveniently in terms of relative states and mass distributions
  - Potential energy: $$\mathcal{U} = -\frac{\mathcal{G}}{2}\int_{\mathcal{B}}\int_{\mathcal{B}} \frac{dm(\mathbf{r})\; dm(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$
  - Kinetic energy: $$T = \frac{1}{4M}\int_{\mathcal{B}}\int_{\mathcal{B}} (\mathbf{v}-\mathbf{v}')\cdot(\mathbf{v}-\mathbf{v}')\, dm(\mathbf{r})dm(\mathbf{r}')$$
  - Lagrangian: $$\mathcal{L} = T - \mathcal{U}$$
  - Total Angular Momentum:

$$\mathbf{H} = \overline{\mathbf{T}}\cdot\frac{1}{2M}\int_{\mathcal{B}}\int_{\mathcal{B}} (\mathbf{r}-\mathbf{r}')\times(\mathbf{v}-\mathbf{v}')\, dm(\mathbf{r})dm(\mathbf{r}')$$

System Reduction

- Transforming to a frame rotating with the total angular momentum:

$$\omega = \frac{1}{I_H}\mathbf{H} = \dot{\theta}\hat{\mathbf{H}}$$

  - Where $I_H = \hat{\mathbf{H}} \cdot \mathbf{I} \cdot \hat{\mathbf{H}}$ is the instantaneous moment of inertia of the entire system about the angular momentum direction.

- The angle θ is ignorable (due to AM conservation), and Routh reduction yields the amended potential:

$$\mathcal{E} = \frac{H^2}{2\, I_H} + \mathcal{U}$$

  - Can be generalized to account for non-holonomic interactions between bodies in contact and rolling on each other
  - Is distinct from, but yields the same results as, the usual amended potential, and is easier to work with (Moeckel, 2017)

$$\mathcal{E}' = \frac{1}{2}\mathbf{H} \cdot \mathbf{I}^{-1} \cdot \mathbf{H} + \mathcal{U}$$

The Full *N*-Body Problem

- Using the amended potential of the form:

$$\mathcal{E} = \frac{H^2}{2I_H} + \mathcal{U}$$

  - where H is the total angular momentum magnitude, a constant
  - I_H is the system moment of inertia about the angular momentum vector, a function of $\boldsymbol{Q}$ and not constant
  - $\mathcal{U}$ is the mutual gravitational potential of the system
  - The total energy of the system is $E = T + \mathcal{E}$, so $\mathcal{E}(\mathbf{Q}) \leq E$
- Thus relative equilibria and their energetic (not spectral) stability can be analyzed through the study of $\mathcal{E}$ alone as a function of a minimal set of coordinates $q_i \in \mathbf{Q}; i = 1, 2, \ldots, 6N$
- Allows us to use the total angular momentum as a parameter, to track how the relative equilibria change as it varies

Point Mass Relative Equilibria

Amended Potential:

$$I_H = \frac{1}{M}\sum_{i<j} m_i m_j \left[r_{ij}^2 - (\mathbf{r}_{ij}\cdot\hat{\mathbf{H}})^2\right]$$

$$\mathcal{E} = \frac{H^2}{2I_H} + \mathcal{U}$$

$$\mathcal{U} = -\mathcal{G}\sum_{i<j}\frac{m_i m_j}{r_{ij}}$$

Equilibrium:

$$\delta\mathcal{E} = \sum_i \frac{\partial \mathcal{E}}{\partial q_i}\delta q_i = 0 \qquad \forall q_i \in Q \qquad \forall \delta q_i$$

Stability:

$$\delta^2\mathcal{E} = \sum_{i,j}\left.\frac{\partial^2\mathcal{E}}{\partial q_i \partial q_j}\right|_* \delta q_i \delta q_j > 0$$

Always true for $N=2$

Never true for $N \geq 3$

Under dissipation a $N \geq 3$ system will always approach singularity

Finite Density Relative Equilibria

Amended Potential:

$$I_H = \frac{1}{M}\sum_{i<j} m_i m_j \left[r_{ij}^2 - (\mathbf{r}_{ij}\cdot\hat{\mathbf{H}})^2 \right] + \hat{\mathbf{H}}\cdot\sum_i \mathbf{T}_i\cdot\mathbf{I}_i\cdot\mathbf{T}_i^T\cdot\hat{\mathbf{H}}$$

$$\mathcal{E} = \frac{H^2}{2I_H} + \mathcal{U}$$

$$\mathcal{U} = \sum_{i<j}\mathcal{U}_{ij}(\mathbf{r}_{ij},\mathbf{T}_{ij}) \qquad \mathcal{U}_{ij} = -\mathcal{G}\int_{\mathcal{B}_i,\mathcal{B}_j}\frac{dm_i\, dm_j}{\rho_{ij}}$$

Equilibrium:

$$\delta\mathcal{E} = \sum_i \frac{\partial\mathcal{E}}{\partial q_i}\delta q_i \geq 0 \qquad \forall q_i \in Q$$

$$\frac{\partial\mathcal{E}}{\partial q_i} = 0 \quad \forall\delta q_i \quad \text{Unconstrained}$$

$$\frac{\partial\mathcal{E}}{\partial q_i} > 0 \quad \forall\delta q_i \geq 0 \quad \text{Constrained}$$

Stability:

$\delta^2\mathcal{E} > 0$ For all unconstrained DOF

$\delta\mathcal{E} \geq 0$ For all constrained DOF

Under dissipation the system will always approach a relative equilibrium

Point Mass vs Finite Density

- The most important differences between these two problems can be reduced to the properties of the amended potential and its relation to the angular momentum *H* and energy *E*

$$\mathcal{E} = \frac{H^2}{2I_H} + \mathcal{U} \leq E$$

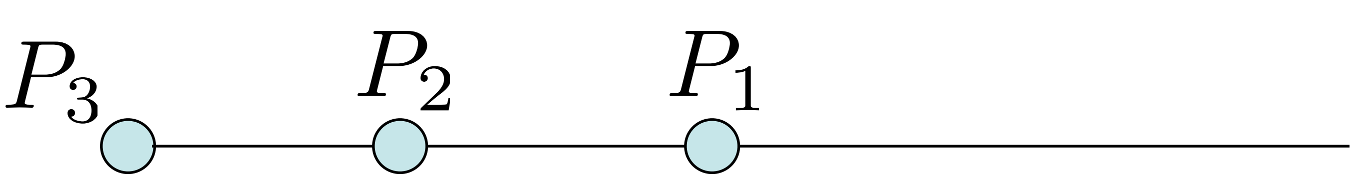
Point Mass Properties:

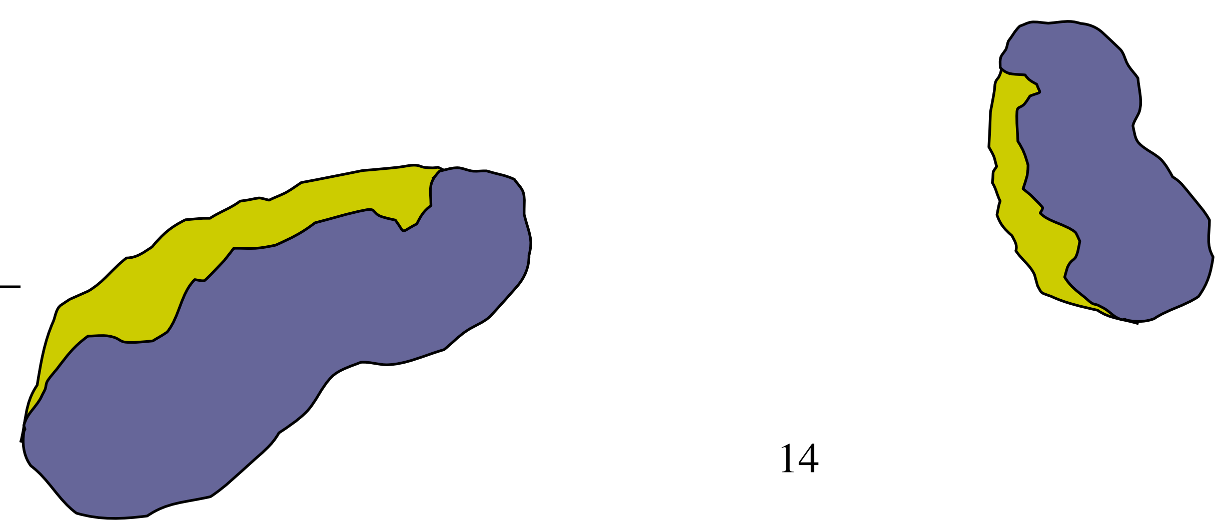
- Unbounded from below as whenever $r_{ij} \to 0$, $\mathcal{U} \rightarrow -\infty$
- Means that there are no minimum energy configurations for $N \geq 3$ as any system can conserve *H* as $E \to -\infty$

Finite Density Properties:

- Compact and bounded as $r_{ij} \geq D_{ij}(\boldsymbol{T}_{ij})$
- Means that minimum energy configurations exist for any *N* as

$$\mathcal{U} \geq D_U > -\infty \qquad I_H \geq D_I > 0$$





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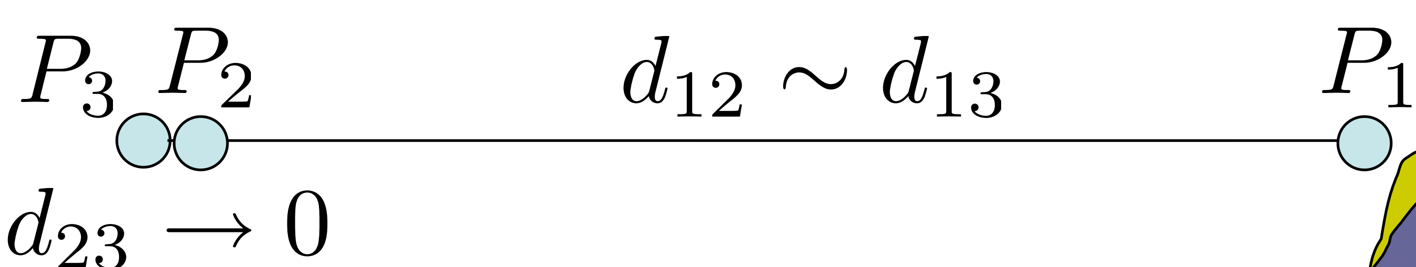

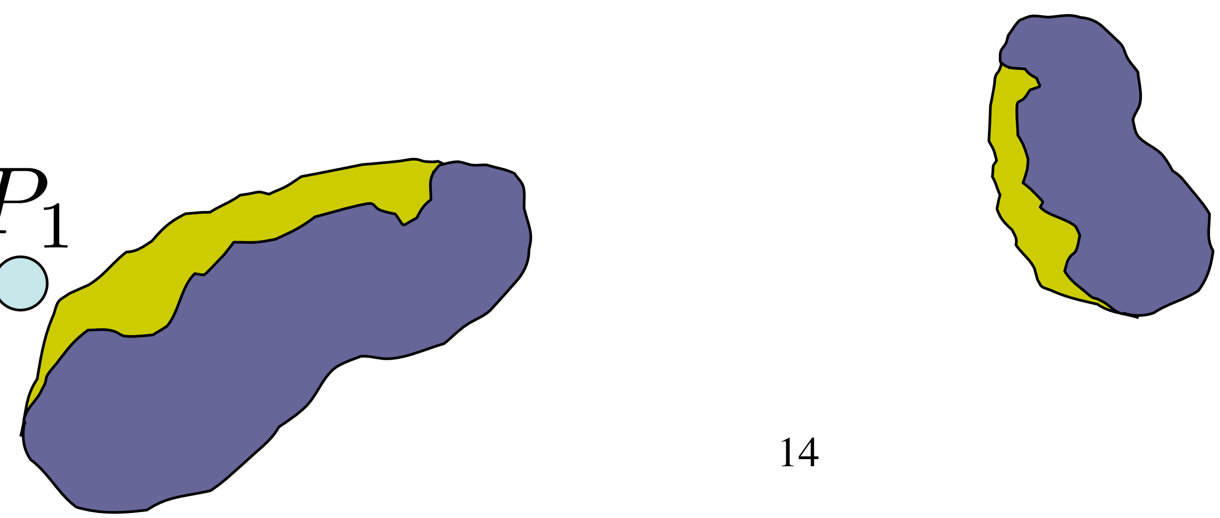

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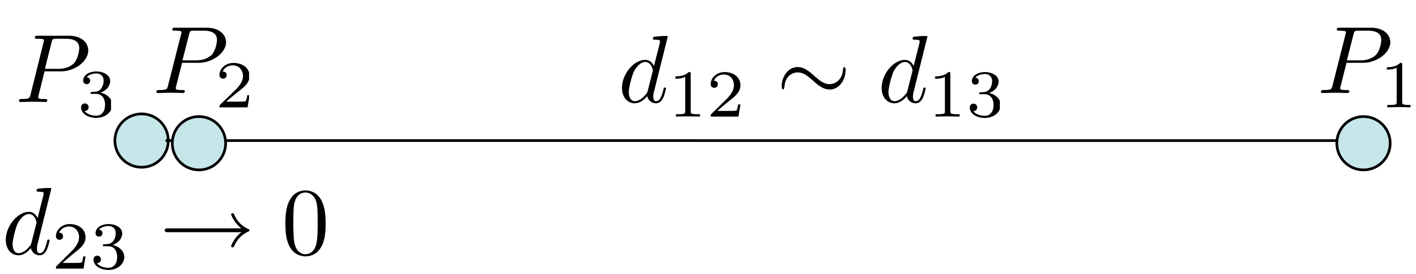
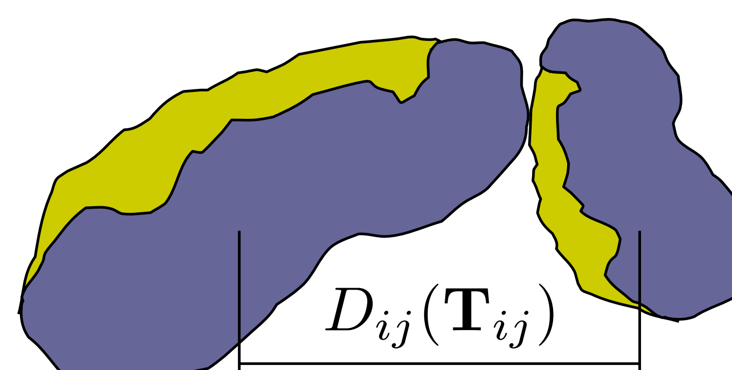




Consider the Simplest Full N-Body problem

Minimum Energy Configurations of the ***Spherical*** *Full Body Problem*

- For definiteness, consider the simple change from point mass to finite spheres (so potential energy is unchanged)
  - For a collection of N spheres of diameter d_i the only change in $\mathscr{E}$ is to I_H

$$I_H = \frac{1}{10}\sum_{i=1}^{N} m_i d_i^2 + \sum_{i=1}^{N} m_i r_i^2$$

- But this dramatically changes the structure of the minimum energy configurations… take the 2-body problem for example with equal size spheres, normalized to unity radius

$$E_m = \frac{h^2}{2d^2} - \frac{1}{d}$$

versus

$$\mathcal{E}_m = \frac{h^2}{2(0.4 + d^2)} - \frac{1}{d}$$

Minimum Energy Configurations of the ***Spherical*** *Full Body Problem*

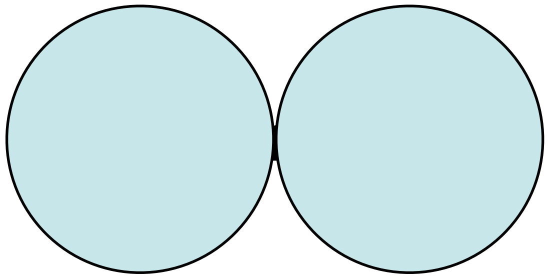
- For definiteness, consider the simple change from point mass to finite spheres (so potential energy is unchanged)
  - For a collection of *N* spheres of diameter d_i the only change in $\mathscr{E}$ is to I_H

$$I_H = \frac{1}{10}\sum_{i=1}^{N} m_i d_i^2 + \sum_{i=1}^{N} m_i r_i^2$$

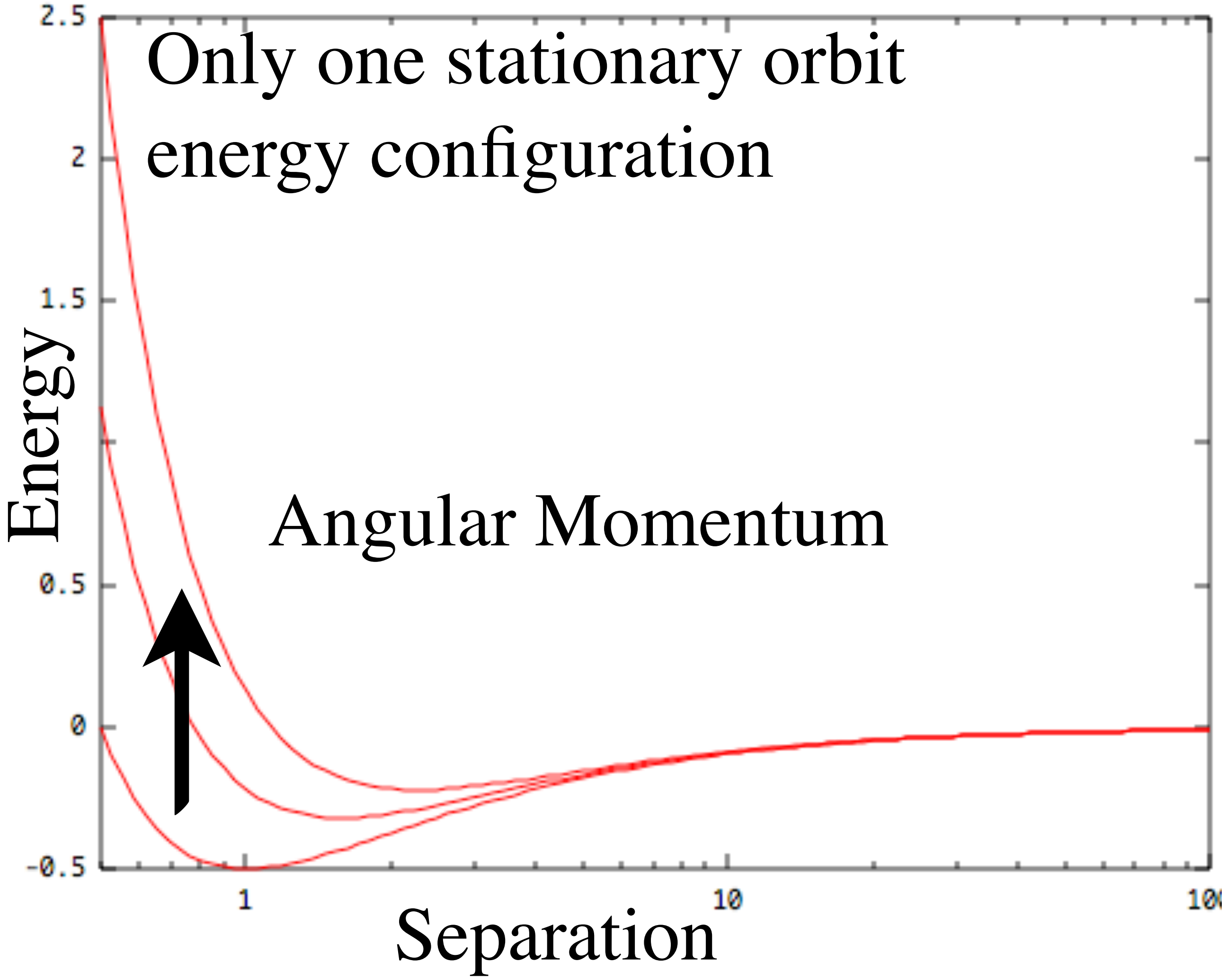
- But this dramatically changes the structure of the minimum energy configurations… take the 2-body problem for example with equal size spheres, normalized to unity radius

$$E_m = \frac{h^2}{2d^2} - \frac{1}{d} \quad \textit{versus} \quad \mathcal{E}_m = \frac{h^2}{2(0.4 + d^2)} - \frac{1}{d}$$

2-Body Problem

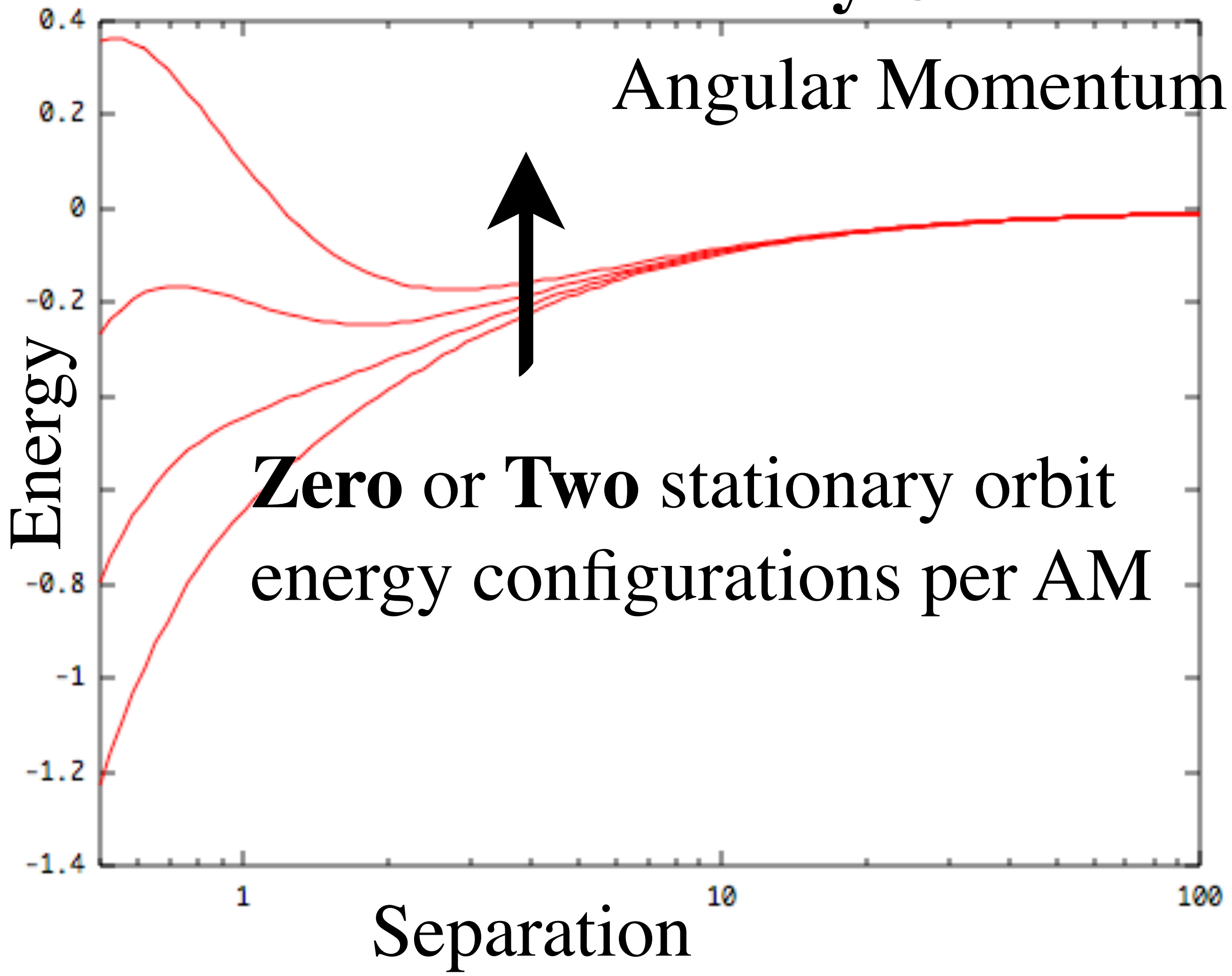


Point Mass Case



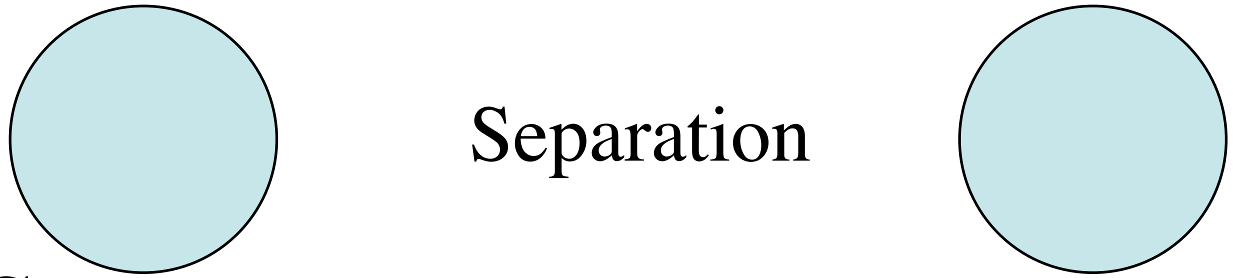
$$E_m = \frac{h^2}{2d^2} - \frac{1}{d}$$

Finite Density Case

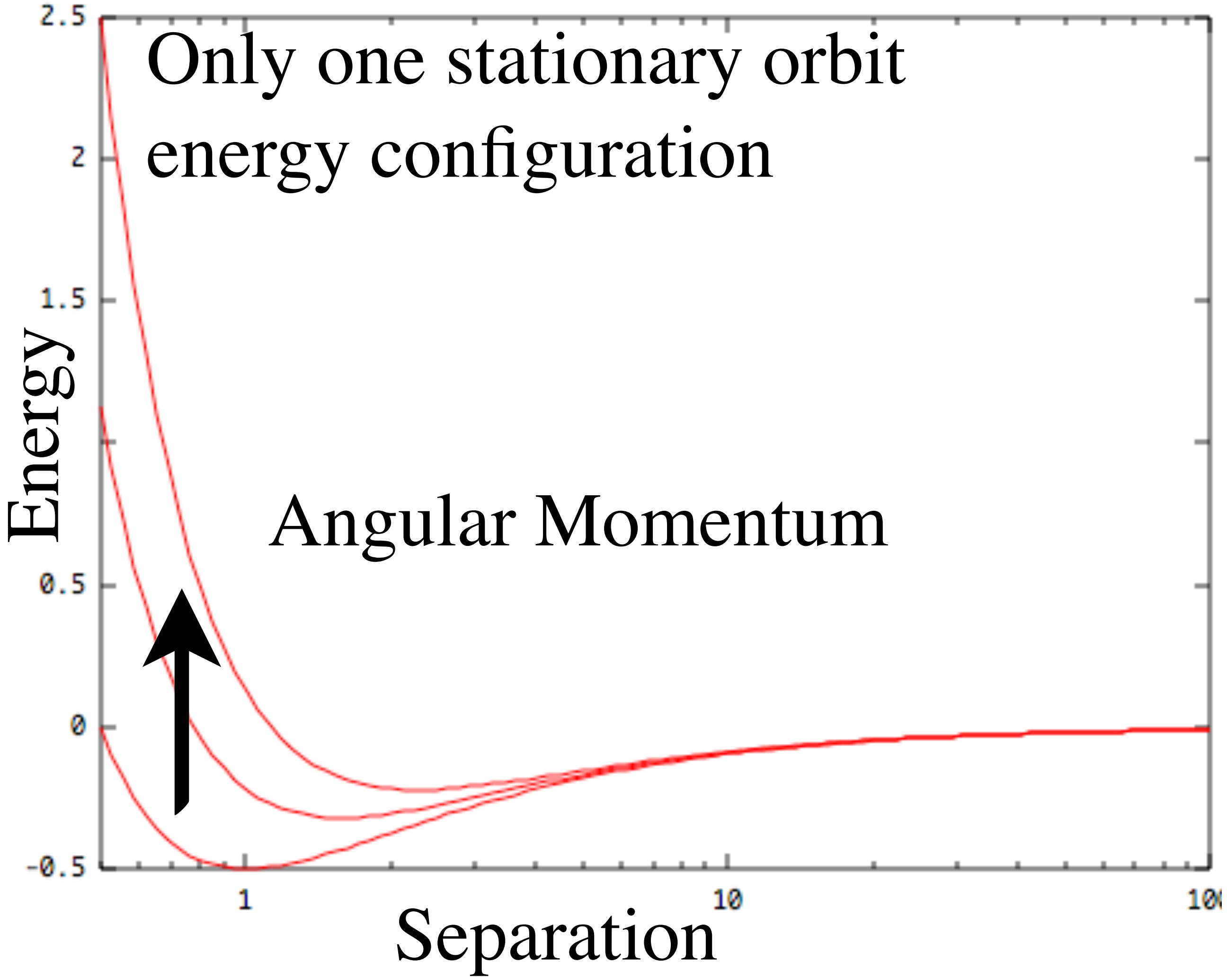


$$\mathcal{E}_m = \frac{h^2}{2(0.4 + d^2)} - \frac{1}{d}$$

2-Body Problem



Point Mass Case



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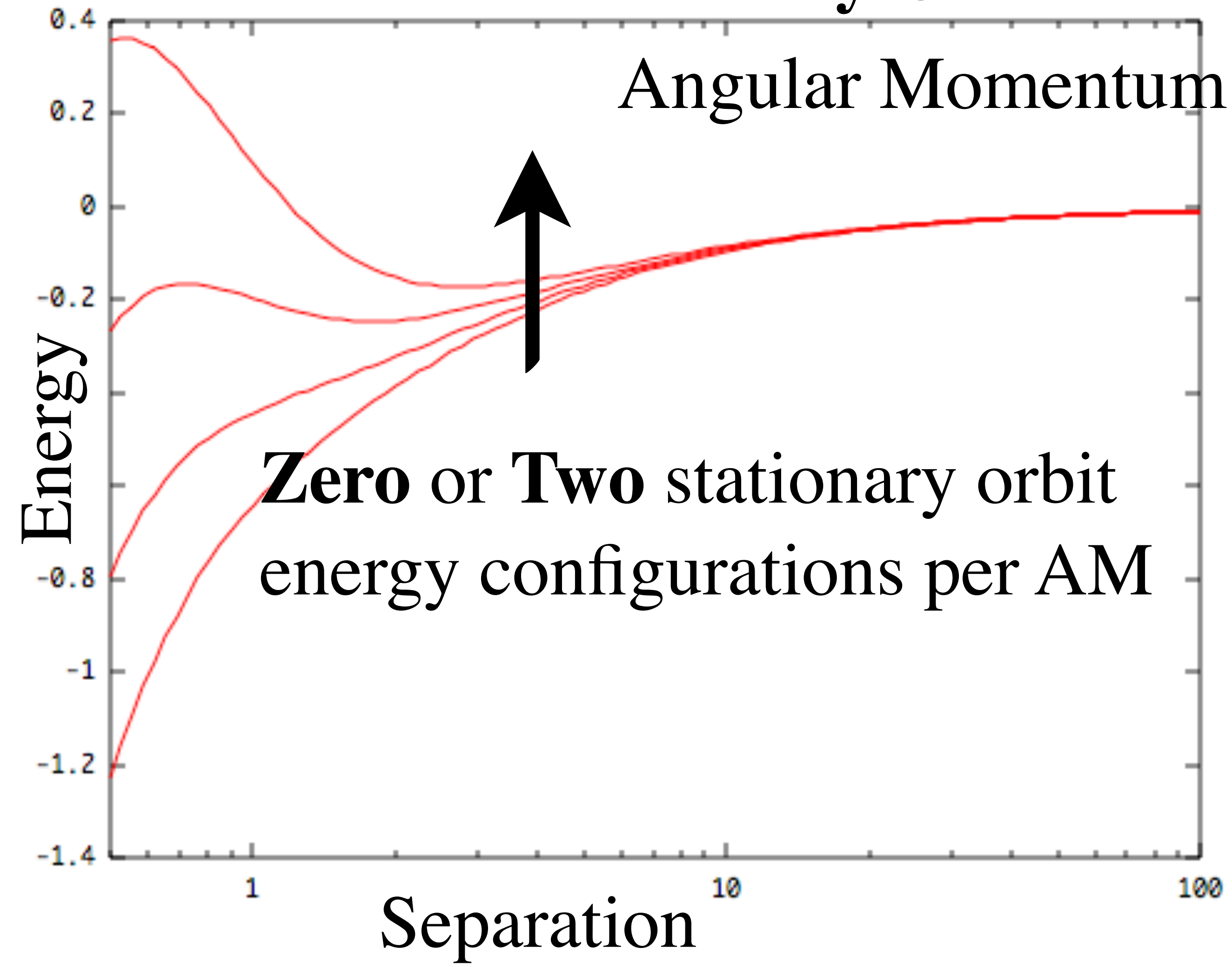
Finite Density Case



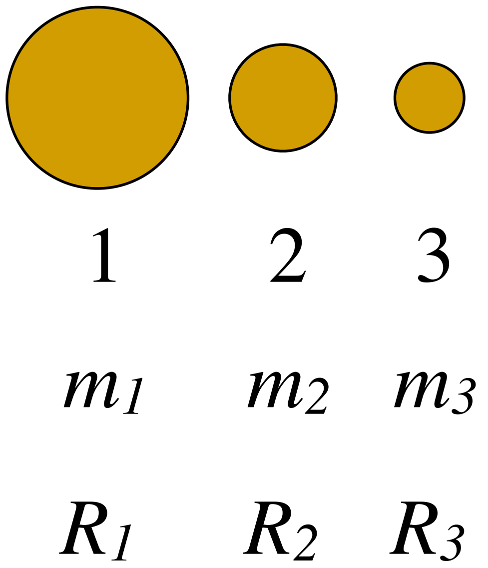
$$\mathcal{E}_m = \frac{h^2}{2(0.4 + d^2)} - \frac{1}{d}$$

Spherical Full 3-Body Problem

- Potential Energy and Total Moment of Inertia (assumed planar)

$$I_H = \frac{2}{5}\left[m_1 R_1^2 + m_2 R_2^2 + m_3 R_3^2\right] + \frac{\left[m_1 m_2 d_{12}^2 + m_2 m_3 d_{23}^2 + m_3 m_1 d_{31}^2\right]}{m_1 + m_2 + m_3}$$

$$\mathcal{U} = -\left[\frac{m_1 m_2}{d_{12}} + \frac{m_2 m_3}{d_{23}} + \frac{m_3 m_1}{d_{31}}\right]$$



$$\mathcal{E} = \frac{H^2}{2I_H} + \mathcal{U}$$

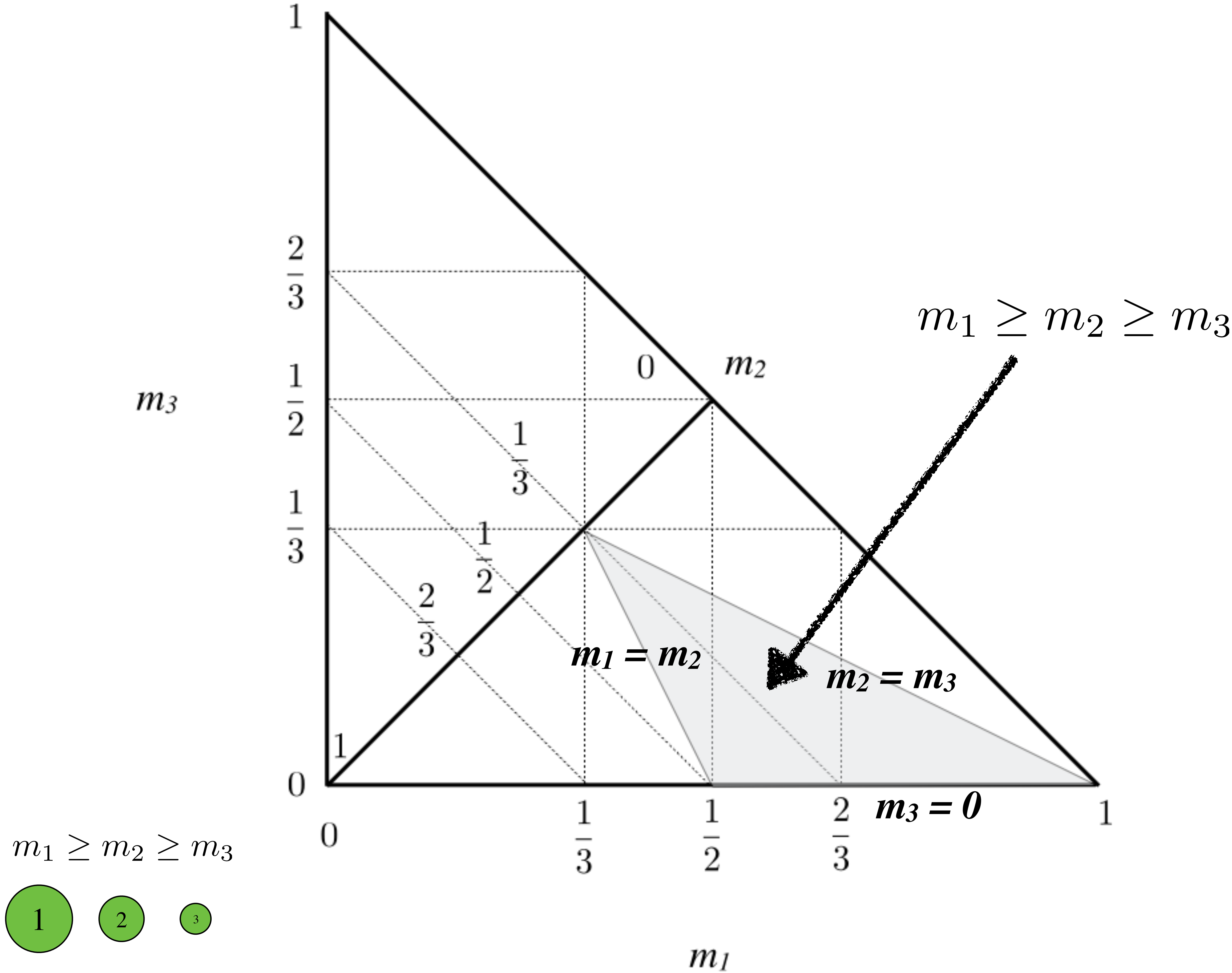
$$d_{ij} \geq R_i + R_j$$

- Normalization:

$$R_1 + R_2 + R_3 = 1$$
$$m_1 + m_2 + m_3 = 1$$

$$m_i = \frac{R_i^3}{R_1^3 + R_2^3 + R_3^3}$$

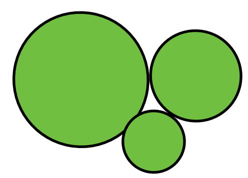
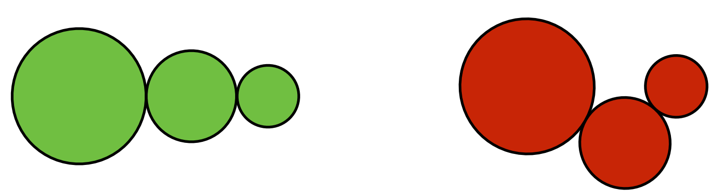
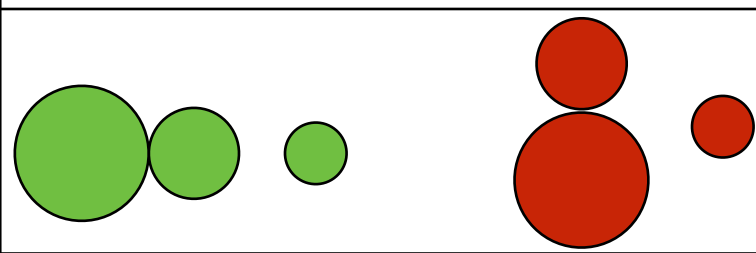
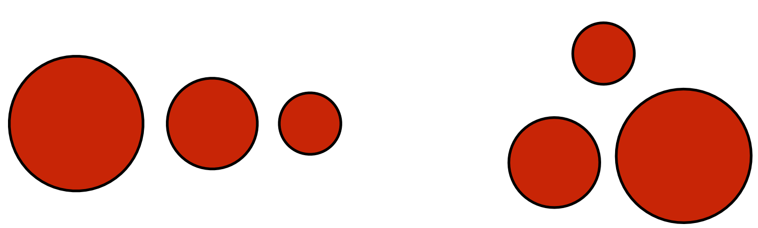
$$R_i = \frac{m_i^{1/3}}{m_1^{1/3} + m_2^{1/3} + m_3^{1/3}}$$



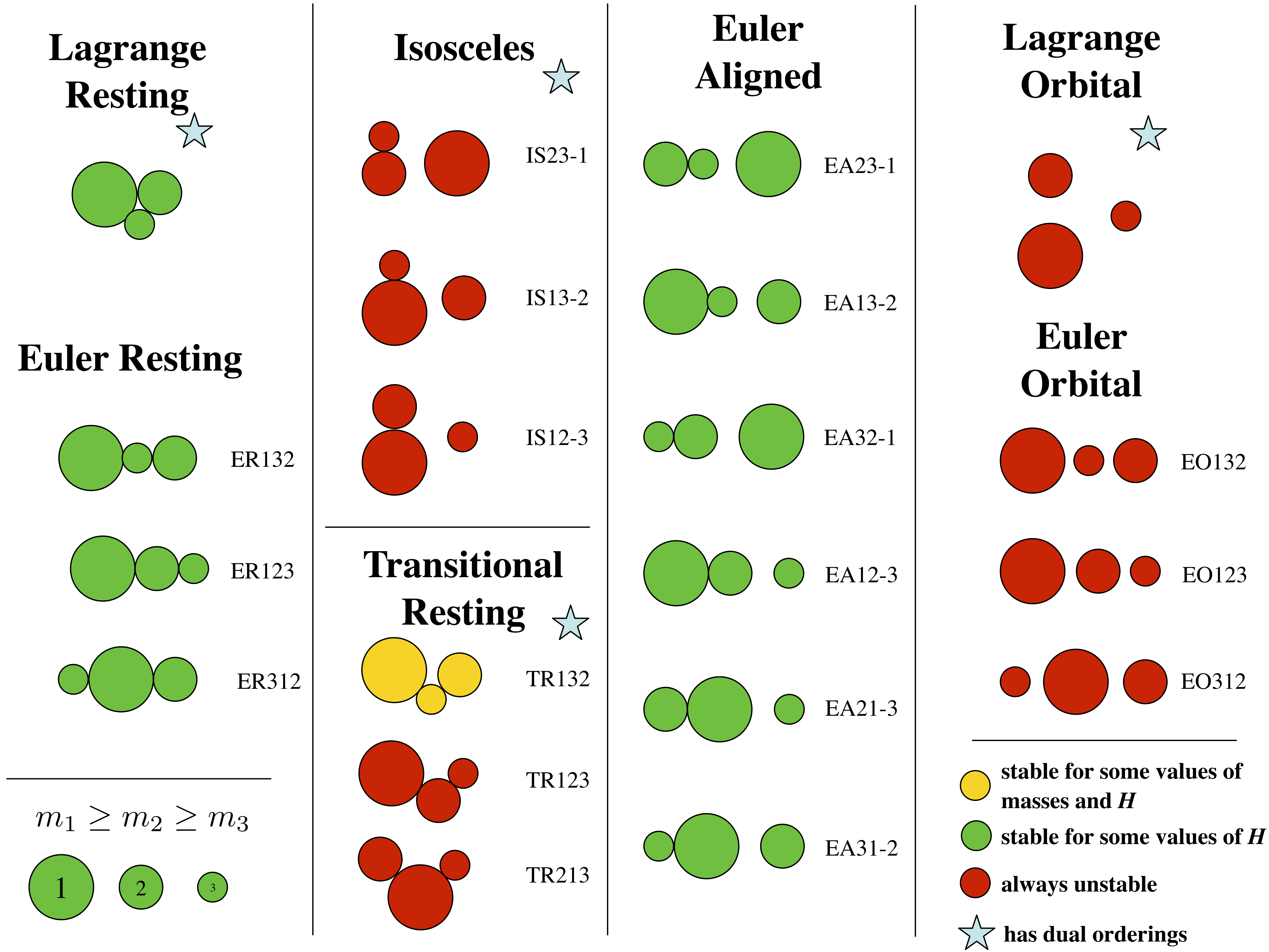
$m_1 \geq m_2 \geq m_3$
m_2
m_3
$m_1 = m_2$
$m_2 = m_3$
$m_3 = 0$
m_1
$m_1 \geq m_2 \geq m_3$
1
2
3

Relative Equilibria in the Full 3BP

- Addition of finite density leads to 23 additional (planar) equilibria in the 3BP
- Relative equilibria as a function of unconstrained ***degrees of freedom*** *(DOF)*

DOF	Configurations	Names	Numbers
0		Lagrange Resting	2
1		Euler Resting Transitional R.	3 6
2		Euler Aligned Isosceles	6 6
3		Euler Lagrange	3 2

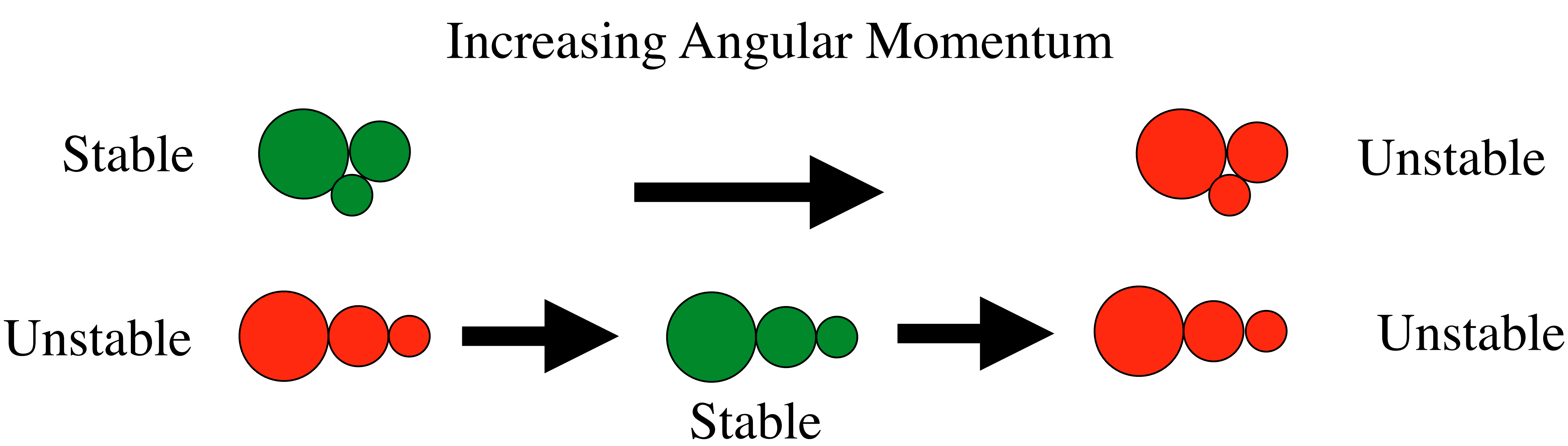
= 28



Lagrange Resting
Euler Resting
ER132
ER123
ER312
$m_1 \geq m_2 \geq m_3$
1
2
3
Isosceles
IS23-1
IS13-2
IS12-3
Transitional Resting
TR132
TR123
TR213
Euler Aligned
EA23-1
EA13-2
EA32-1
EA12-3
EA21-3
EA31-2
Lagrange Orbital
Euler Orbital
EO132
EO123
EO312
stable for some values of masses and *H*
stable for some values of *H*
always unstable
has dual orderings

Relationships Between Equilibria

- The equilibria are connected to each other through bifurcation pathways as *H* is varied — none are isolated
  - These bifurcation pathways can also represent stability transitions in “rubble pile asteroids” as their angular momentum increases
  - Especially significant are transitions from stable relative equilibria to unstable relative equilibria, as additional stable relative equilibria will exist at a lower energy, leading to an abrupt on-sent of dynamical evolution



Relationships Between Equilibria

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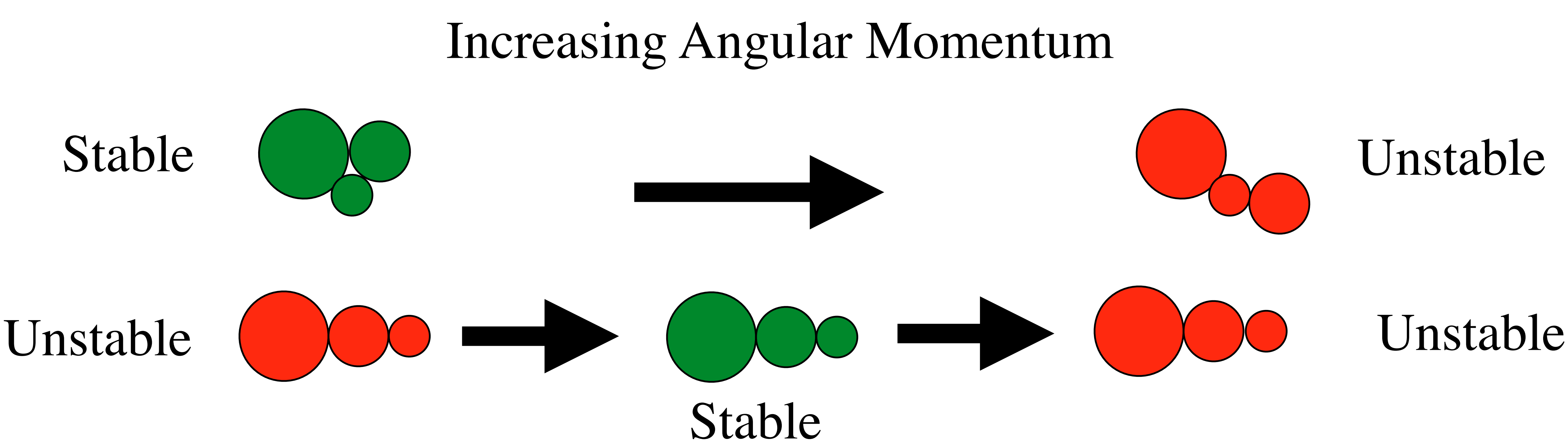

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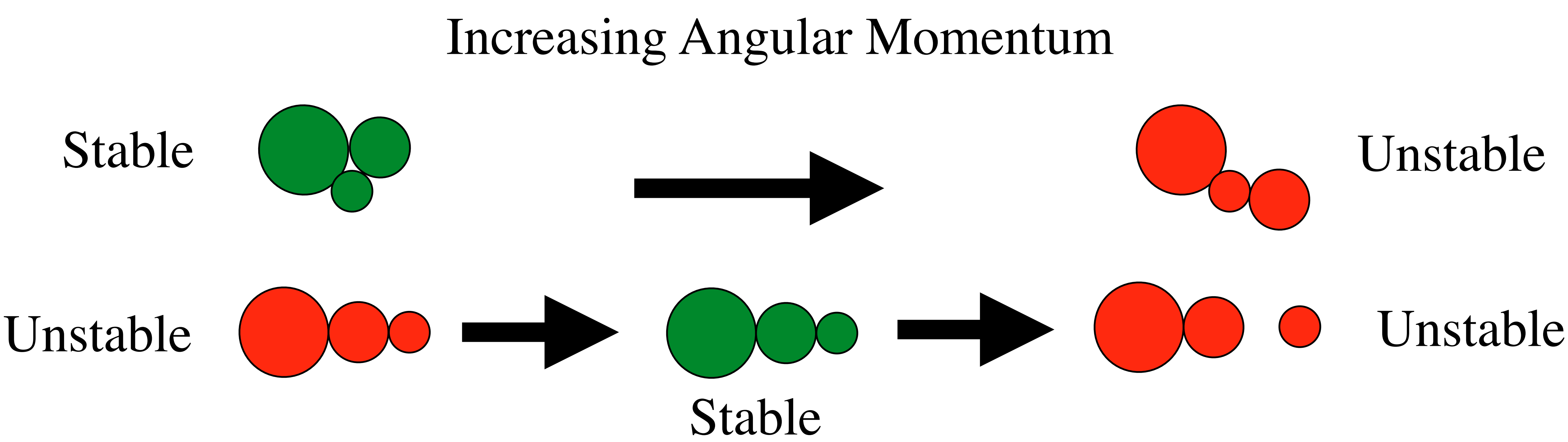
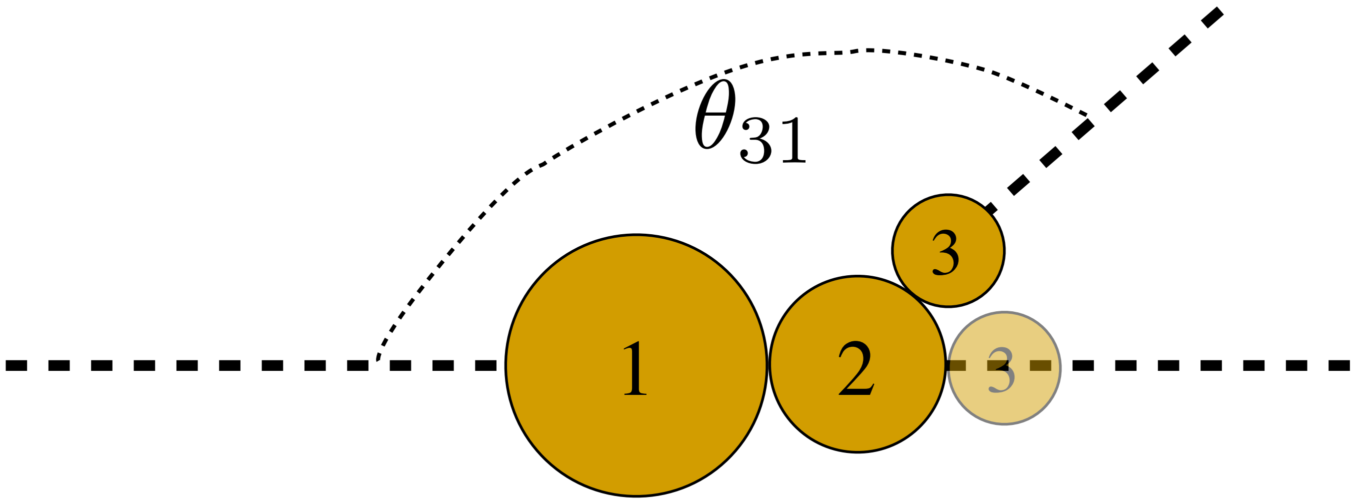


Stability Transitions

- Euler Resting transition from unstable to stable:

$$d_{12} = R_1 + R_2$$

$$d_{23} = R_2 + R_3$$



$$d_{31}^2 = d_{12}^2 + d_{23}^2 - 2d_{12}d_{23}\cos\theta_{31}$$

$$\delta\mathcal{E} = -\frac{H^2}{2I_H^2}\delta I_H + \delta U = 0$$

$$\delta I_H = 2m_1 m_3 d_{12} d_{23} \sin\theta_{31}\delta\theta_{31}$$

$$\delta U = \frac{m_1 m_3}{d_{31}^3} d_{12} d_{23} \sin\theta_{31}\delta\theta_{31}$$

Resting Euler Equilibria become ***stable*** when:

$$H^2 \geq \frac{I_H^2}{(R_1 + 2R_2 + R_3)^3}$$

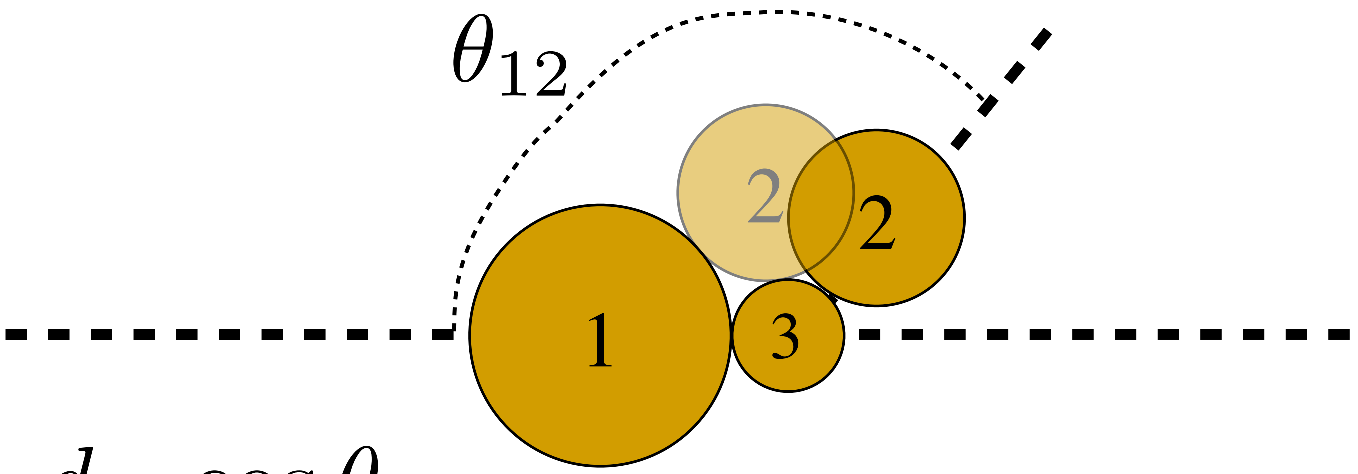
Stability Transitions

- Lagrange configuration transition from stable to unstable:

$$d_{31} = R_3 + R_1$$

$$d_{23} = R_2 + R_3$$

$$d_{12}^2 = d_{23}^2 + d_{31}^2 - 2 d_{23} d_{31} \cos\theta_{12}$$



$$\delta \mathcal{E} = -\frac{H^2}{2 I_H^2} \delta I_H + \delta \mathcal{U} = 0$$

$$\delta I_H = 2 m_1 m_2 d_{23} d_{31} \sin\theta_{12} \delta\theta_{12}$$

$$\delta \mathcal{U} = \frac{m_1 m_2}{d_{12}^3} d_{23} d_{31} \sin\theta_{12} \delta\theta_{12}$$

Resting Lagrange Equilibria become ***unstable*** when:

$$H^2 \geq \frac{I_H^2}{(R_1 + R_2)^3}$$

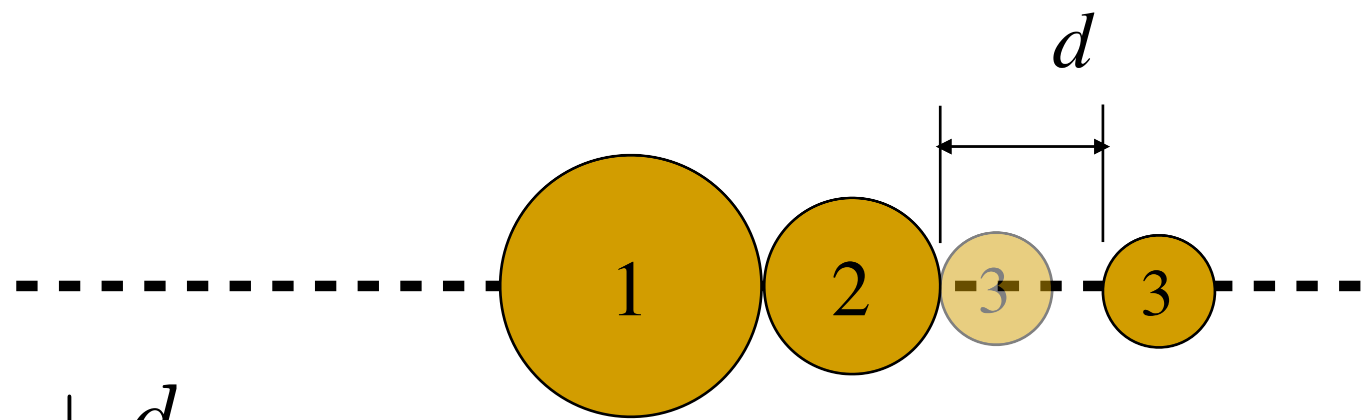
Stability Transitions

- Euler configurations transition from stable to unstable:

$$d_{12} = R_1 + R_2$$

$$d_{23} = R_2 + R_3 + d$$

$$d_{31} = R_1 + 2R_2 + R_3 + d$$

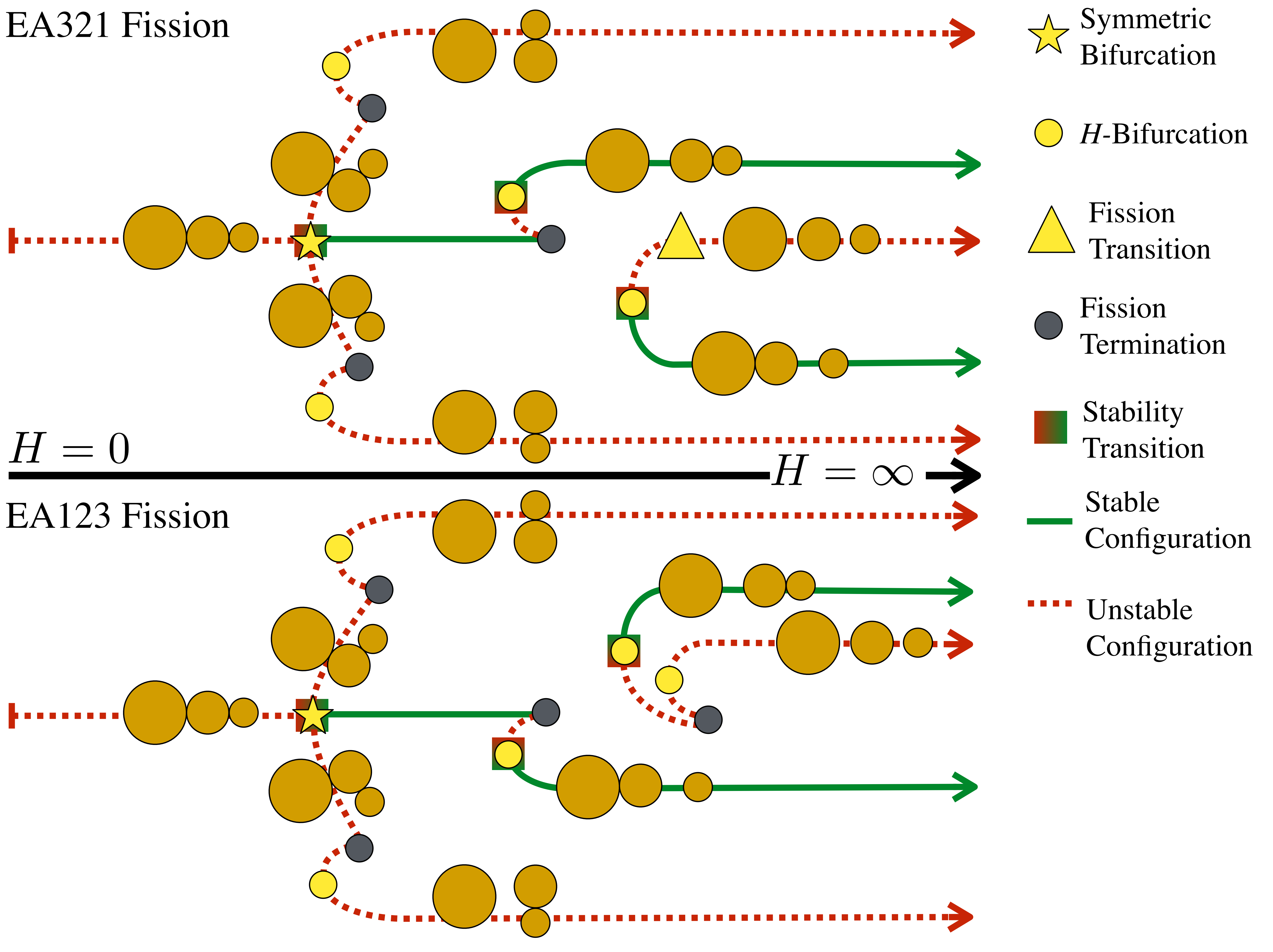


$$\delta I_H = 2\left[m_1 m_3 (d_{12} + d_{23} + d) + m_2 m_3 (d_{23} + d)\right]\delta d$$

$$\delta U = \left[\frac{m_1 m_3}{d_{31}^3}(d_{12} + d_{23} + d) + \frac{m_2 m_3}{d_{23}^3}(d_{23} + d)\right]\delta d$$

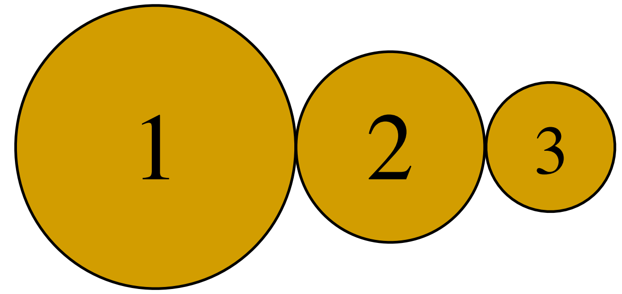
Resting Euler Equilibria become ***unstable*** when:

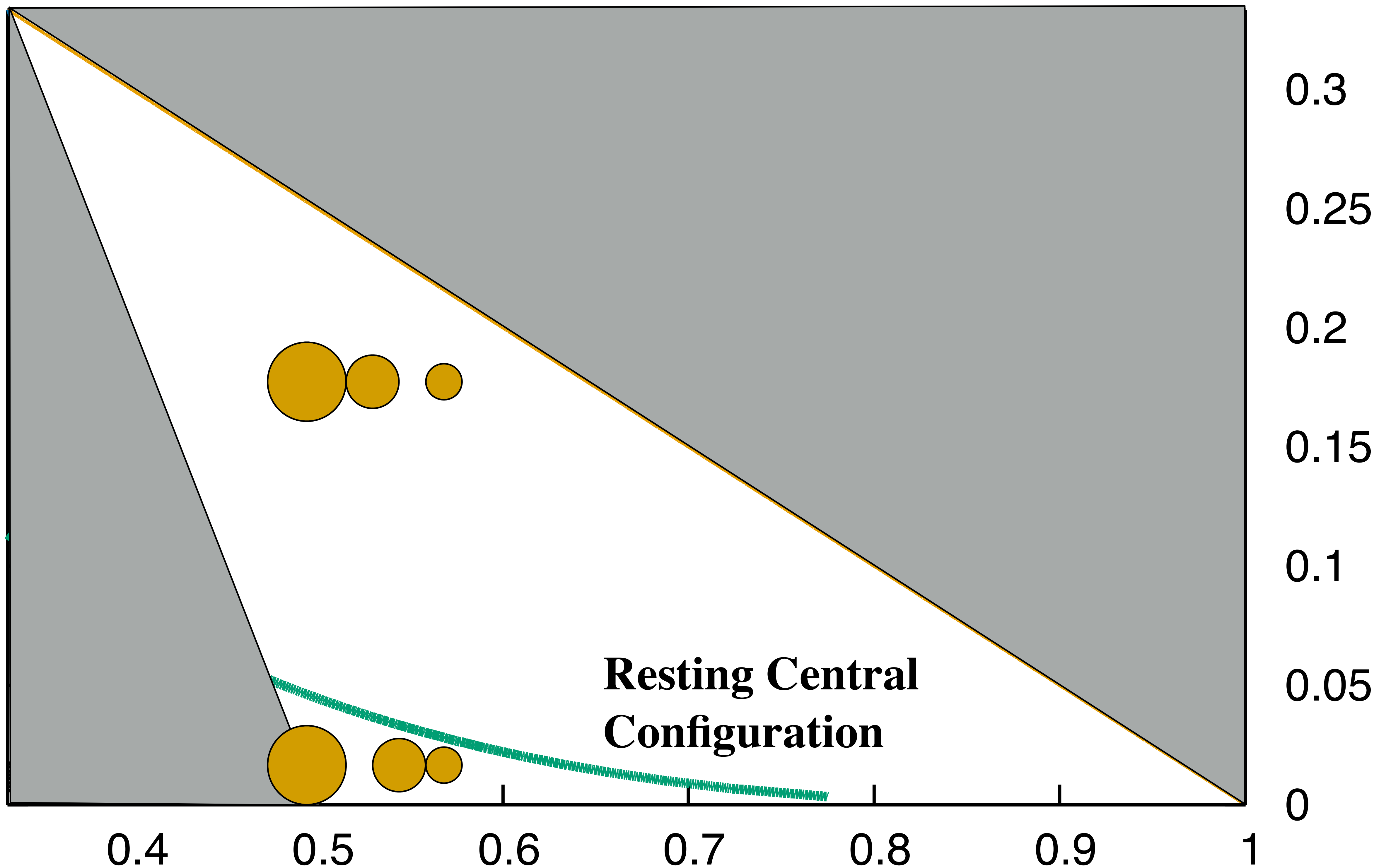
$$H^2 \geq I_H^2 \frac{\left[\frac{m_1}{(R_1+2R_2+R_3)^2} + \frac{m_2}{(R_2+R_3)^2}\right]}{m_1(R_1 + 2R_2 + R_3) + m_2(R_2 + R_3)}$$



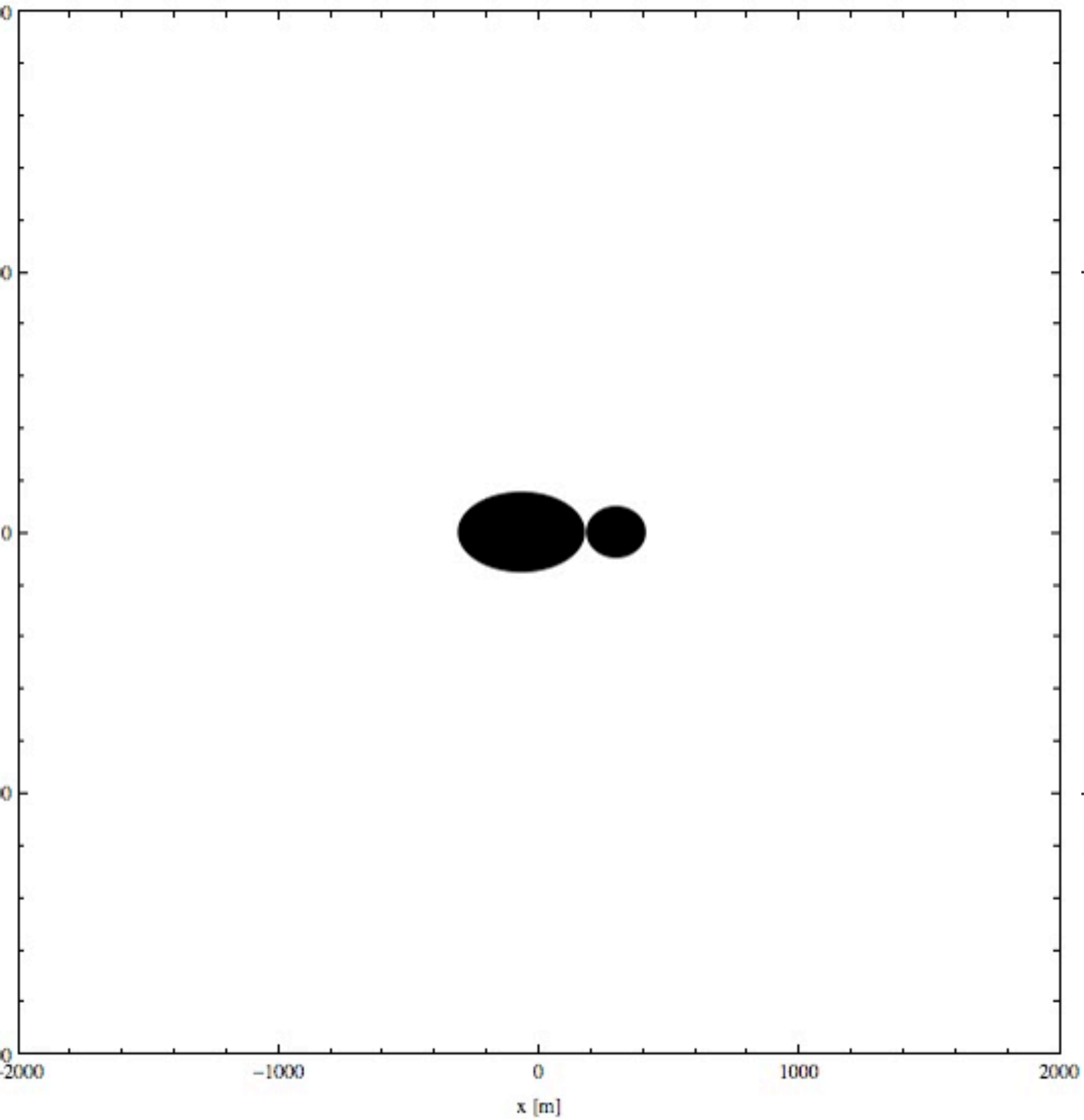
EA321 Fission
$H = 0$
$H = \infty$
EA123 Fission
Symmetric Bifurcation
H-Bifurcation
Fission Transition
Fission Termination
Stability Transition
Stable Configuration
Unstable Configuration

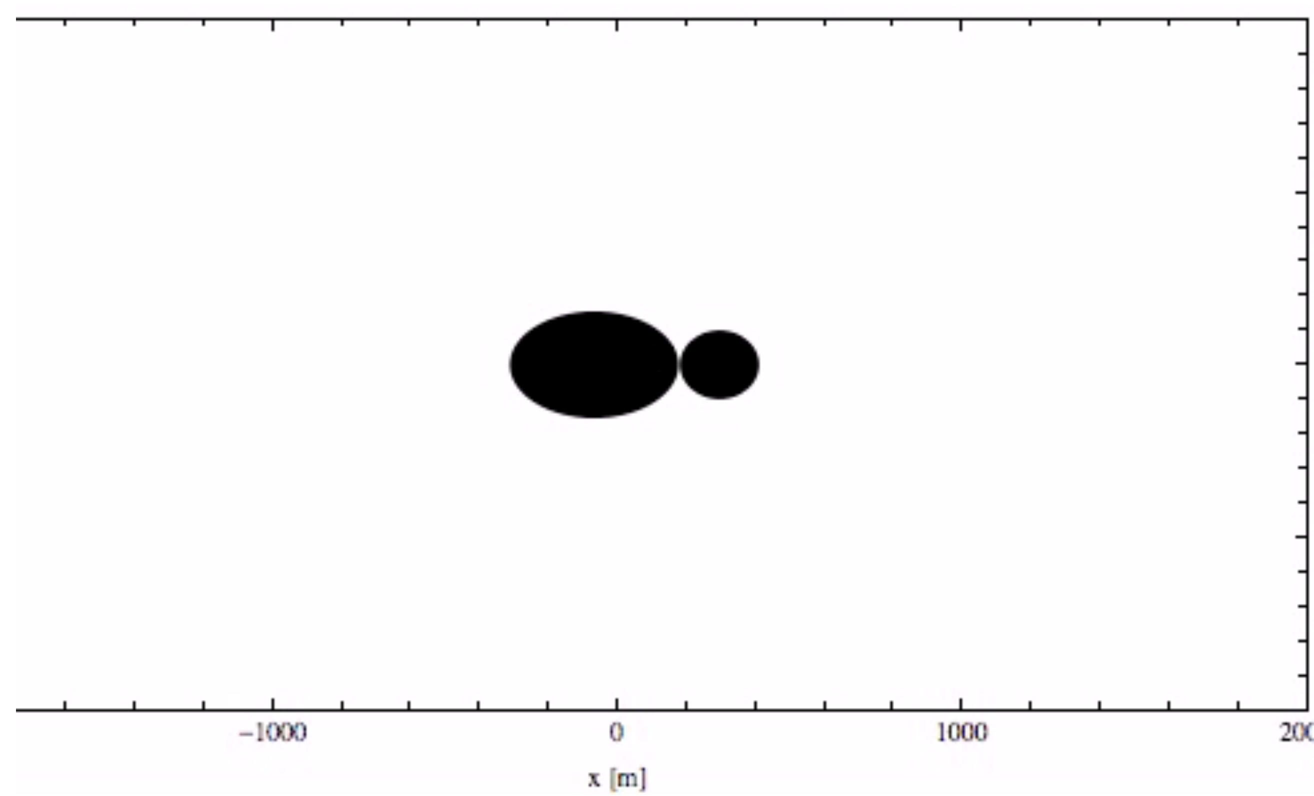
Euler Fission: EA123



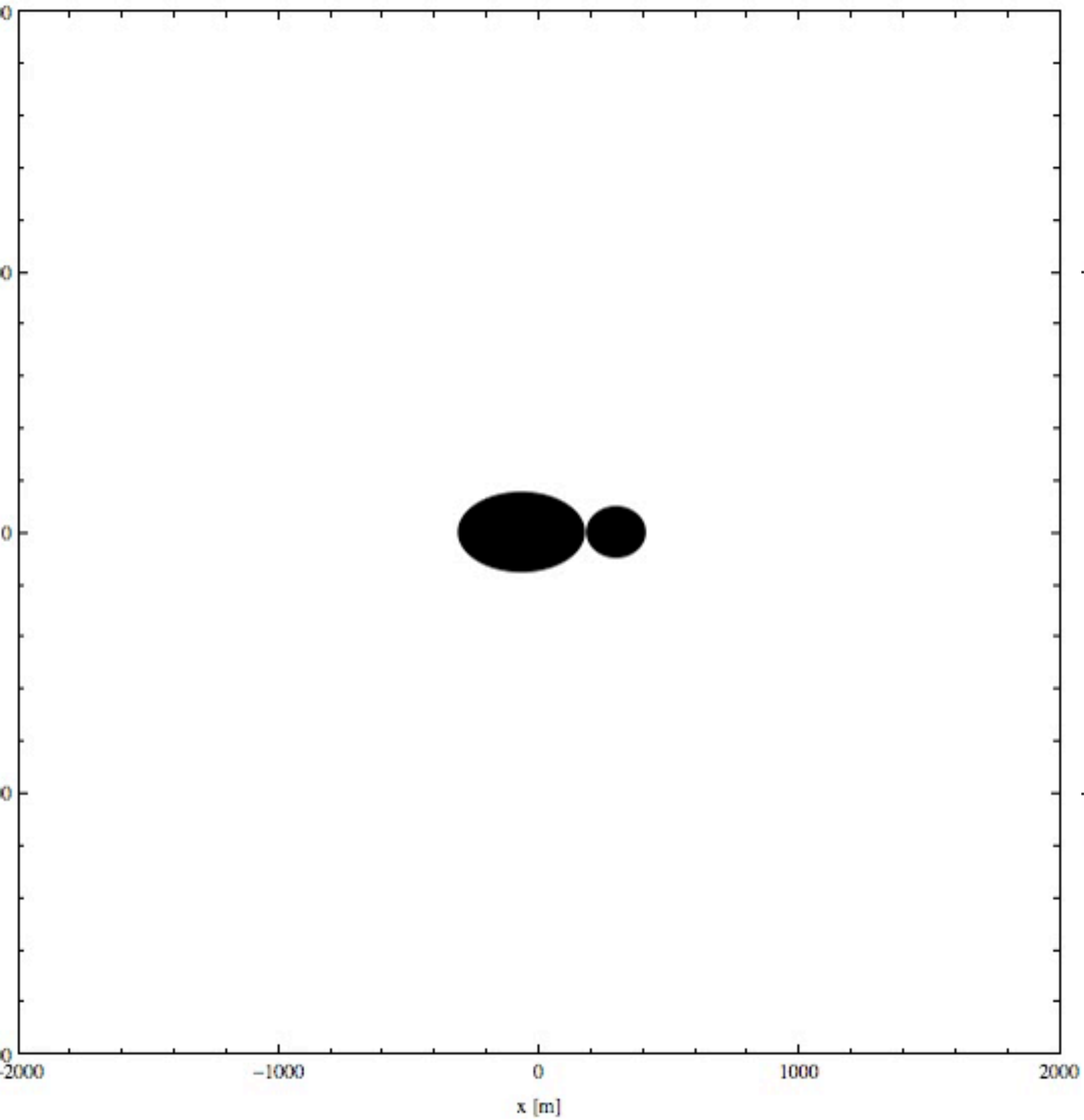


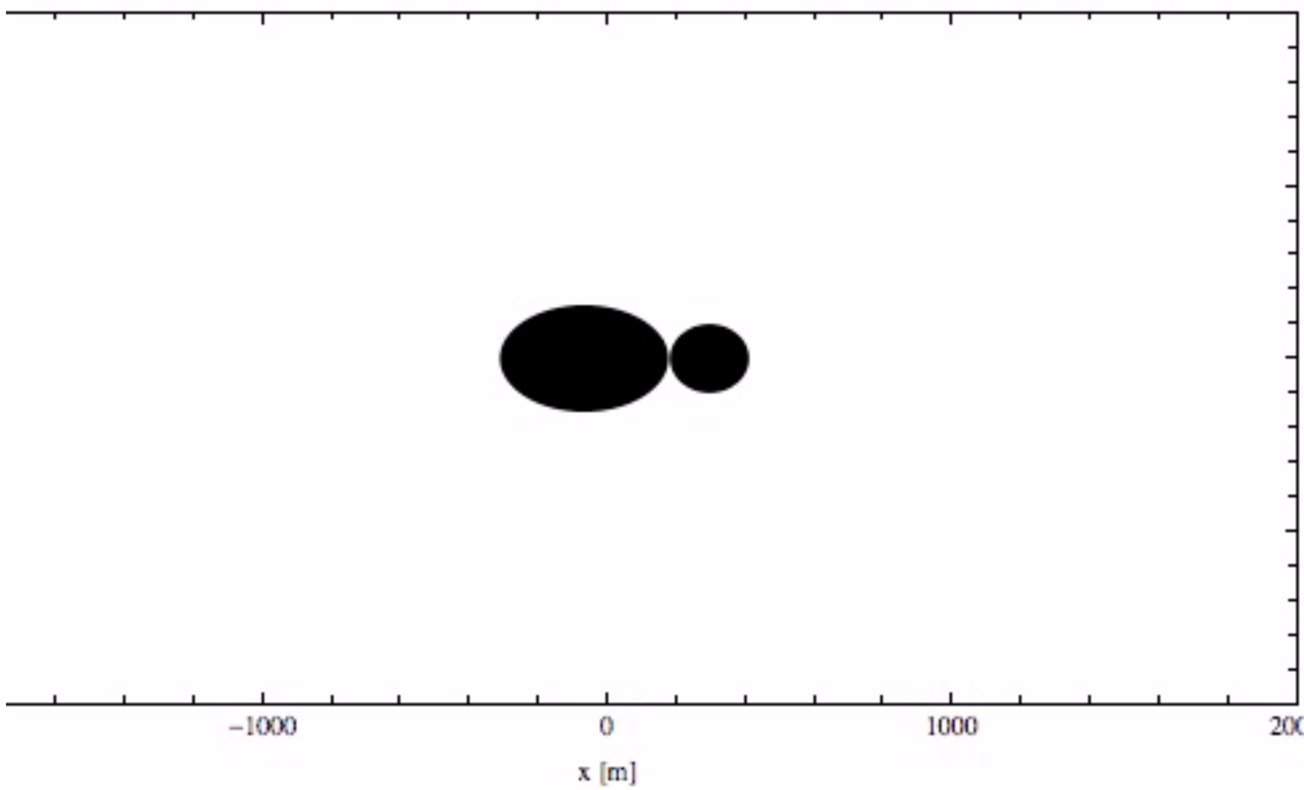
Dynamics post-fission



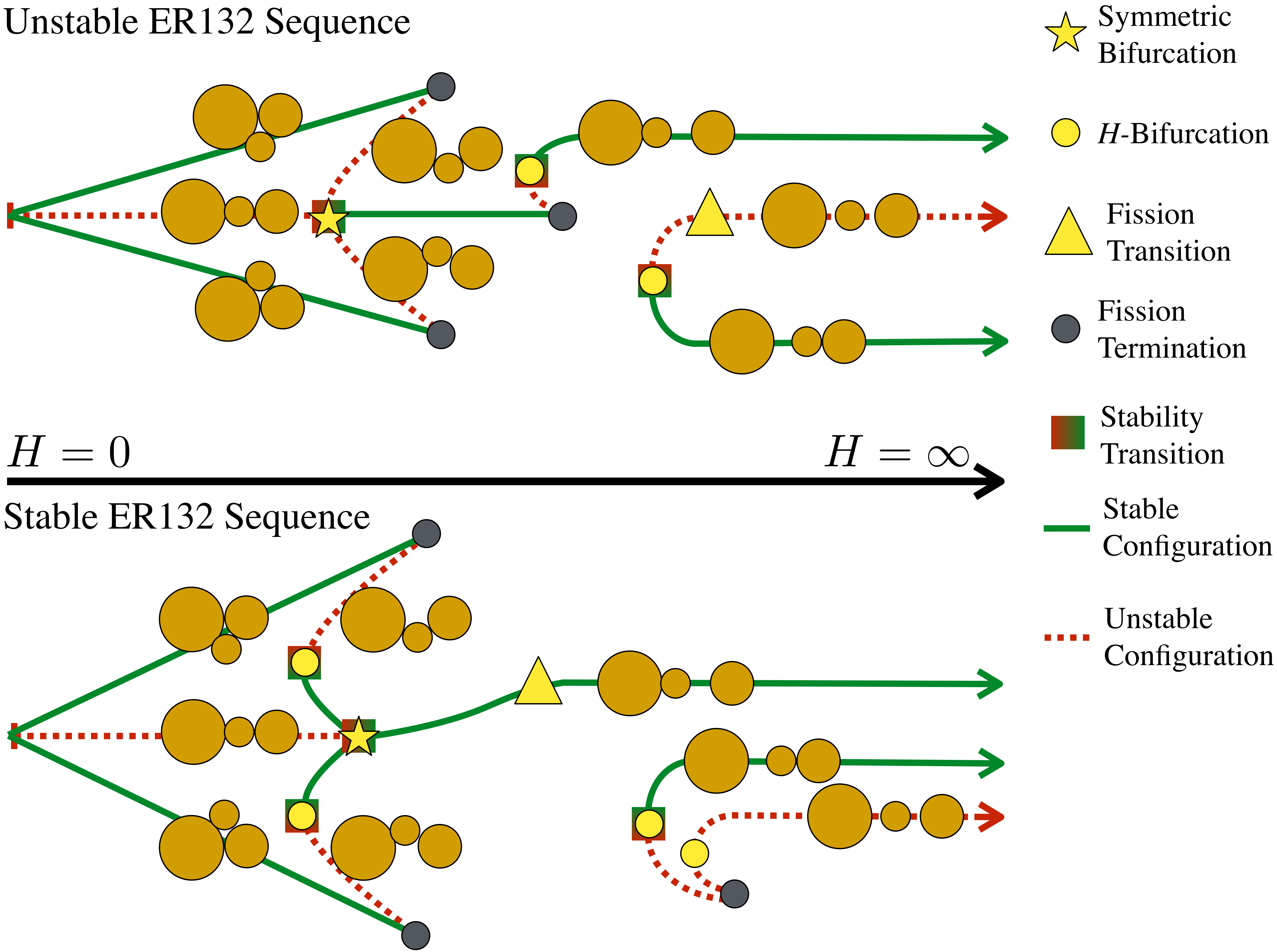


Dynamics post-fission

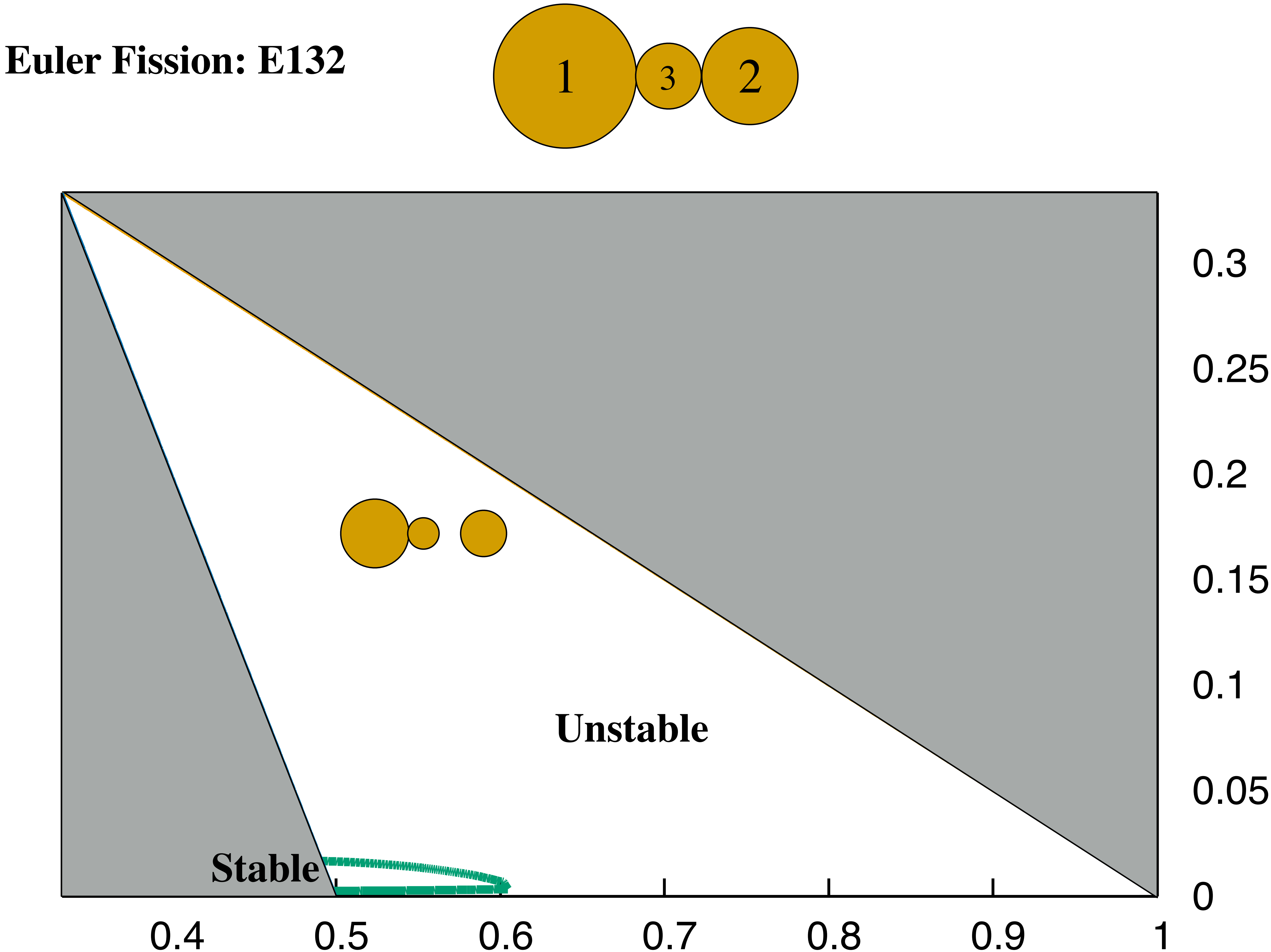




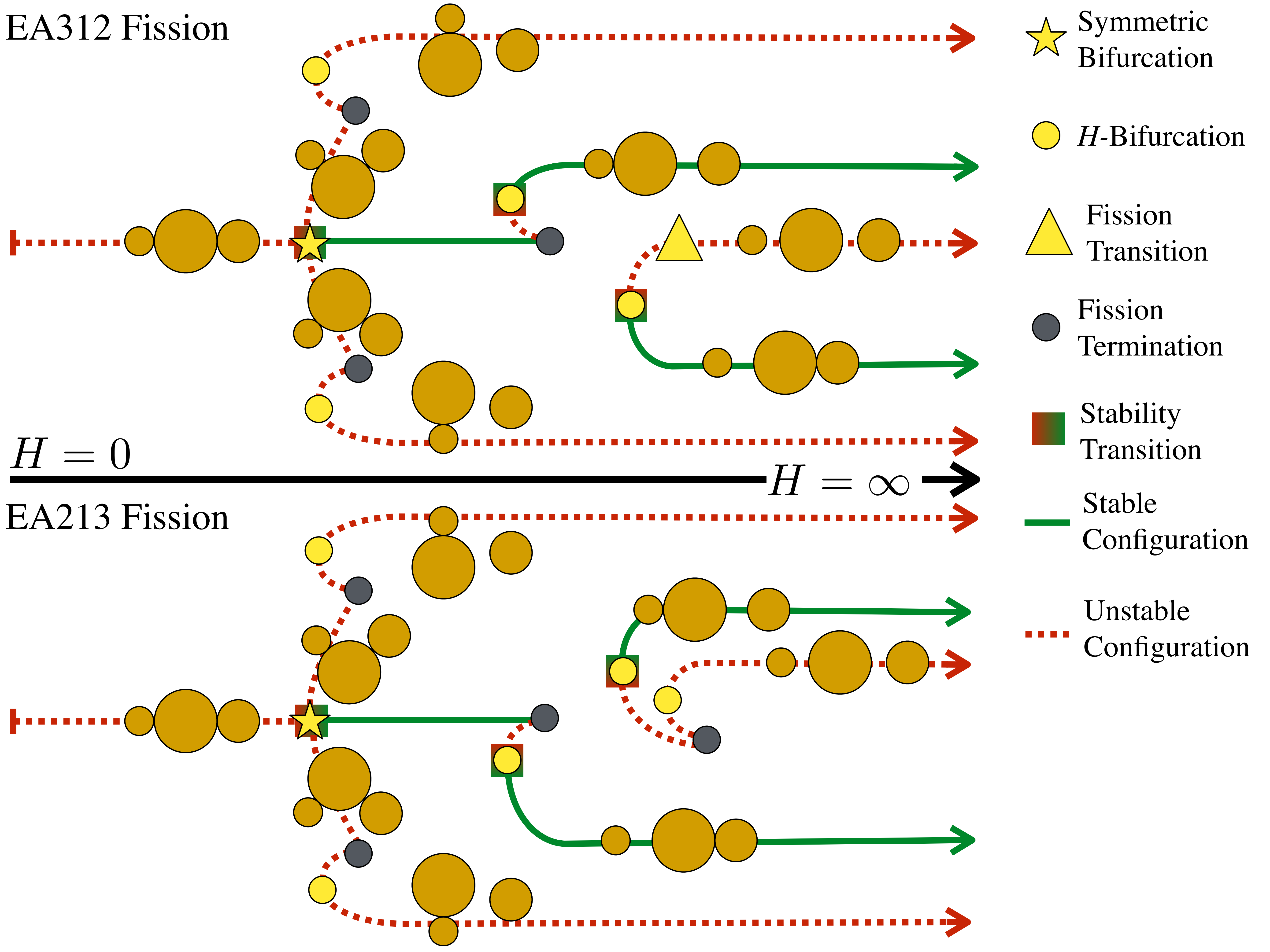
Movies by S.A. Jacobson



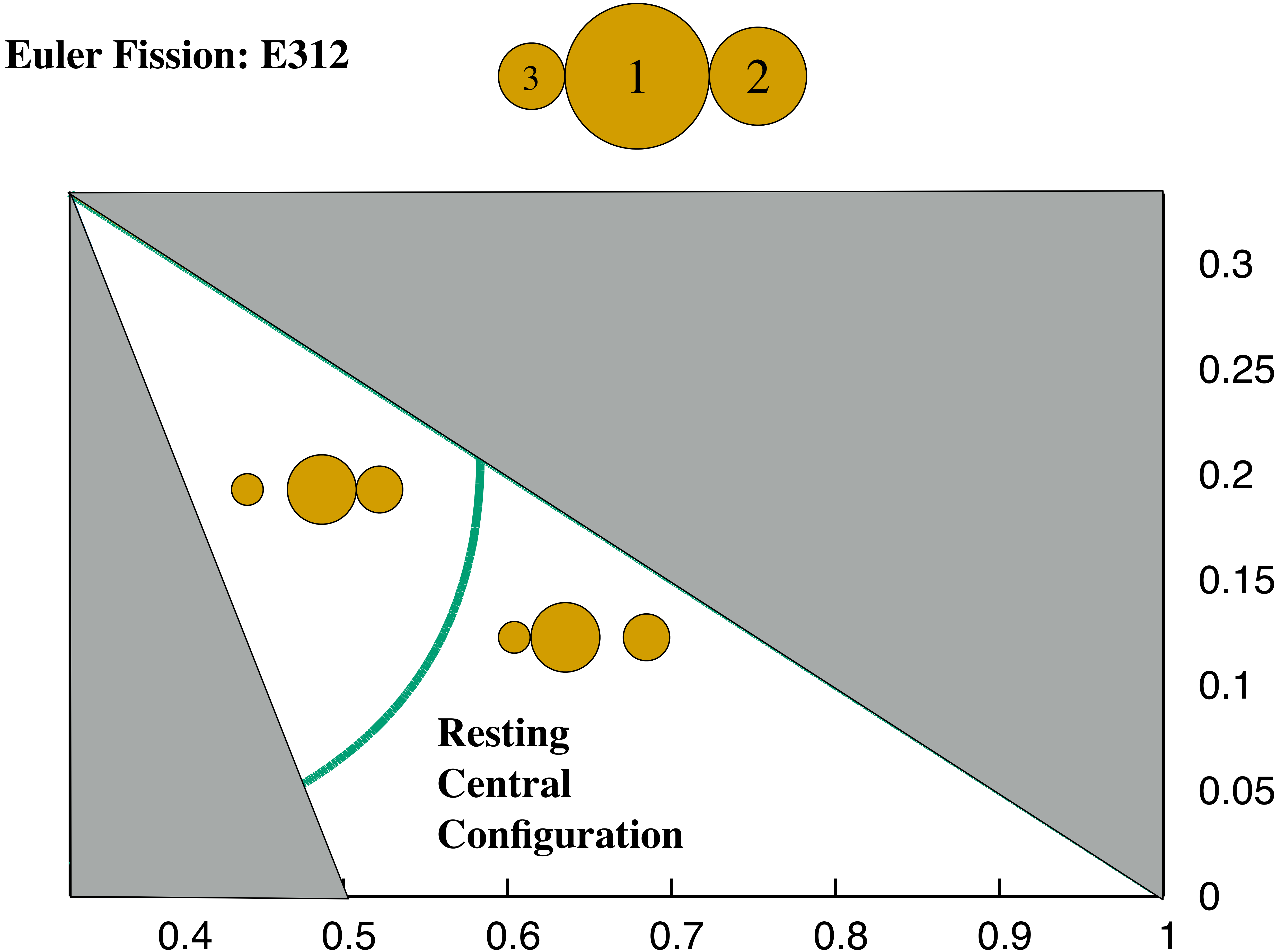
Unstable ER132 Sequence
$H = 0$
$H = \infty$
Stable ER132 Sequence
Symmetric Bifurcation
H-Bifurcation
Fission Transition
Fission Termination
Stability Transition
Stable Configuration
Unstable Configuration

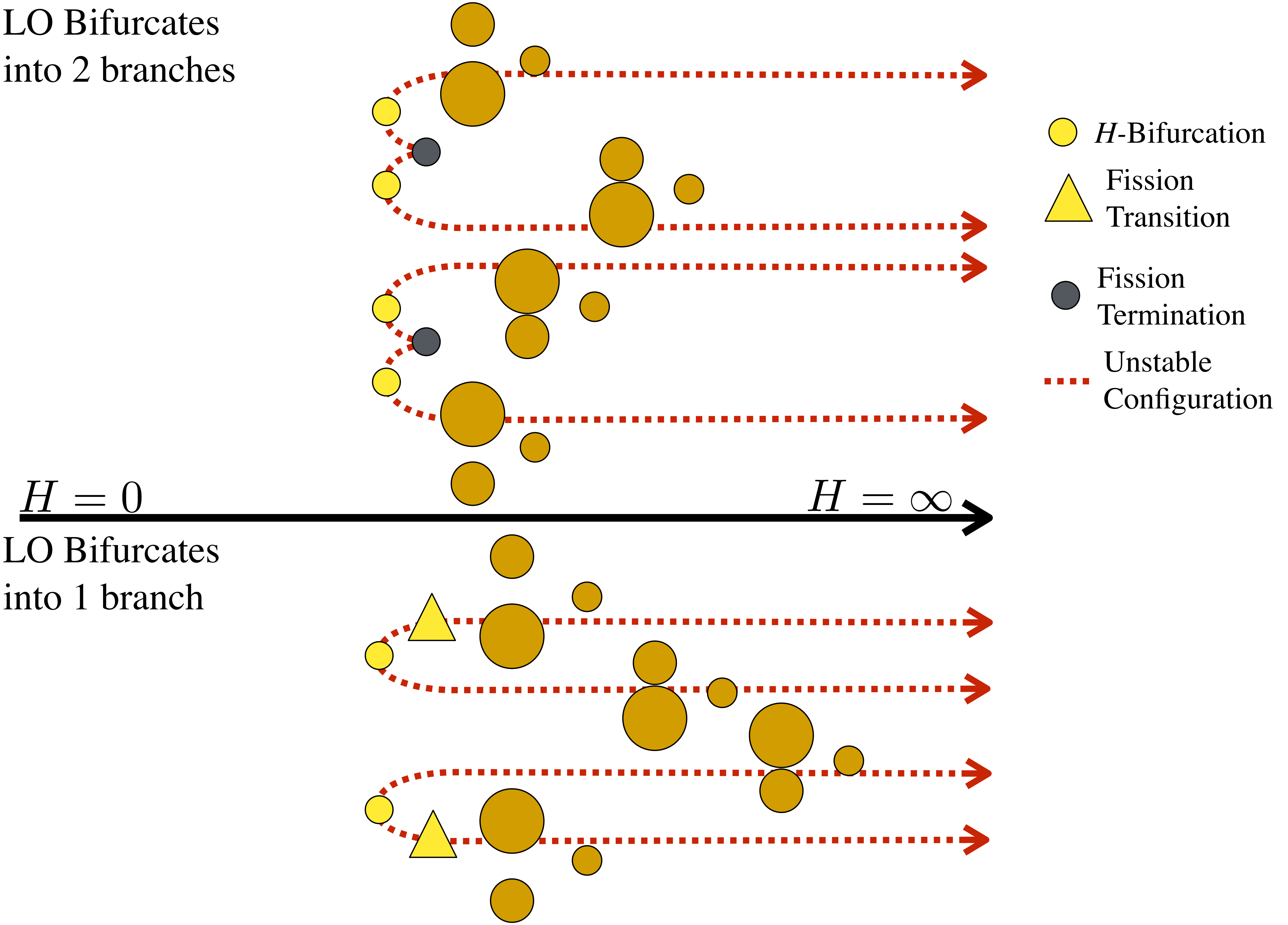


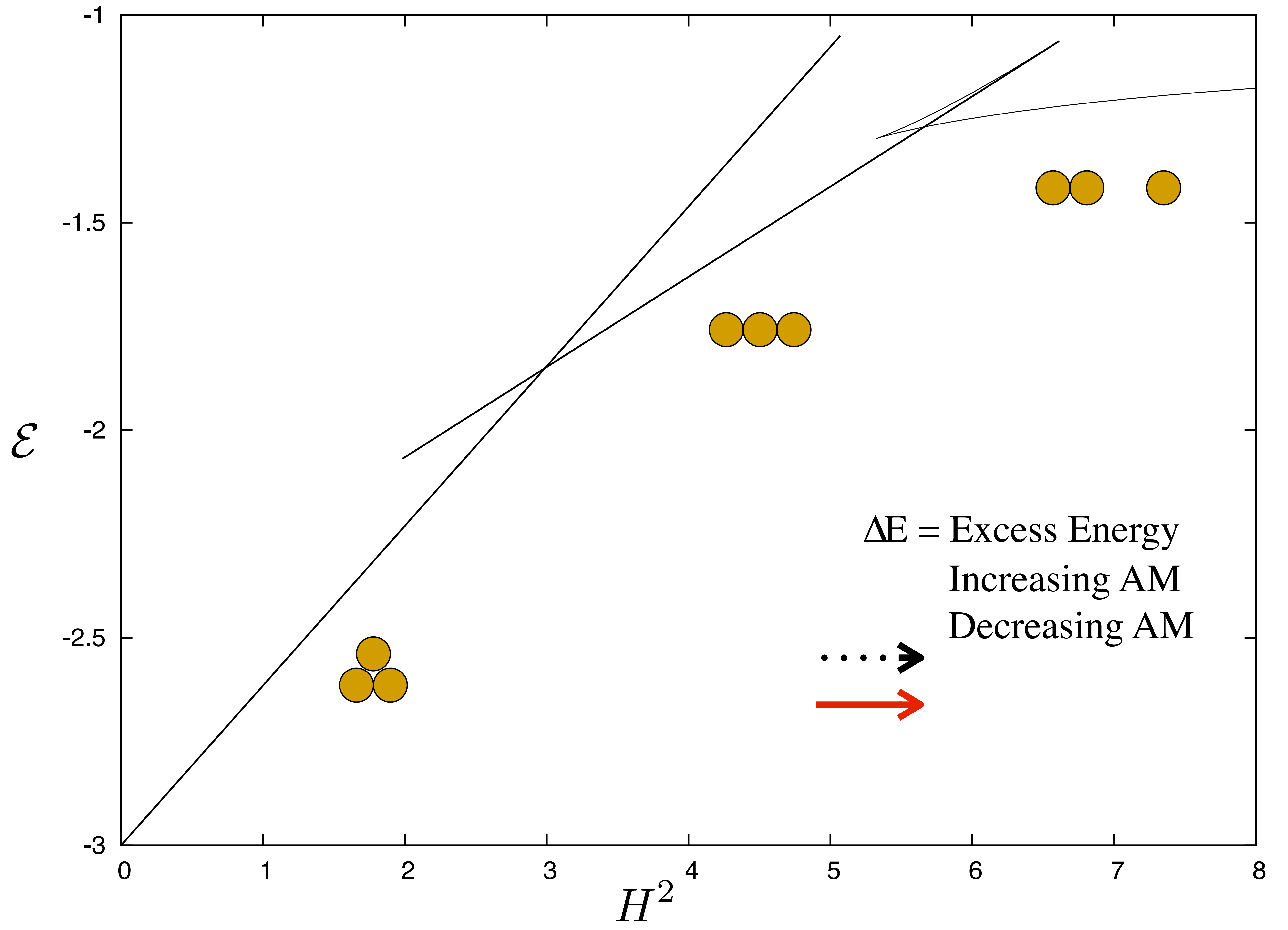
Euler Fission: E132
1
3
2
0.3
0.25
0.2
0.15
0.1
0.05
0
Unstable
Stable
0.4
0.5
0.6
0.7
0.8
0.9
1



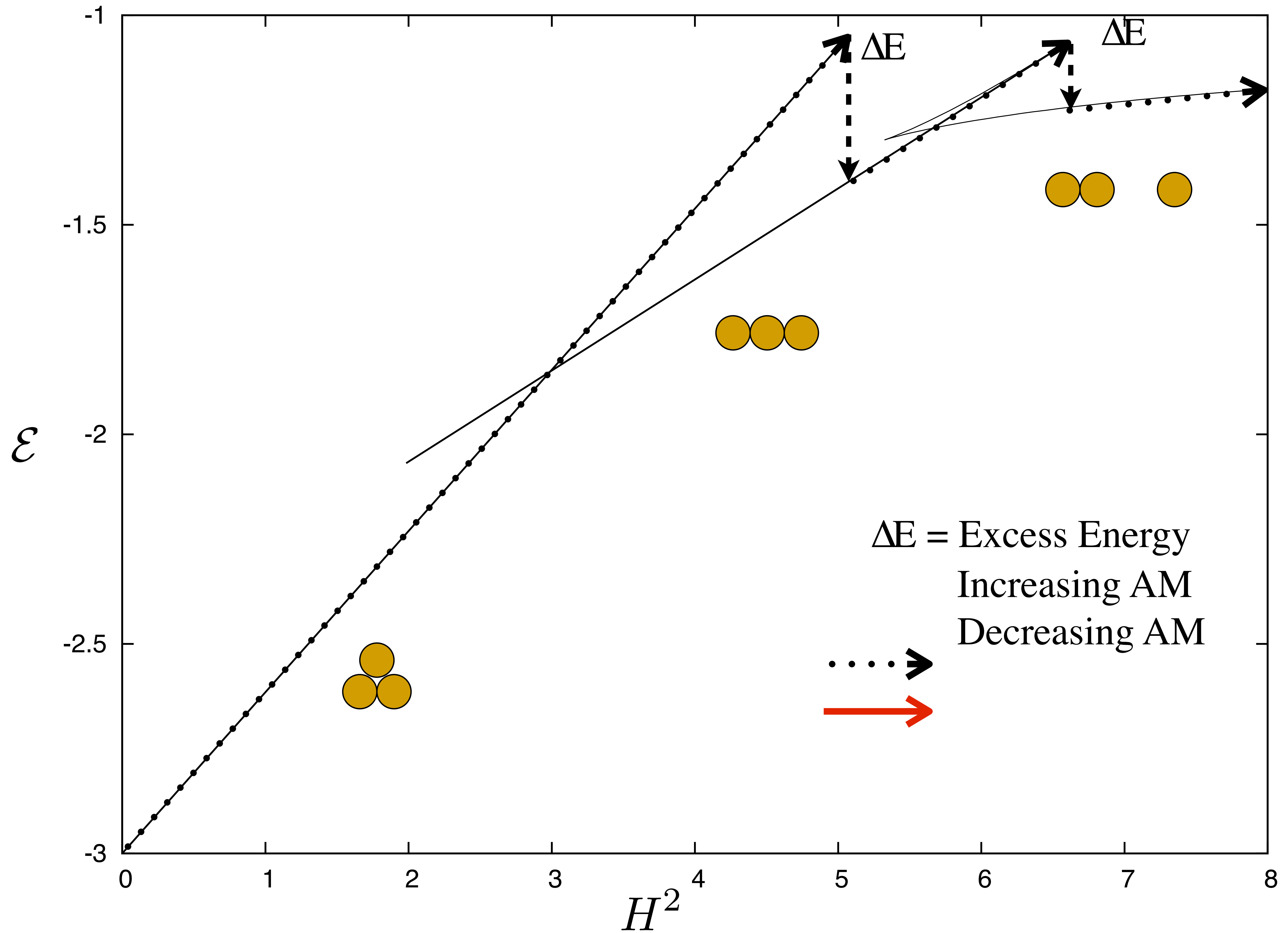
EA312 Fission
EA213 Fission
$H = 0$
$H = \infty$
Symmetric Bifurcation
H-Bifurcation
Fission Transition
Fission Termination
Stability Transition
Stable Configuration
Unstable Configuration


Euler Fission: E312
3
1
2
0.3
0.25
0.2
0.15
0.1
0.05
0
0.4
0.5
0.6
0.7
0.8
0.9
1
Resting
Central
Configuration

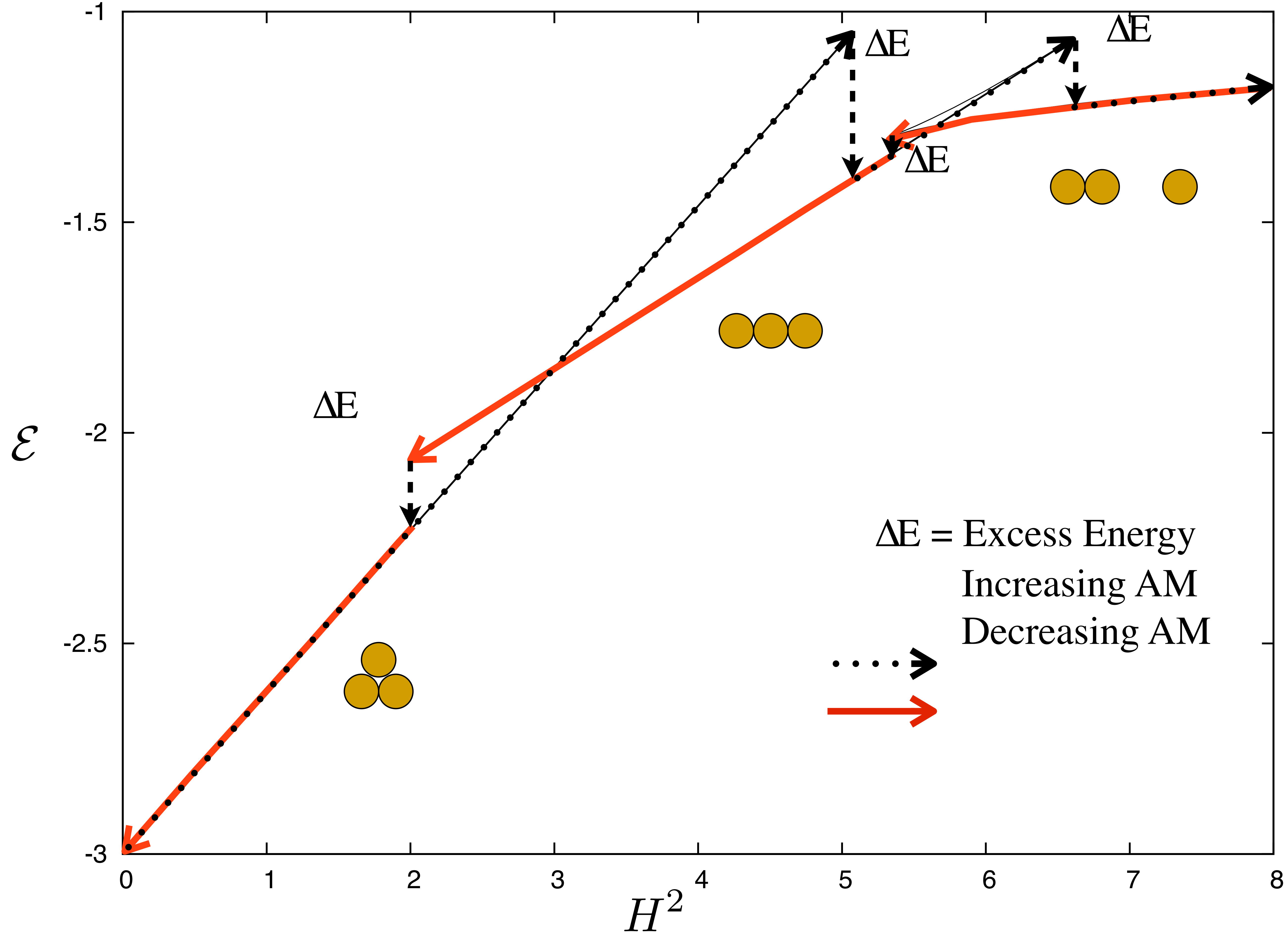

LO Bifurcates
into 2 branches
$H = 0$
$H = \infty$
LO Bifurcates
into 1 branch
H-Bifurcation
Fission
Transition
Fission
Termination
Unstable
Configuration



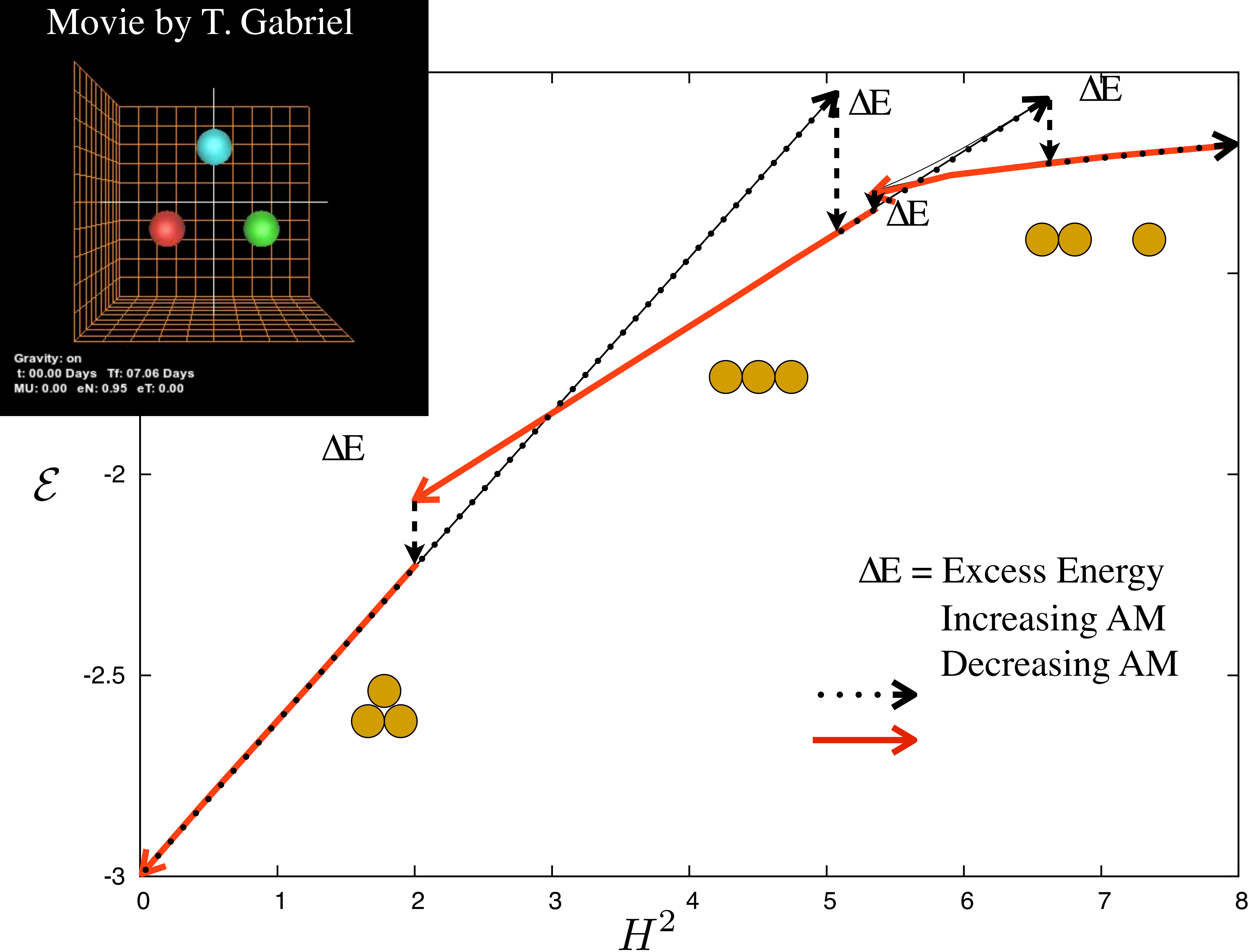
-1
-1.5
-2
-2.5
-3
0
1
2
3
4
5
6
7
8
$\mathcal{E}$
H^2
ΔE = Excess Energy
Increasing AM
Decreasing AM



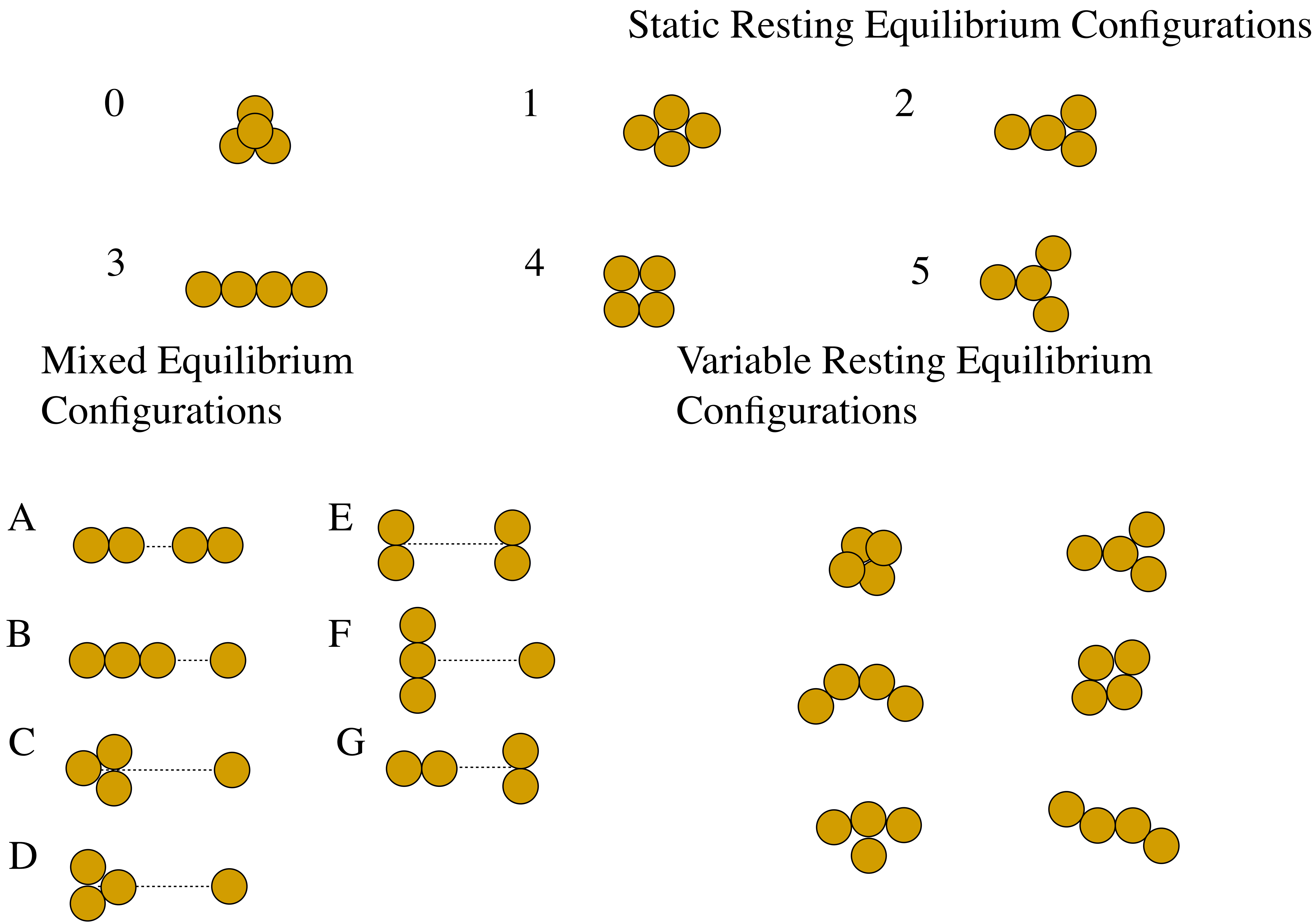
ΔE
ΔE
ΔE = Excess Energy
Increasing AM
Decreasing AM
$\mathcal{E}$
H^2
-1
-1.5
-2
-2.5
-3
0
1
2
3
4
5
6
7
8

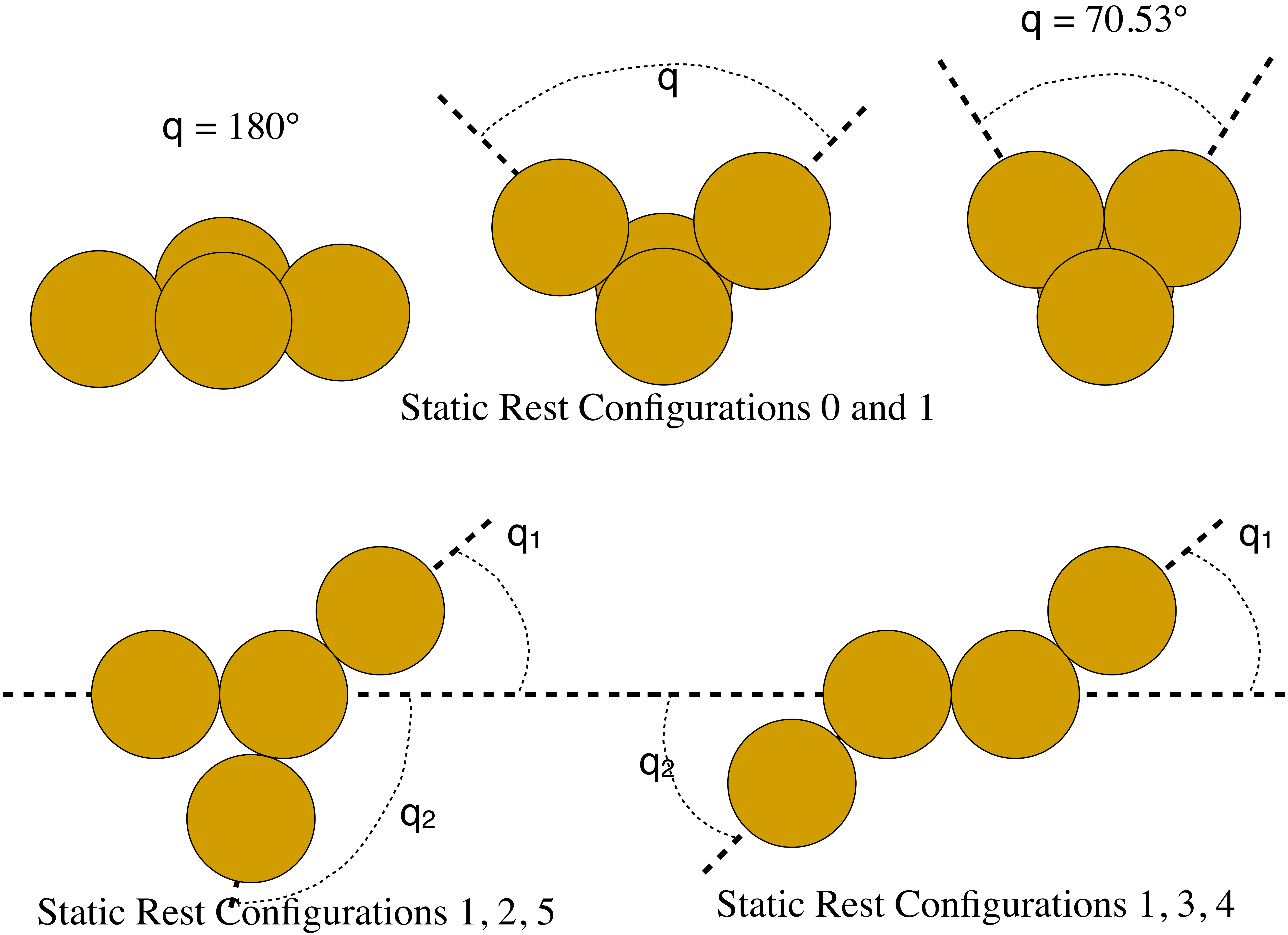


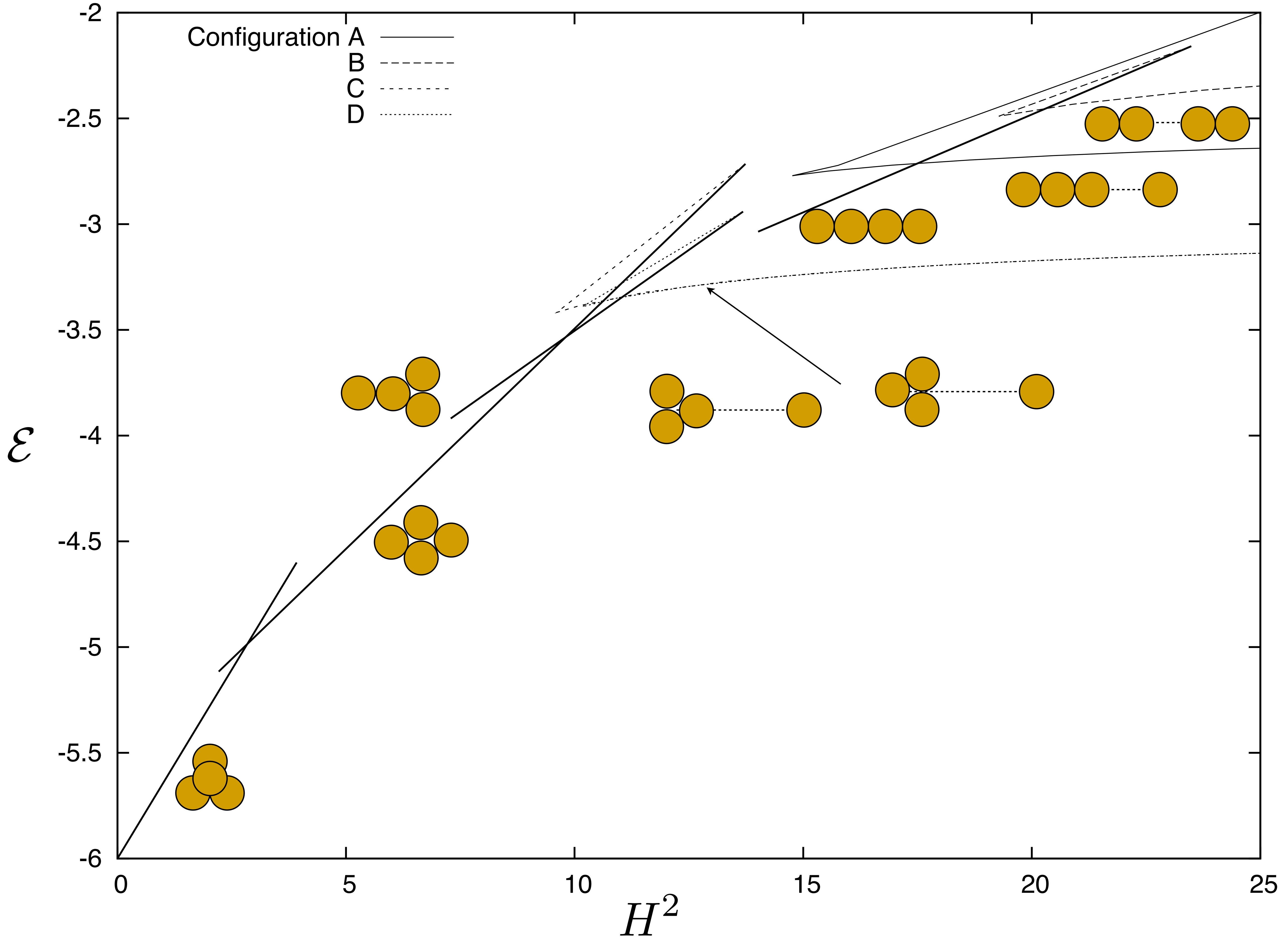
-1
-1.5
-2
-2.5
-3
0
1
2
3
4
5
6
7
8
$\mathcal{E}$
H^2
ΔE
ΔE
ΔE
ΔE
ΔE = Excess Energy
Increasing AM
Decreasing AM


Movie by T. Gabriel
Gravity: on
t: 00.00 Days Tf: 07.06 Days
MU: 0.00 eN: 0.95 eT: 0.00
ΔE
ΔE
ΔE
ΔE
$\mathcal{E}$
-2
-2.5
-3
0
1
2
3
4
5
6
7
8
H^2
ΔE = Excess Energy
Increasing AM
Decreasing AM

Finite Density 4 Body Problem




q = 180°
q
q = 70.53°
Static Rest Configurations 0 and 1
q1
q2
q1
q2
Static Rest Configurations 1, 2, 5
Static Rest Configurations 1, 3, 4



Configuration A
B
C
D
-2
-2.5
-3
-3.5
-4
-4.5
-5
-5.5
-6
0
5
10
15
20
25
$\mathcal{E}$
H^2

What Happens After Fission?

2-Component Rotational Fission

- Fission can be a smooth transition for a rubble pile
- Energy and AM are ideally conserved, but are decomposed:
  - Kinetic Energy

$$\frac{1}{2}\omega \cdot I_0 \cdot \omega = \frac{1}{2}\omega \cdot I_1 \cdot \omega + \frac{1}{2}\omega \cdot I_2 \cdot \omega + \frac{1}{2}\frac{M_1 M_2}{M_1 + M_2}(R\omega)^2$$

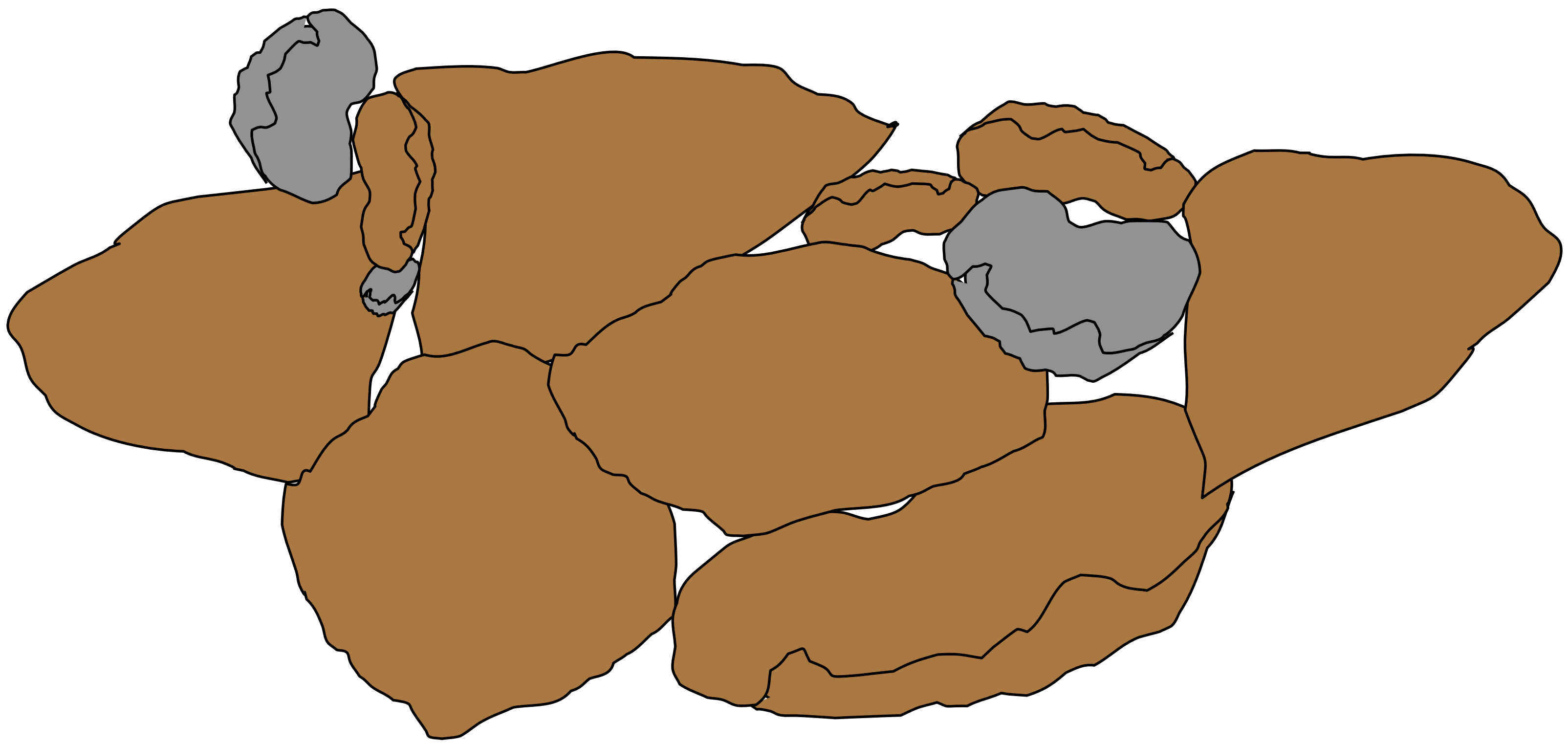
  - Potential Energy

$$\mathcal{U}_{00} = \mathcal{U}_{11} + \mathcal{U}_{22} + \mathcal{U}_{12}$$

  - The mutual potential energy is “liberated” and serves as a conduit to transfer rotational and translational KE

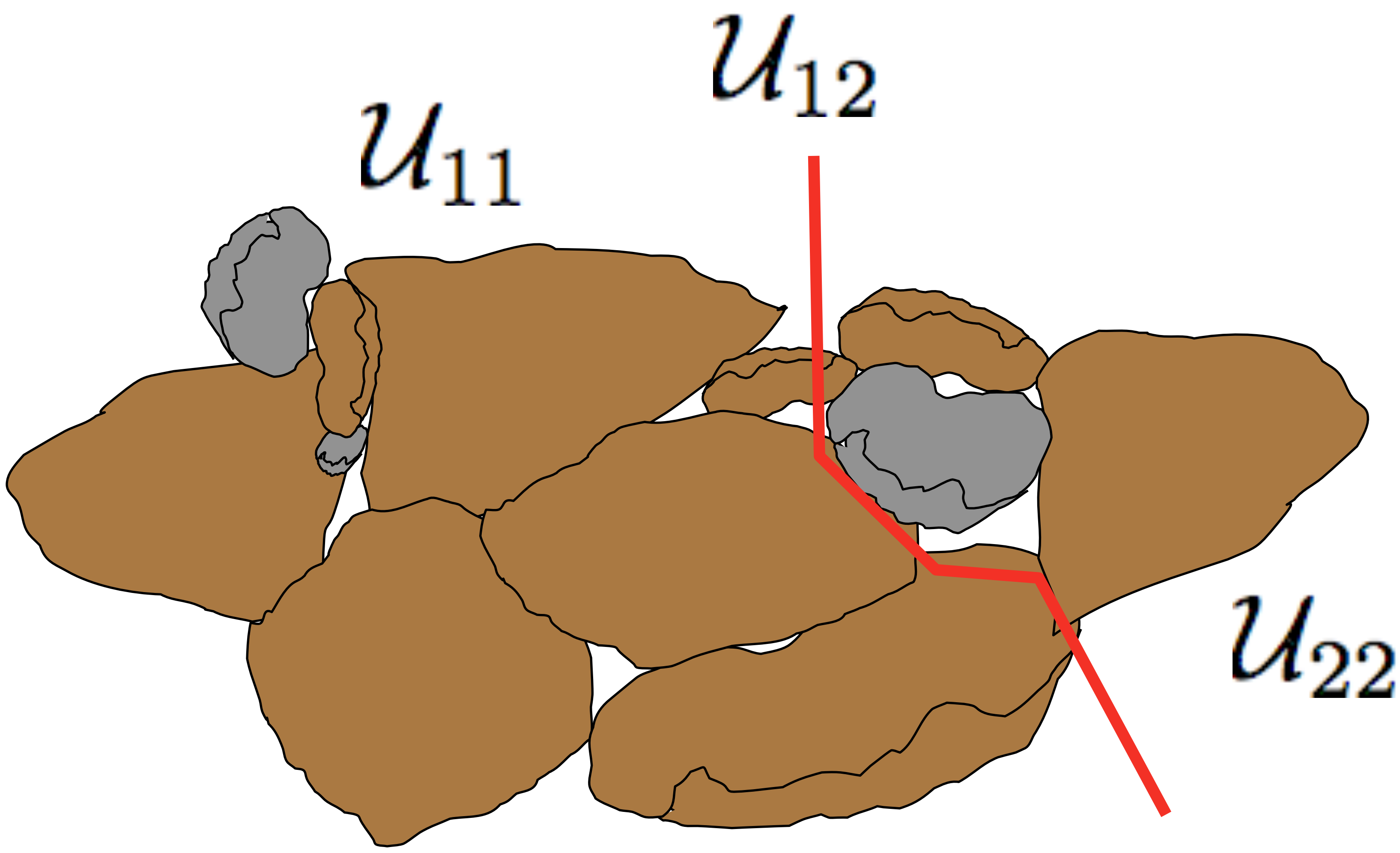
Rotational Fission

$\mathcal{U}_{00}$



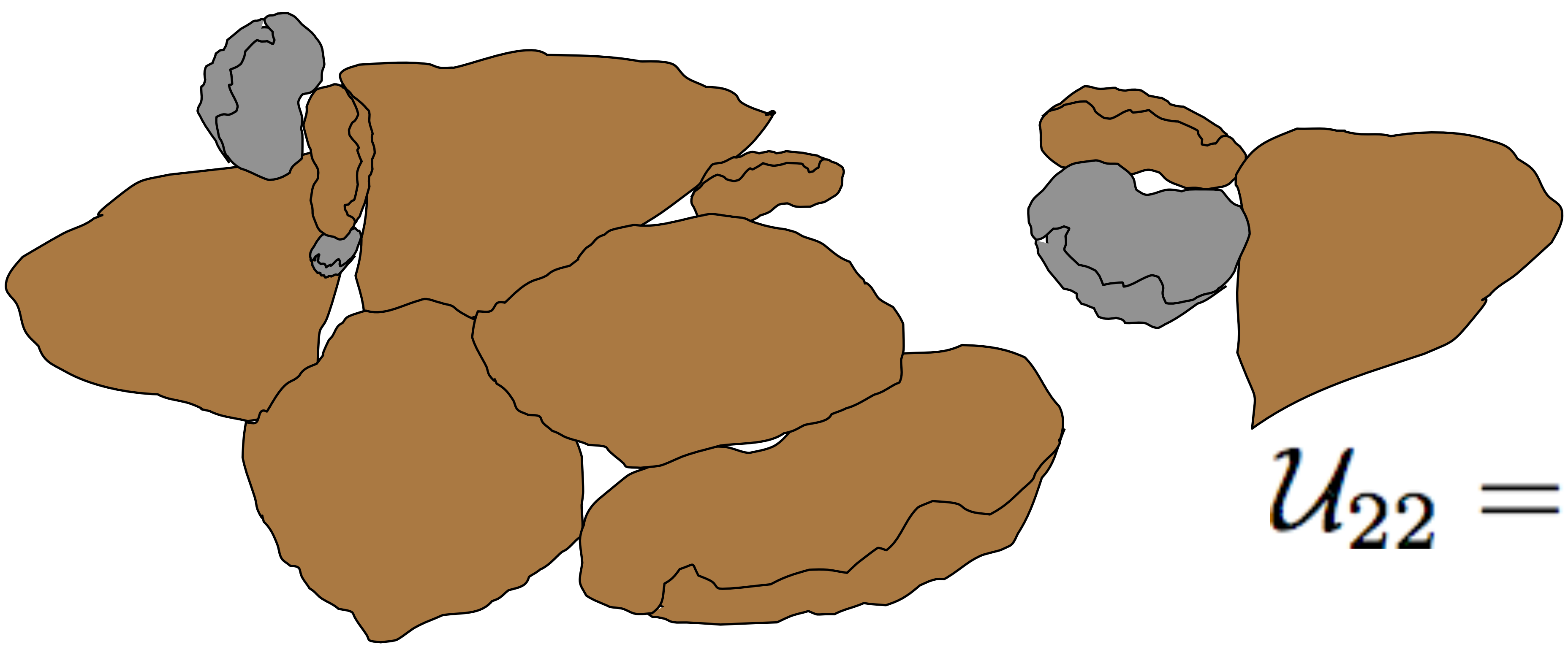
Rotational Fission

$$\mathcal{U}_{00} = \mathcal{U}_{11} + \mathcal{U}_{22} + \mathcal{U}_{12}$$



Orbital Evolution

$$\Delta T_{\text{rot}} + \Delta T_{\text{trans}} + \Delta \mathcal{U}_{12} = 0$$



$$\mathcal{U}_{22} = \text{ Constant}$$

$$\mathcal{U}_{11} = \text{ Constant}$$

Free Energy

- The "free energy" of the system controls the final system state:

$$E_{\text{Free}} = E - \mathcal{U}_{11} - \mathcal{U}_{22}$$

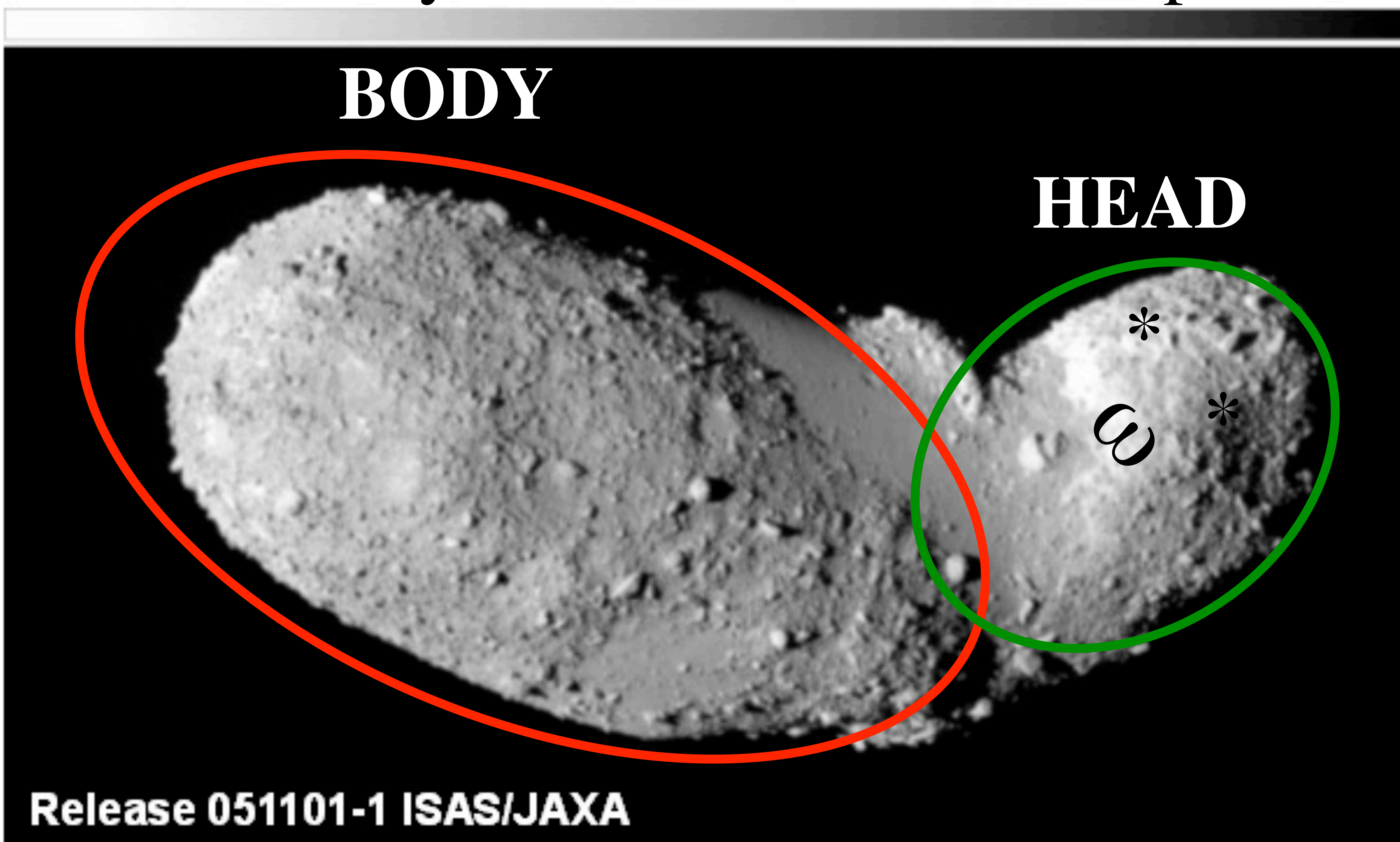
$$E_{\text{Free}} = \frac{1}{2}\left[\omega_1 \cdot I_1 \cdot \omega_1 + \omega_2 \cdot I_2 \cdot \omega_2 + \frac{M_1 M_2}{M_1 + M_2} V \cdot V\right] + \mathcal{U}_{12}$$

  - If disruption occurs, the mutual potential goes to 0: $\mathcal{U}_{12} \rightarrow 0$
  - If $E_{\text{Free}} > 0$, system can "catastrophically disrupt"
  - If $E_{\text{Free}} < 0$, system cannot "catastrophically disrupt" — Hill Stable
  - If $0 < E_{\text{Free}} << 1$ escape leads to a slowly rotating primary

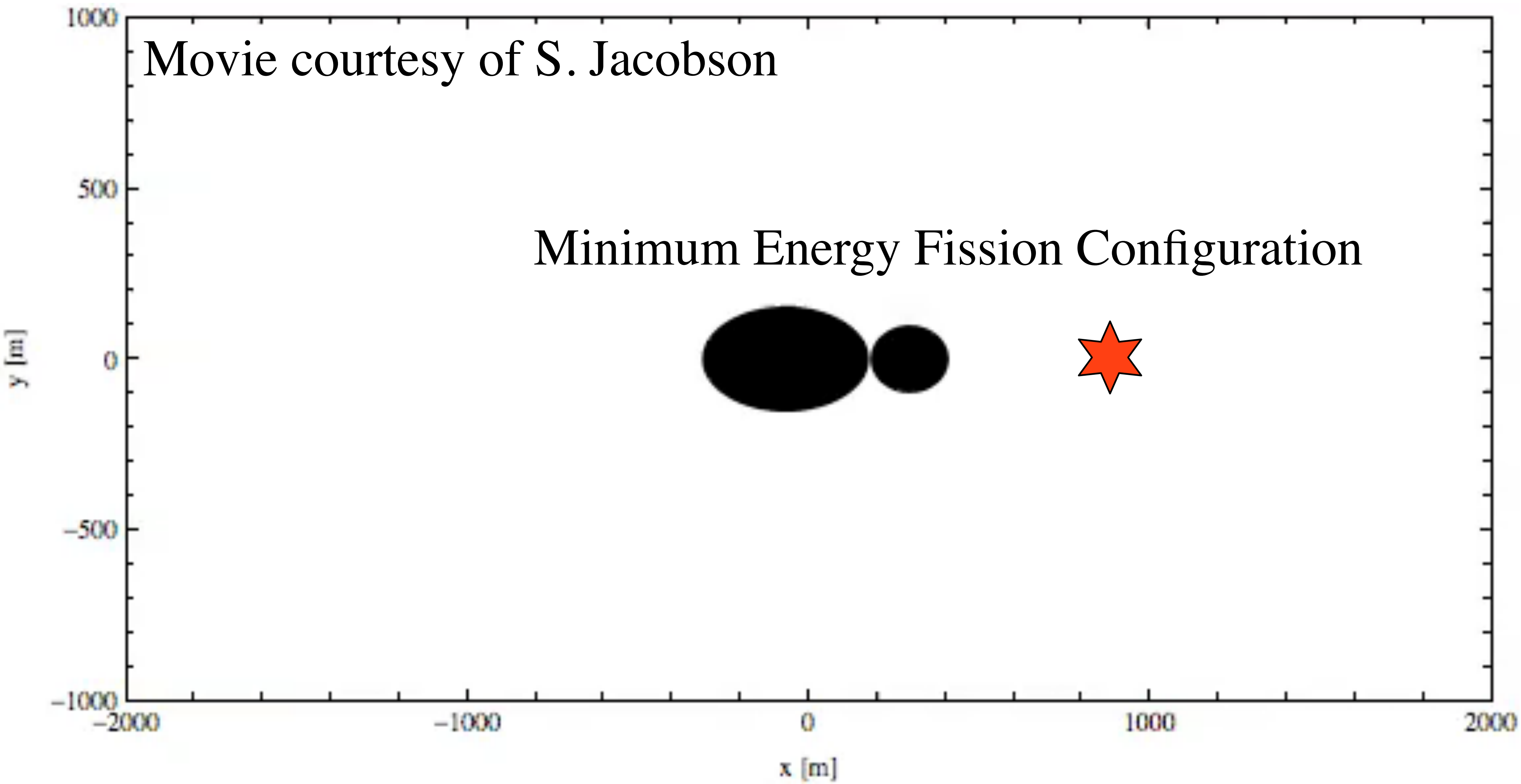
$$\frac{1}{2}\omega_1 \cdot I_1 \cdot \omega_1 \ll 1$$

Itokawa

Head and Body will orbit at a ~ 6 hour period

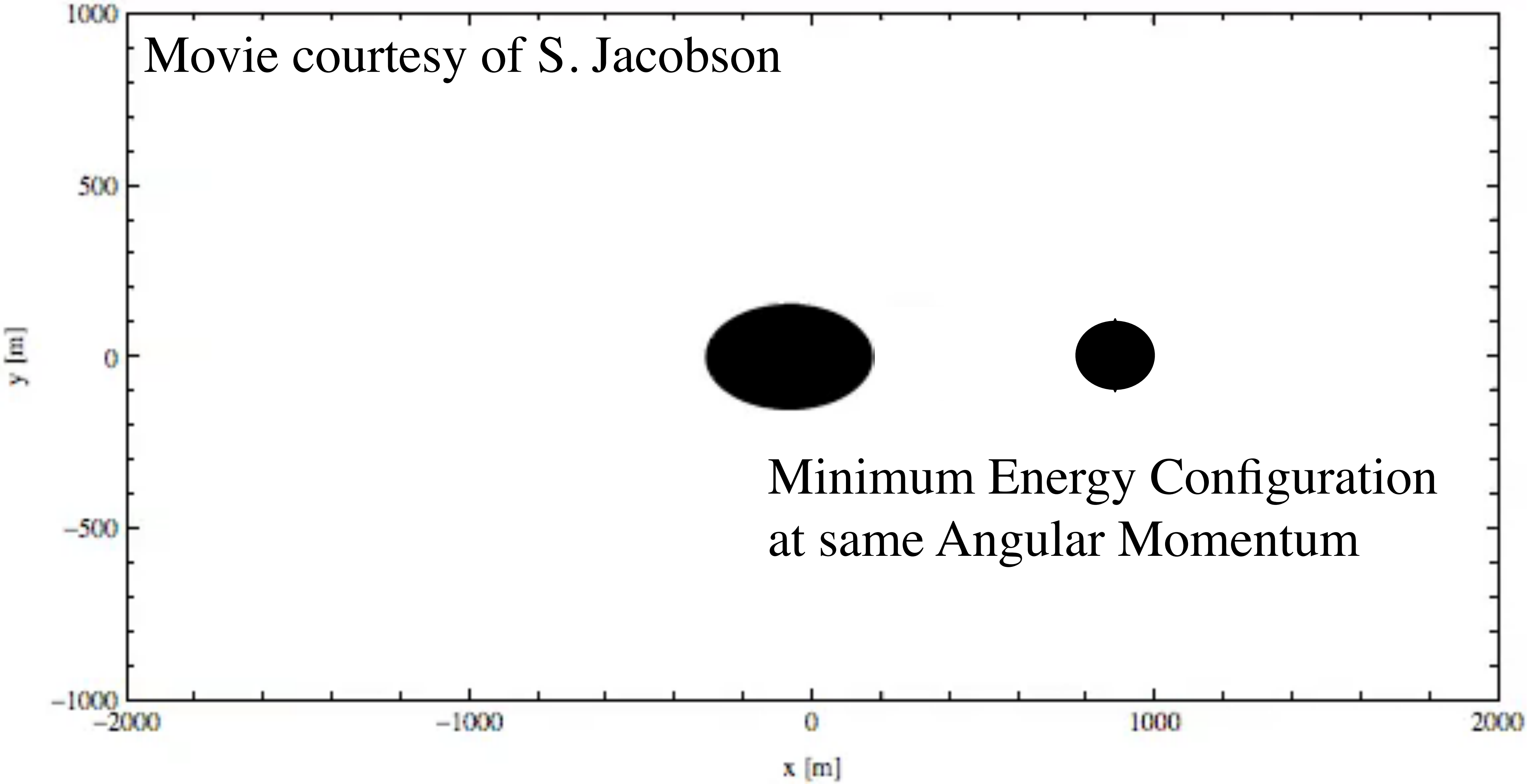


Itokawa Post-Fission Dynamics



- Total system energy is negative but near zero, disruption impossible
- Re-impact is possible if initial Energy is larger than fission energy
- Relative speeds on the order of cm/s only, allows non-catastrophic re-impacts

Itokawa Post-Fission Dynamics



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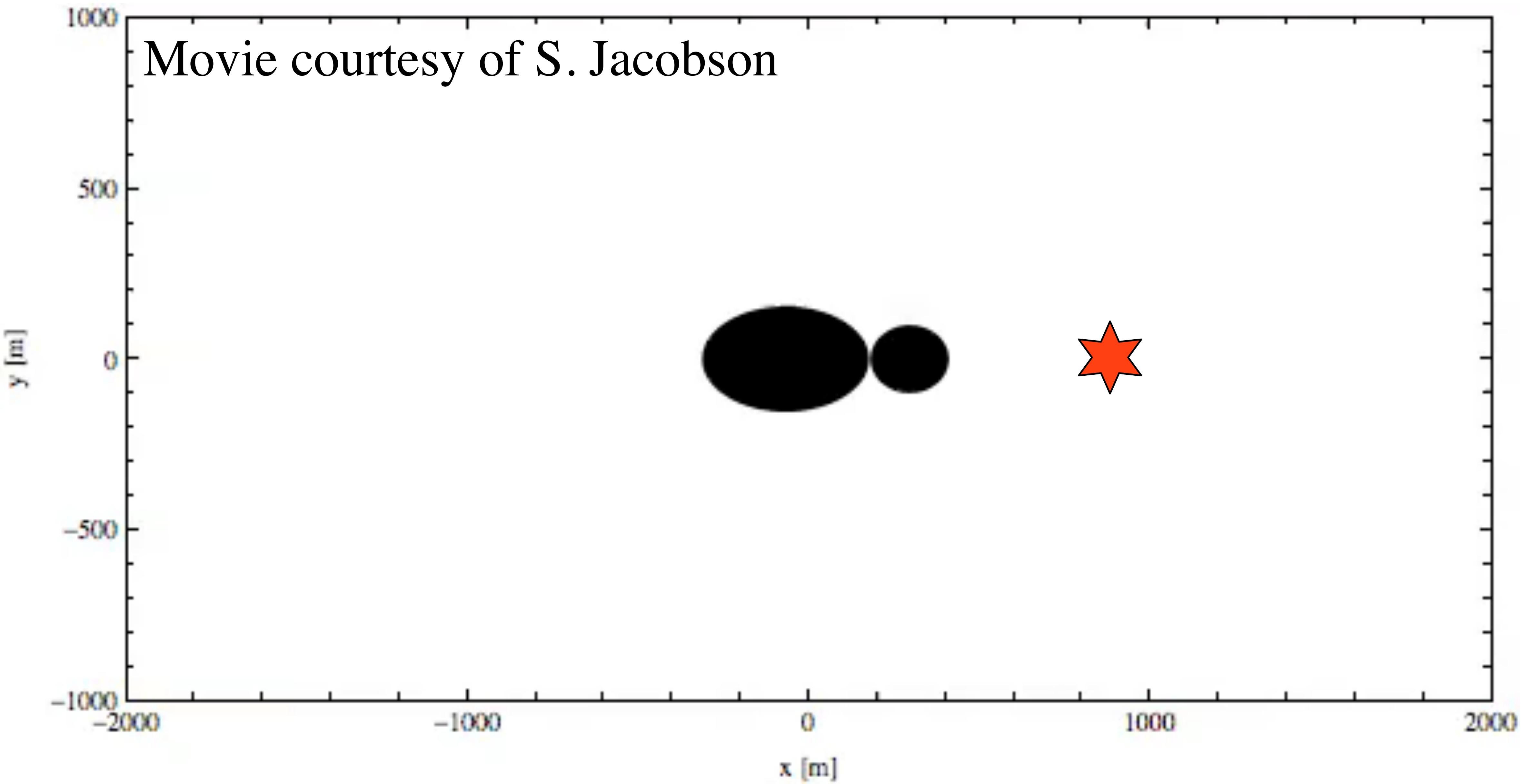
Itokawa Post-Fission Dynamics



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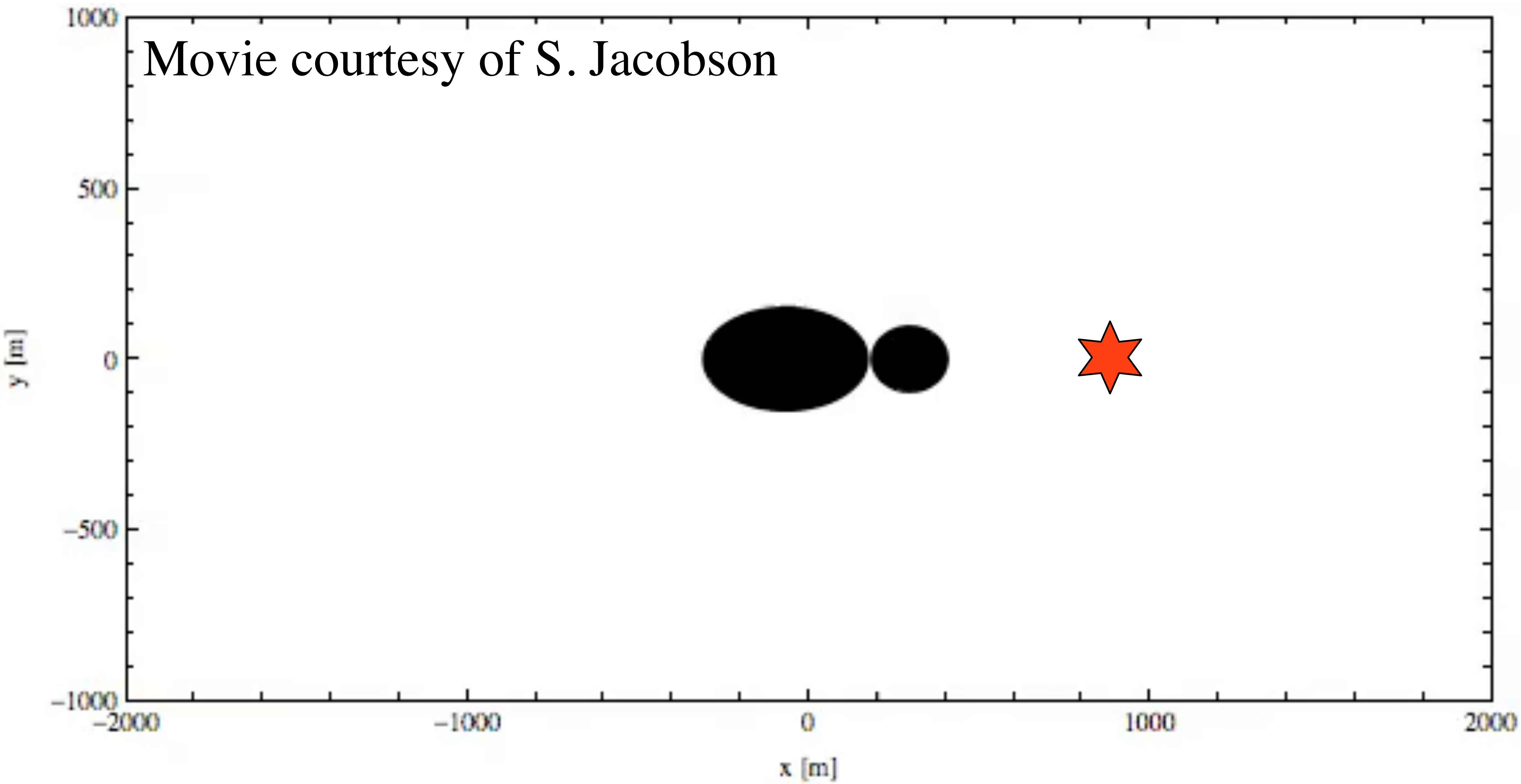
Itokawa Post-Fission Dynamics



- Total system energy is negative but near zero, disruption impossible
- Re-impact is possible if initial Energy is larger than fission energy
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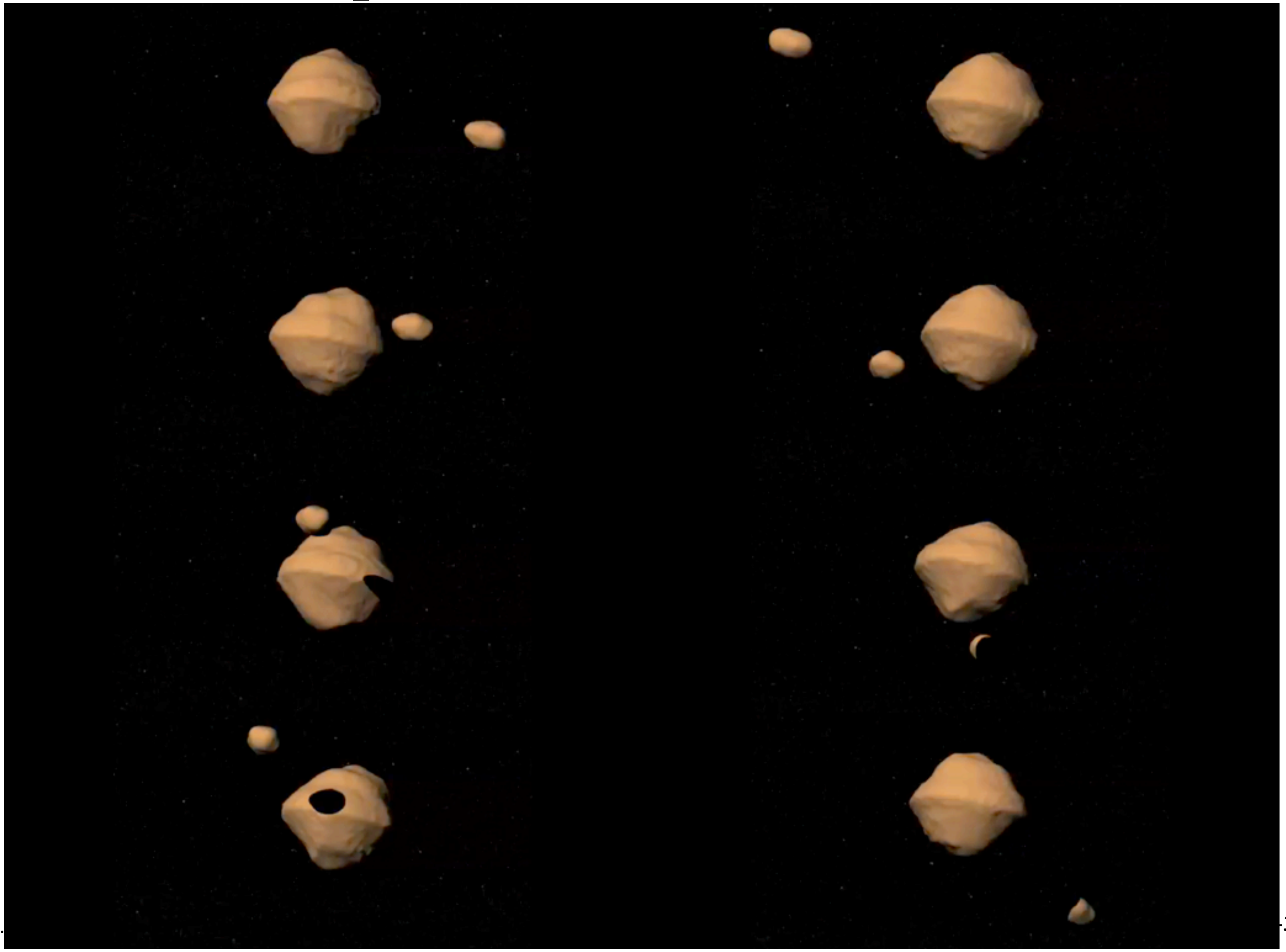
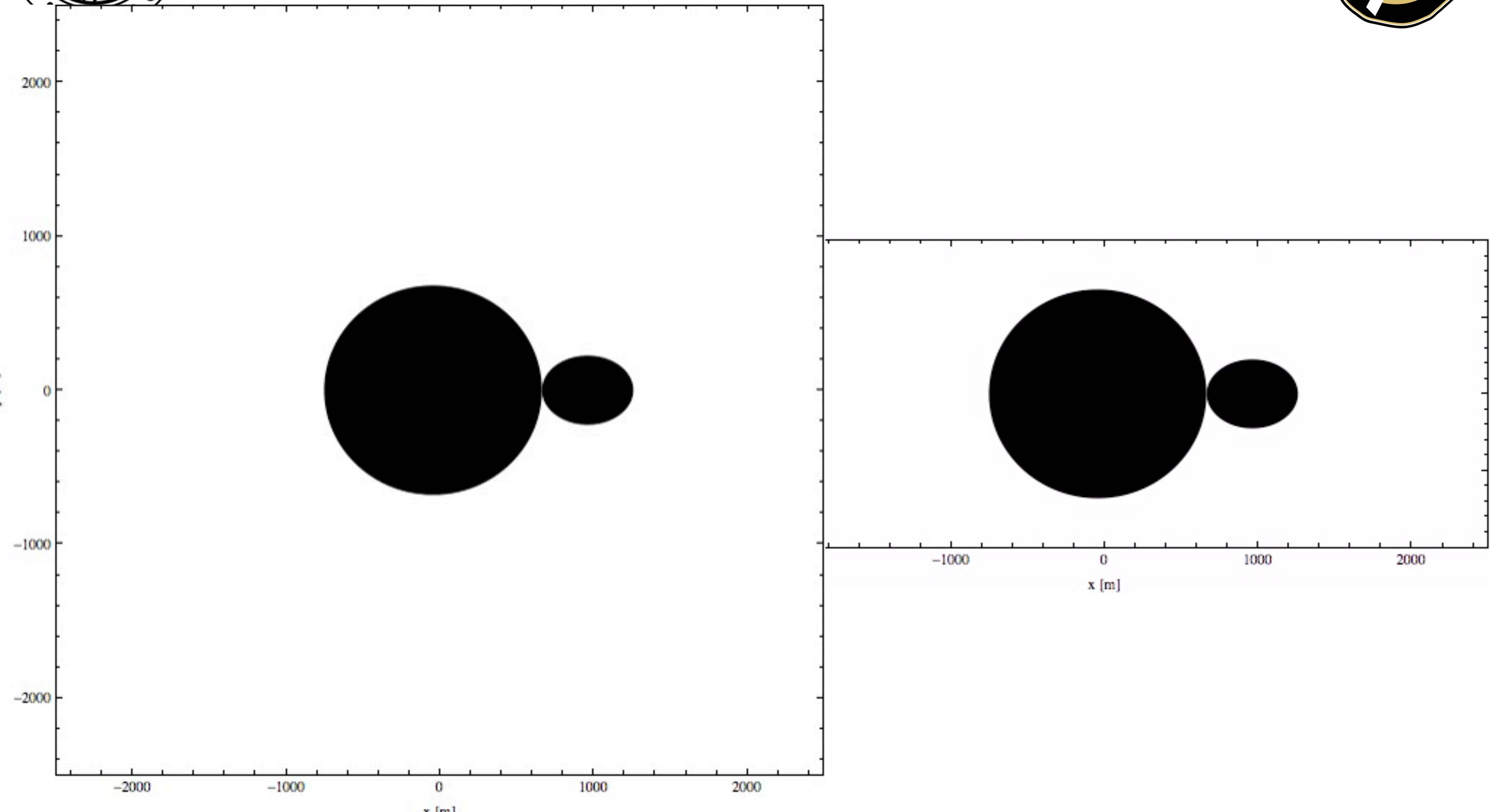
1999 KW4

- Contact binary with Alpha and Beta resting on each other will fission at a spin rate > 4 hours

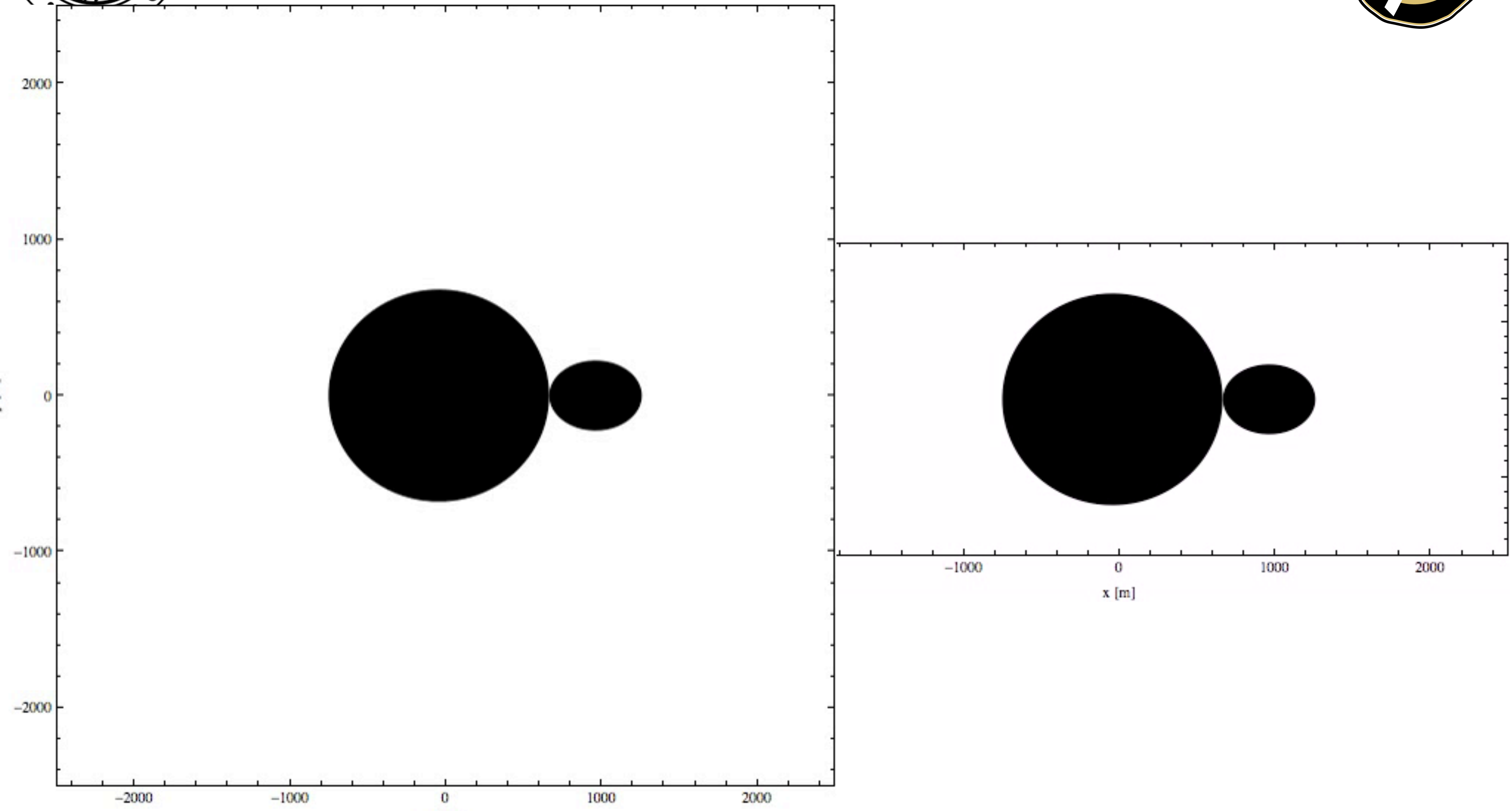


1999 KW4 Fission



- Total system energy is positive, disruption occurs after a “few months”
- If energy dissipation is “fast”, can also settle into a minimum energy orbit

1999 KW4 Fission



- Total system energy is positive, disruption occurs after a "few months"
- If energy dissipation is "fast", can also settle into a minimum energy orbit

Hill Stability in the 3BP

- In the point-mass 3-body problem Hill Stability (i.e., completely bounded motion) is difficult to establish
  - Due in part to the unbounded amended potential
  - Examples arise from KAM theory and the Circular Restricted 3-BP
- For the finite density 3-body problem, conditions for Hill Stability can be proven easily for multiple situations
  - Arise as the mutual potential between 2 bodies is bounded from below:

$$\mathcal{U} = \sum_{i<j} \mathcal{U}_{ij} \qquad \mathcal{U}_{ij} = -\frac{m_i m_j}{d_{ij}} \geq -\frac{m_i m_j}{r_i + r_j}$$

  - Then if body *k* escapes, $r_{ik} \rightarrow \infty$ and:

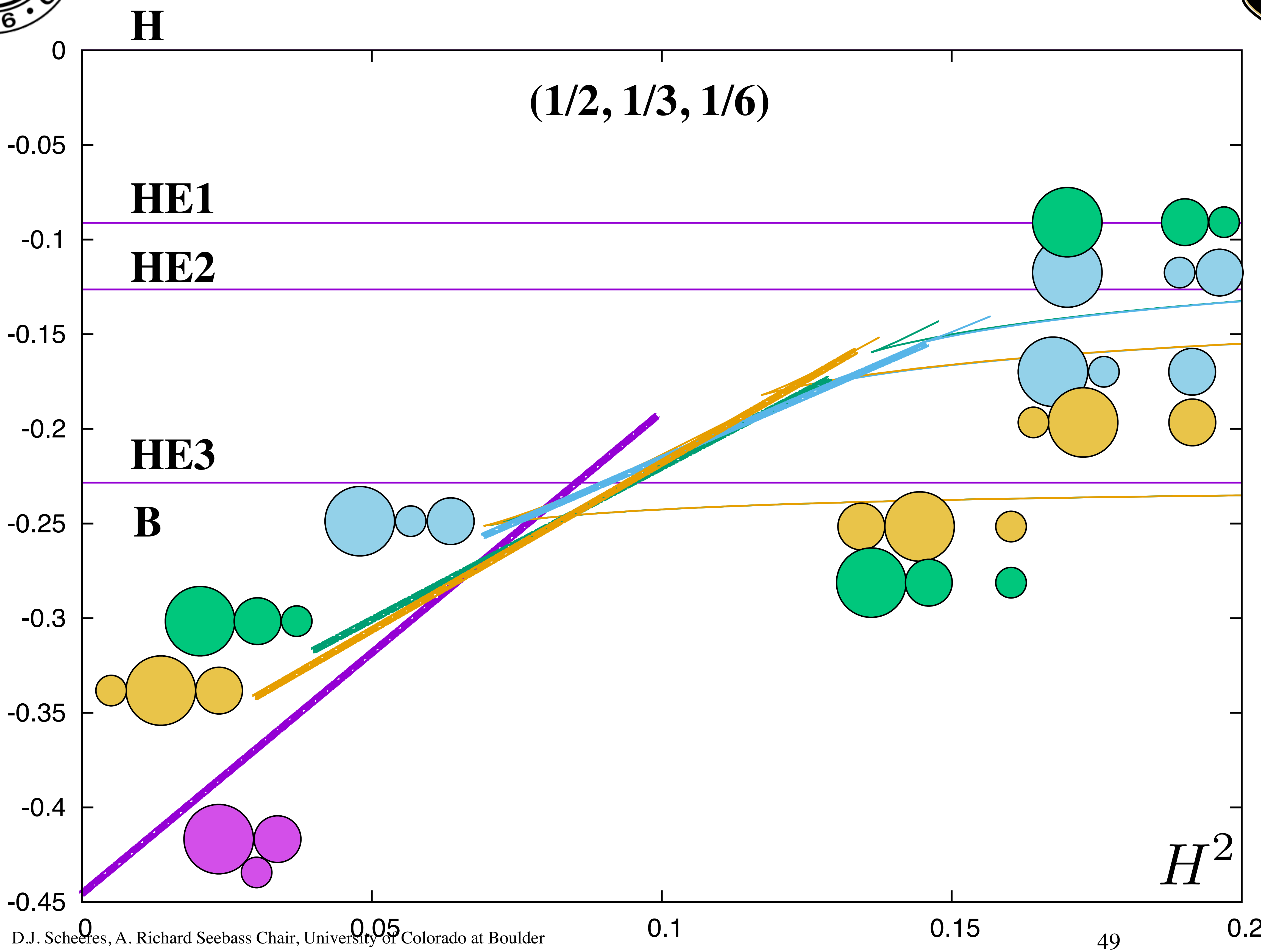
$$I_H \rightarrow \infty \qquad \mathcal{U}_{ik} \rightarrow 0 \ \forall i \qquad \mathcal{E} \rightarrow \sum_{(i<j)\neq k} \mathcal{U}_{ij}$$

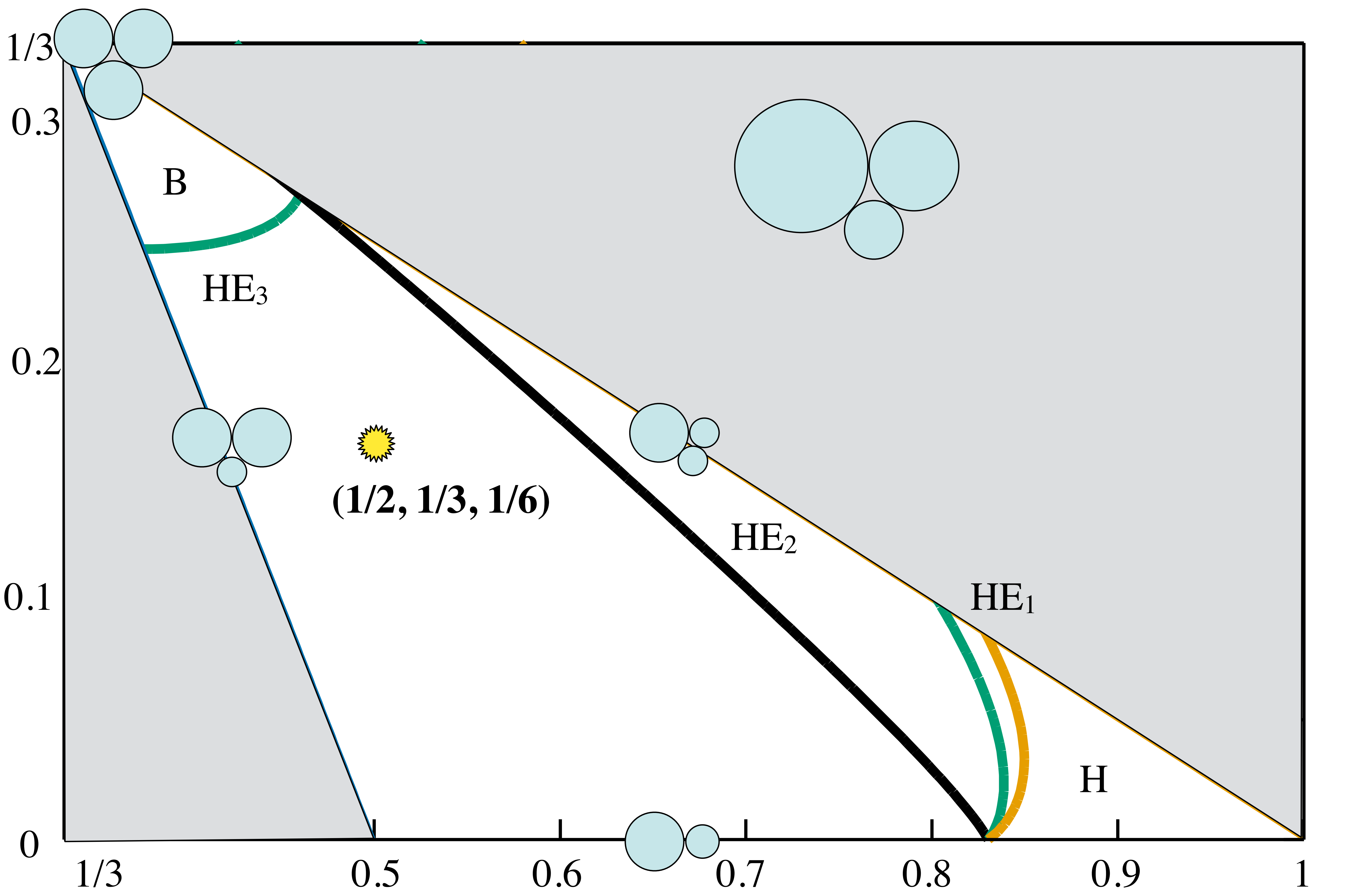
Thus if $E < -\sum_{(i<j)\neq k} \frac{m_i m_j}{r_i + r_j}$ body *k* can't escape!

Hill Stability for the F3BP

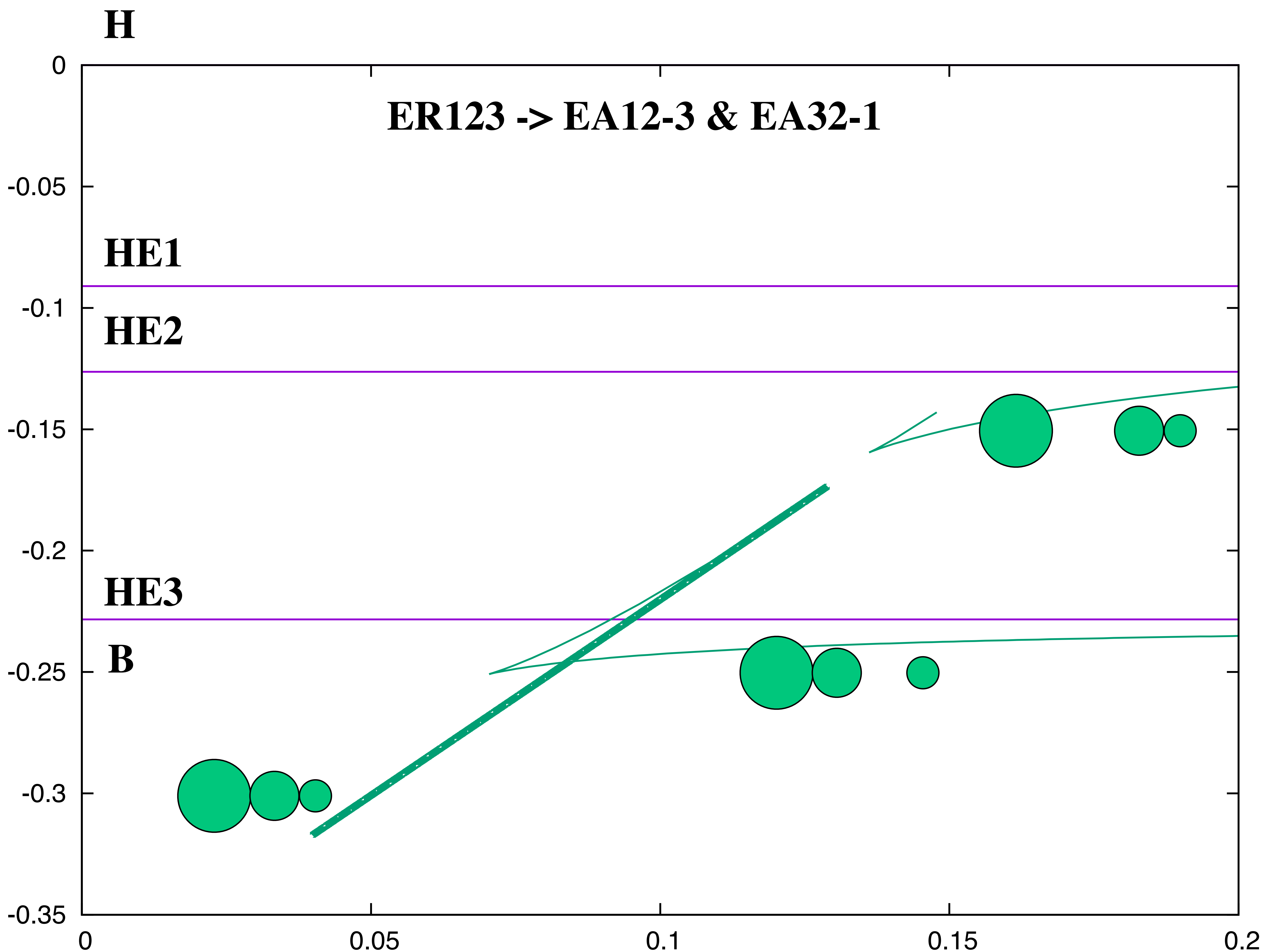
- Assuming $m_3 \le m_2 \le m_1$ it is easy to prove that:
  - **B**: If $E < -\frac{m_1 m_2}{r_1 + r_2}$ All motion is bounded
  - **HE3**: If $-\frac{m_1 m_2}{r_1 + r_2} < E$ Body 3 can escape
  - **HE2**: If $-\frac{m_1 m_3}{r_1 + r_3} < E$ Body 2 or 3 can escape
  - **HE1**: If $-\frac{m_2 m_3}{r_2 + r_3} < E$ Body 1, 2 or 3 can escape
  - **H**: If $0 < E$ All bodies can mutually escape

Energy / Angular Momentum Chart

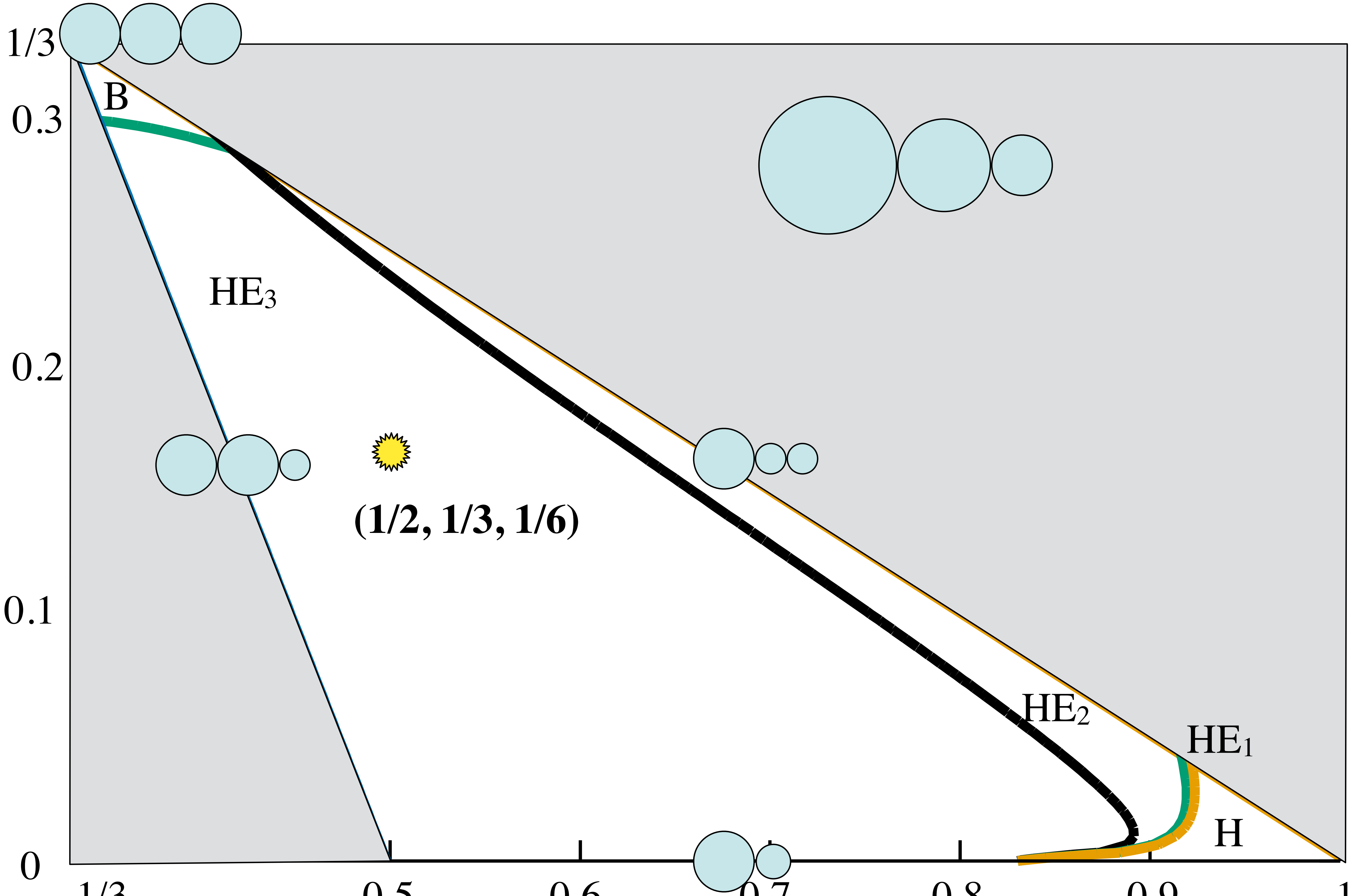




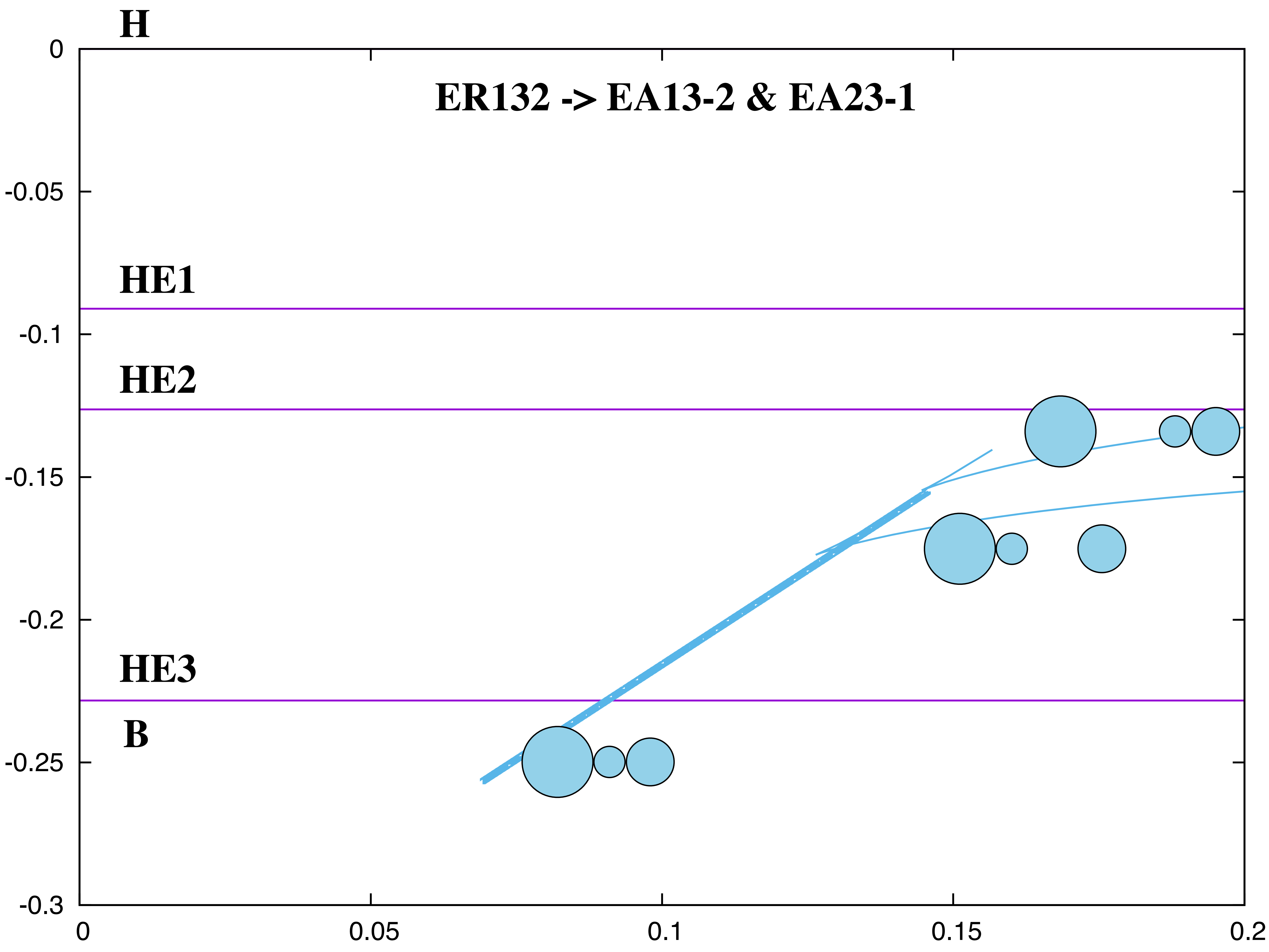
1/3
0.3
0.2
0.1
0
1/3
0.5
0.6
0.7
0.8
0.9
1
B
HE3
HE2
HE1
H
(1/2, 1/3, 1/6)



H
ER123 -> EA12-3 & EA32-1
HE1
HE2
HE3
B
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
0
0.05
0.1
0.15
0.2



1/3
0.3
0.2
0.1
0
B
HE3
HE2
HE1
H
(1/2, 1/3, 1/6)
1/3
0.5
0.6
0.7
0.8
0.9
1


H
ER132 -> EA13-2 & EA23-1
HE1
HE2
HE3
B
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
0
0.05
0.1
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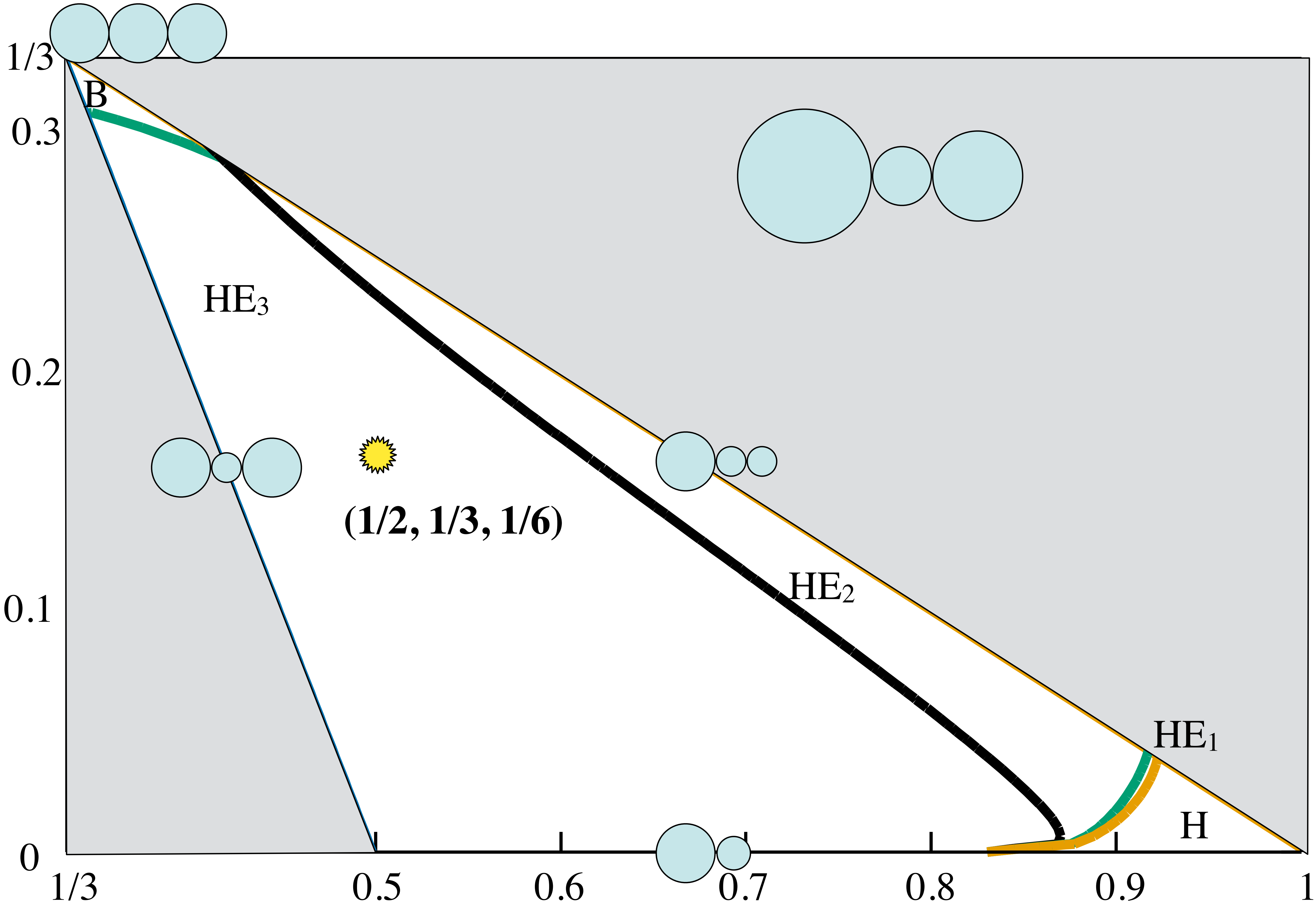
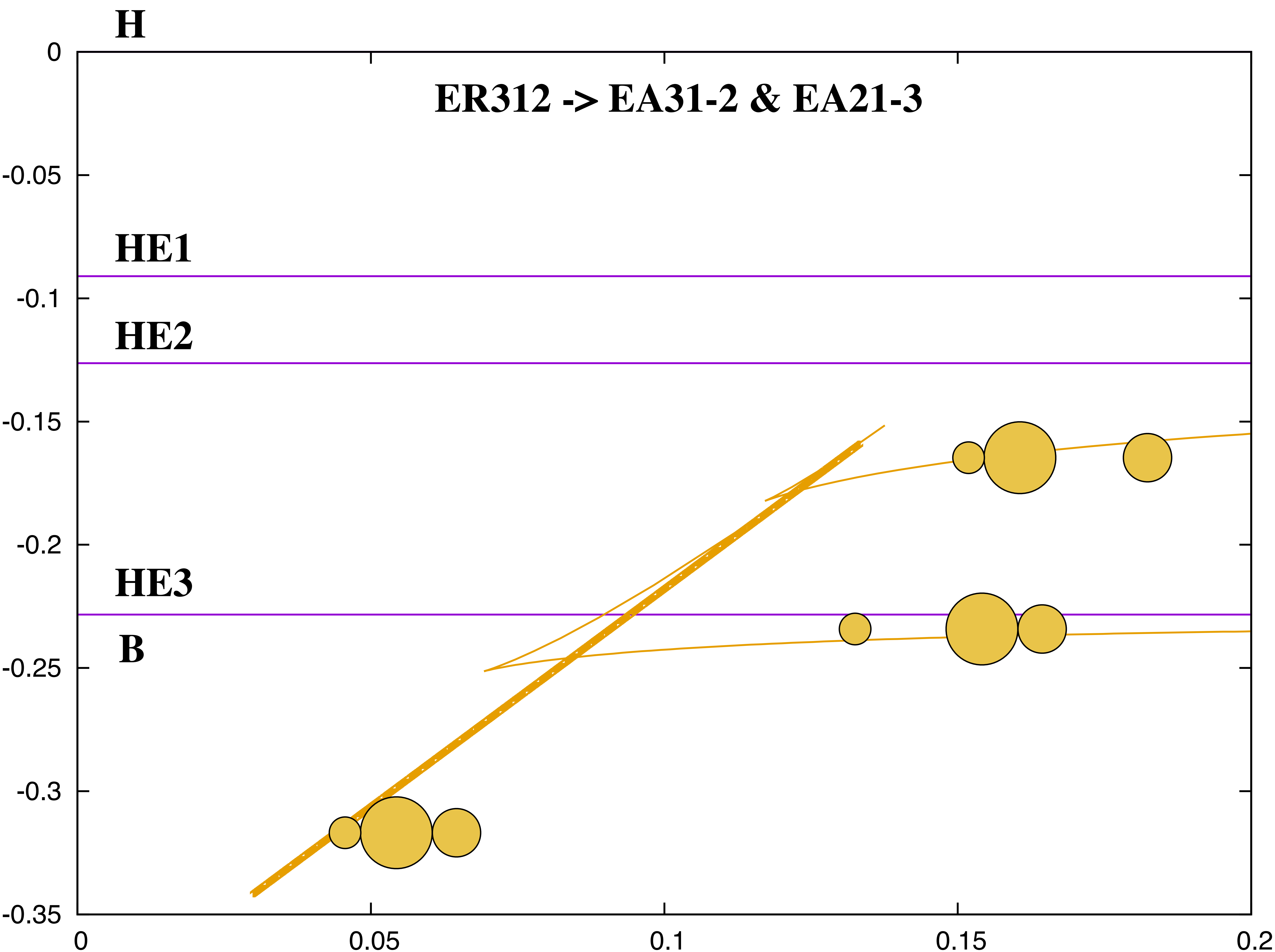


1/3
0.3
0.2
0.1
0
B
HE3
HE2
HE1
H
(1/2, 1/3, 1/6)
1/3
0.5
0.6
0.7
0.8
0.9
1


H
ER312 -> EA31-2 & EA21-3
HE1
HE2
HE3
B
0
-0.05
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-0.25
-0.3
-0.35
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0.05
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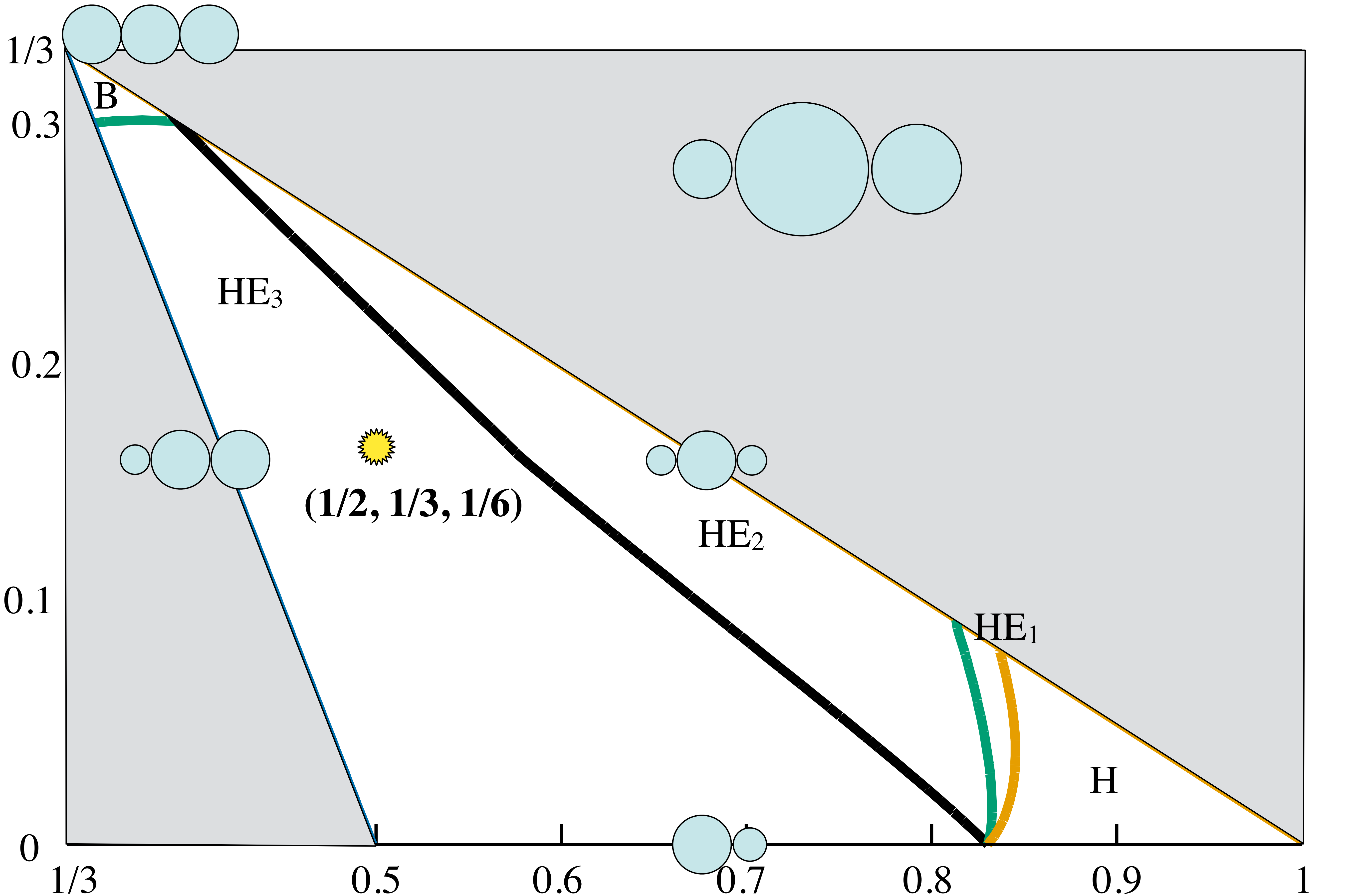
1/3
0.3
0.2
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1/3
0.5
0.6
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0.8
0.9
1
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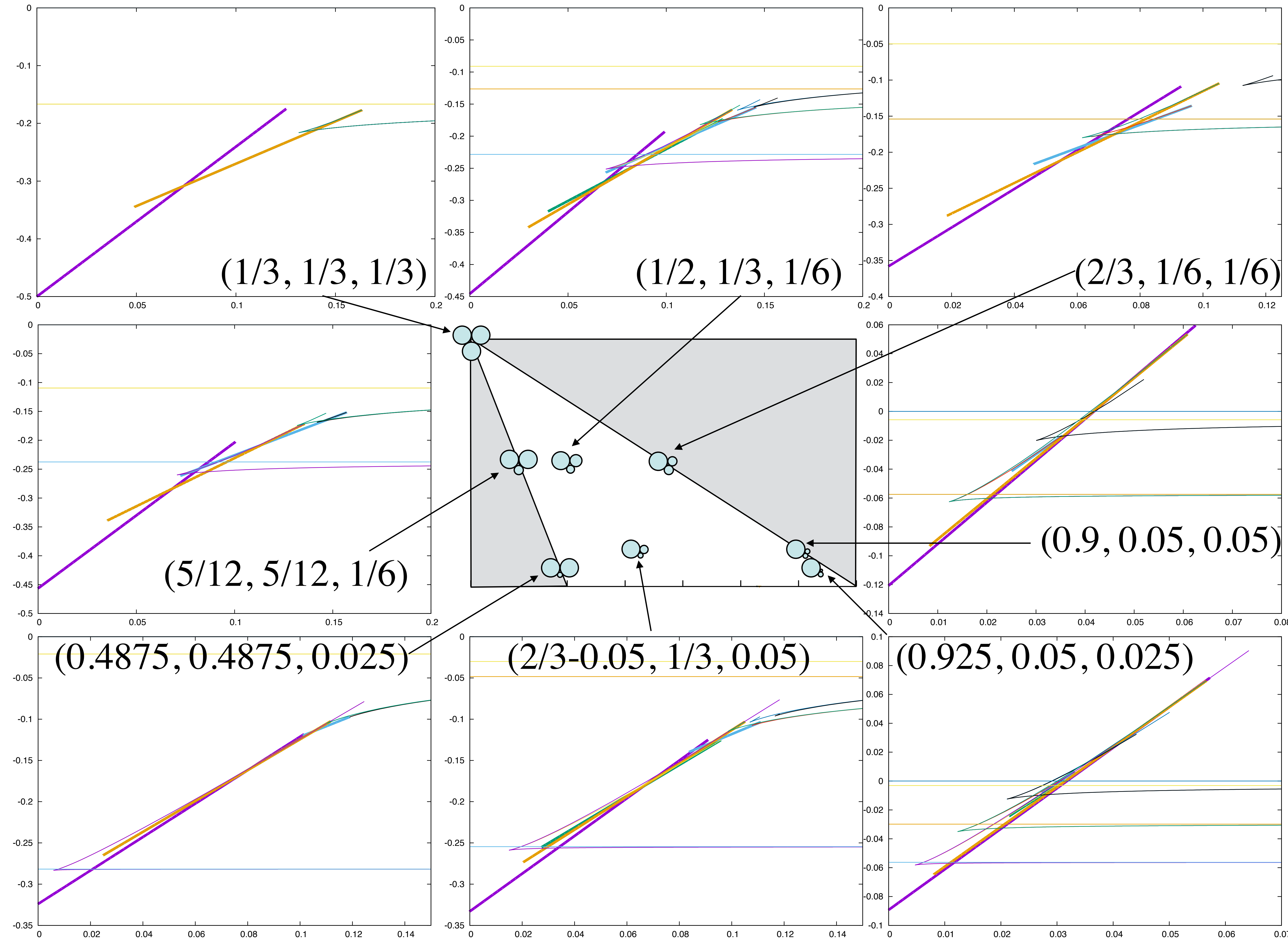
HE$_2$
HE$_1$
H
(1/2, 1/3, 1/6)



(1/3, 1/3, 1/3)
(1/2, 1/3, 1/6)
(2/3, 1/6, 1/6)
(5/12, 5/12, 1/6)
(0.9, 0.05, 0.05)
(0.4875, 0.4875, 0.025)
(2/3-0.05, 1/3, 0.05)
(0.925, 0.05, 0.025)

Hill Stability for N body systems

System Energy of an N-Body Problem

$$E = T + \mathcal{U}$$

Gravitational Potential Energy for Finite Density Bodies

$$\mathcal{U} = -\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{1}{r_{ij}}$$

Minimum Potential

$$\mathcal{U}_m(N) = \min_{r_{ij} \geq 1} \mathcal{U}$$

- A *Configuration* of P components with q_i; $i = 1, 2, \ldots P$ bodies in each component is ***Configuration Hill Stable*** if none of these components can mutually escape to infinity.
- A given configuration of P components with q_i bodies in each component is Configuration Hill Stable if

$$E < \sum_i^P \mathcal{U}_m(q_i)$$

$$\mathcal{U}_m(2) = -1 \qquad \mathcal{U}_m(4) = -6$$

$$\mathcal{U}_m(3) = -3 \qquad \mathcal{U}_m(5) > -10$$

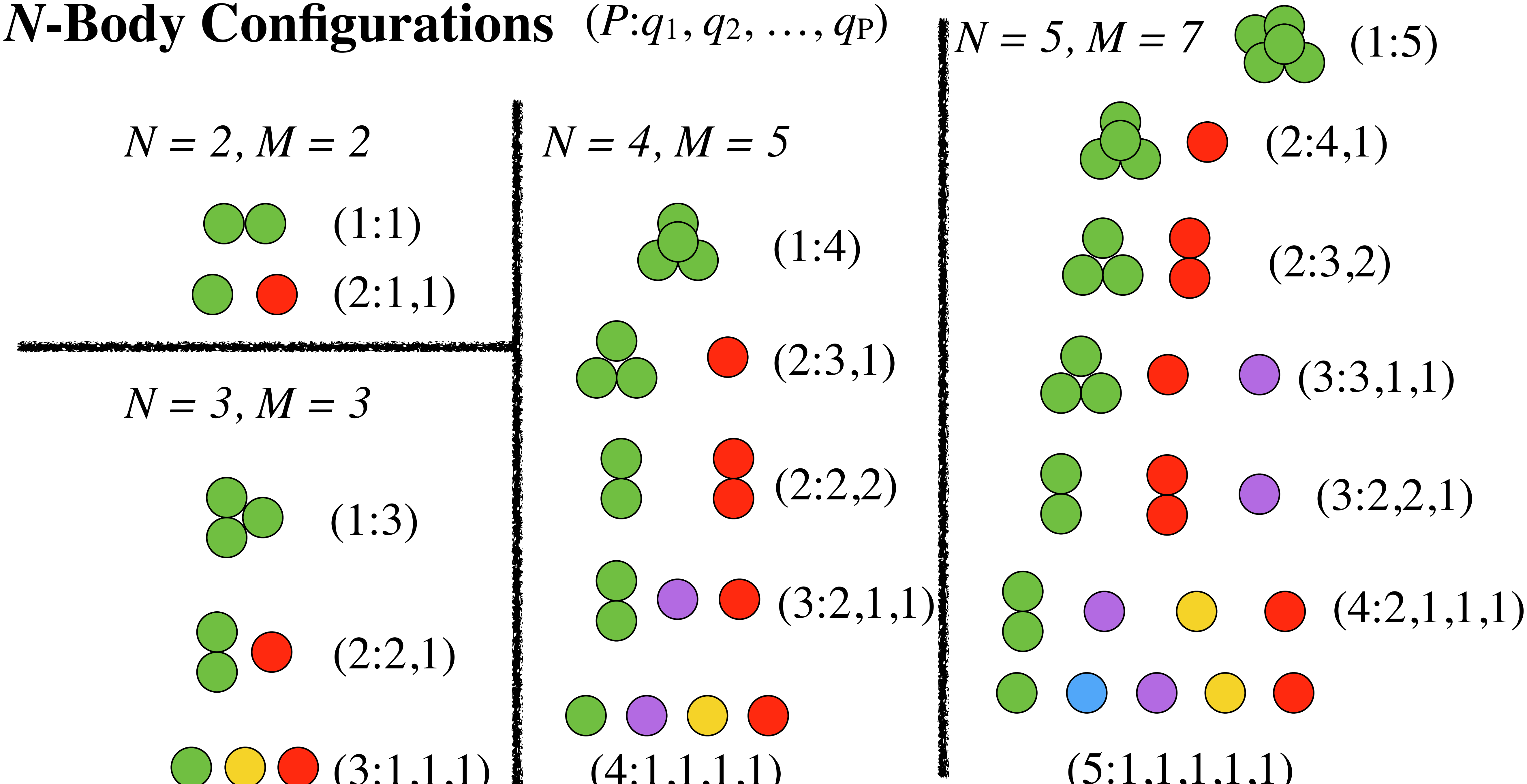
  - Since the distance between bodies is bounded, there is a minimum gravitational potential energy, which restricts escape.

See: D.J. Scheeres. 2016. "Hill Stability of Configurations in the Full N-Body Problem," in Asteroids: New observations, New models, Proceedings of the International Astronomical Union 318: 128–134.

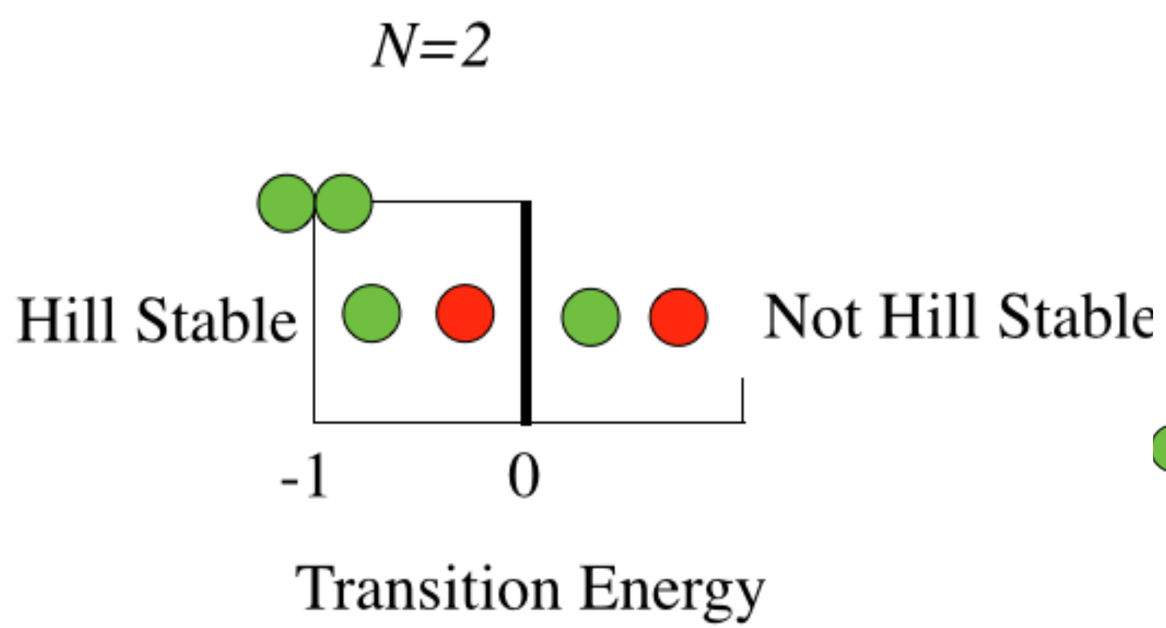
Fission and Escape for *N* Bodies

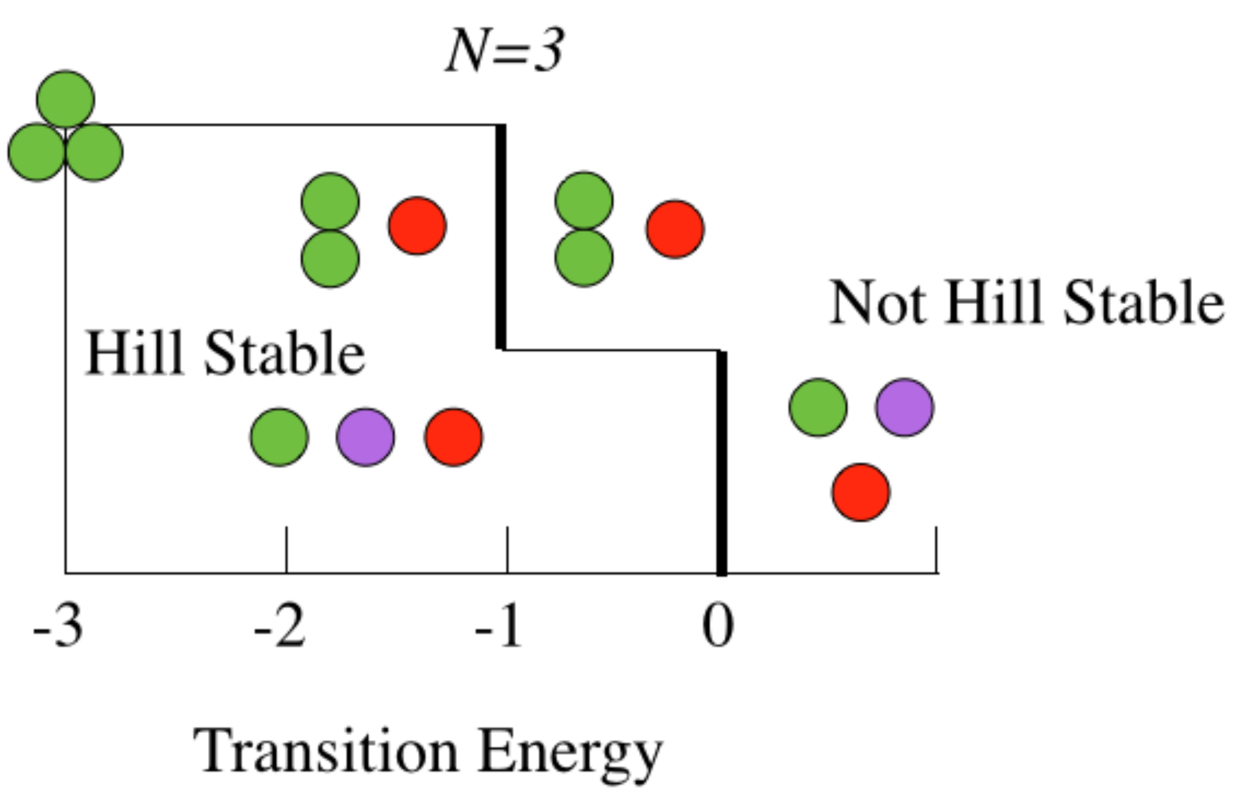
- Given *N* bodies, a *Configuration* is a collection of these bodies into *P* groups, with each group having a set number of bodies q_i.

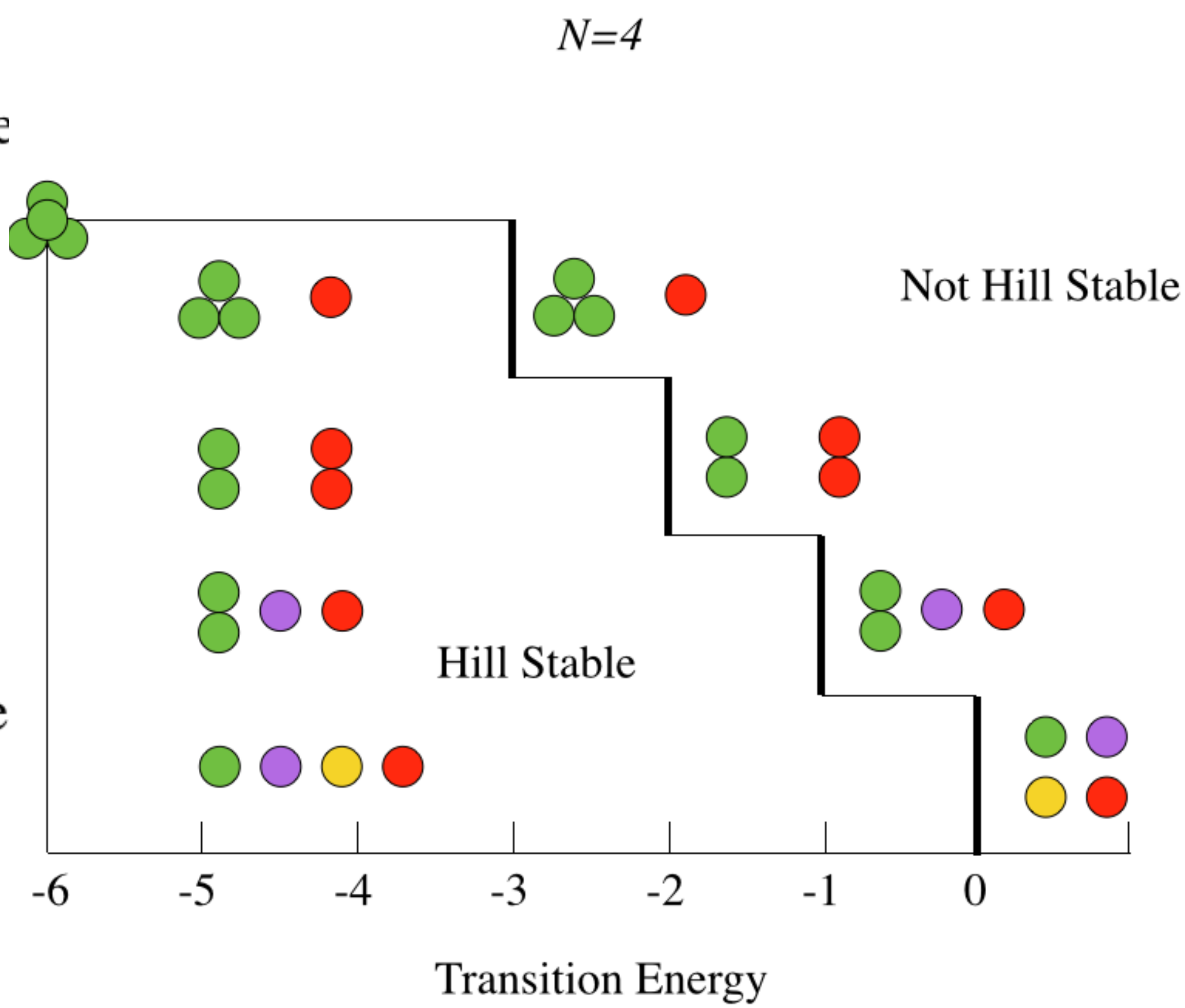
N-Body Configurations $(P{:}q_1, q_2, \ldots, q_P)$



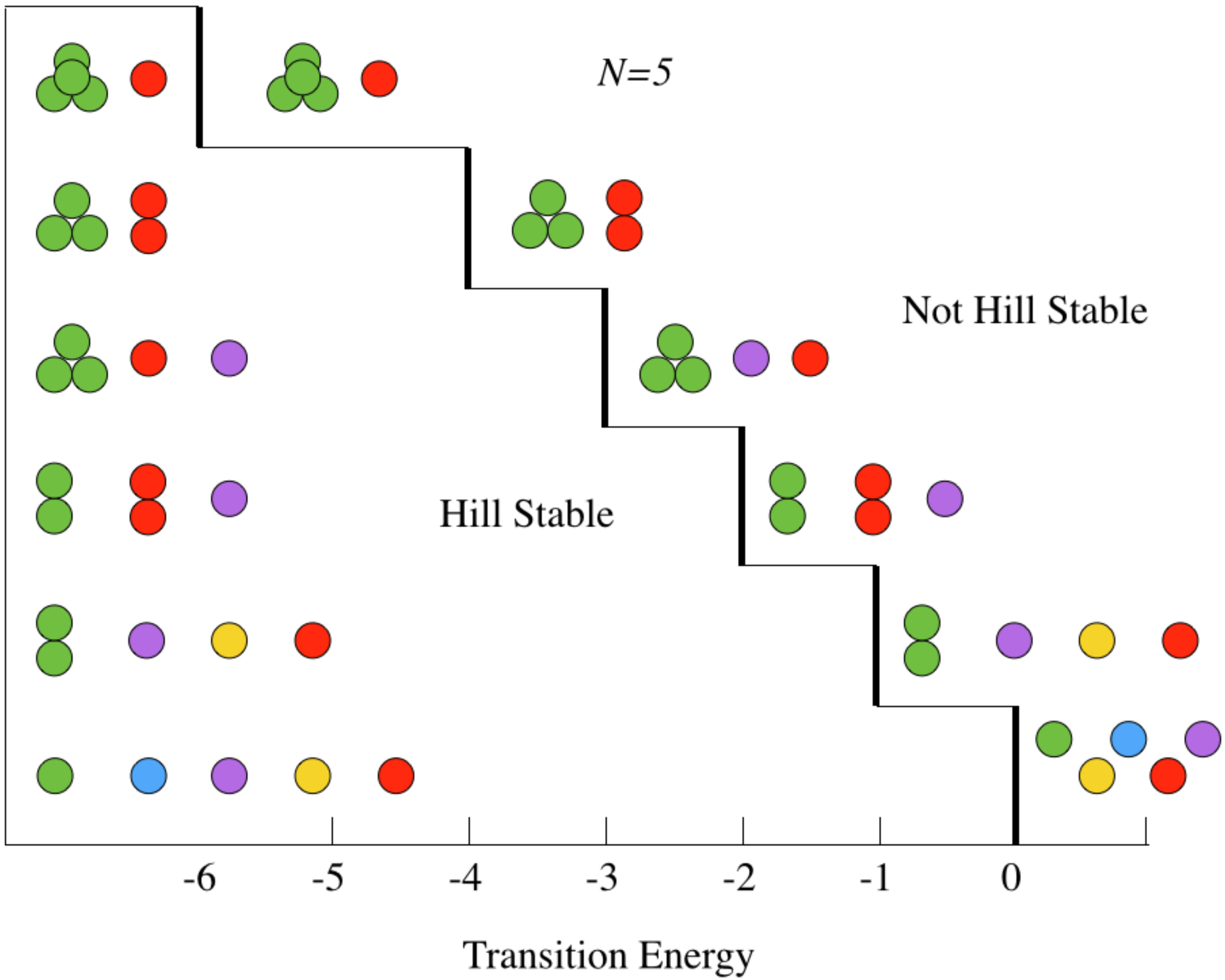
Energy Limits for N = 2, 3, 4





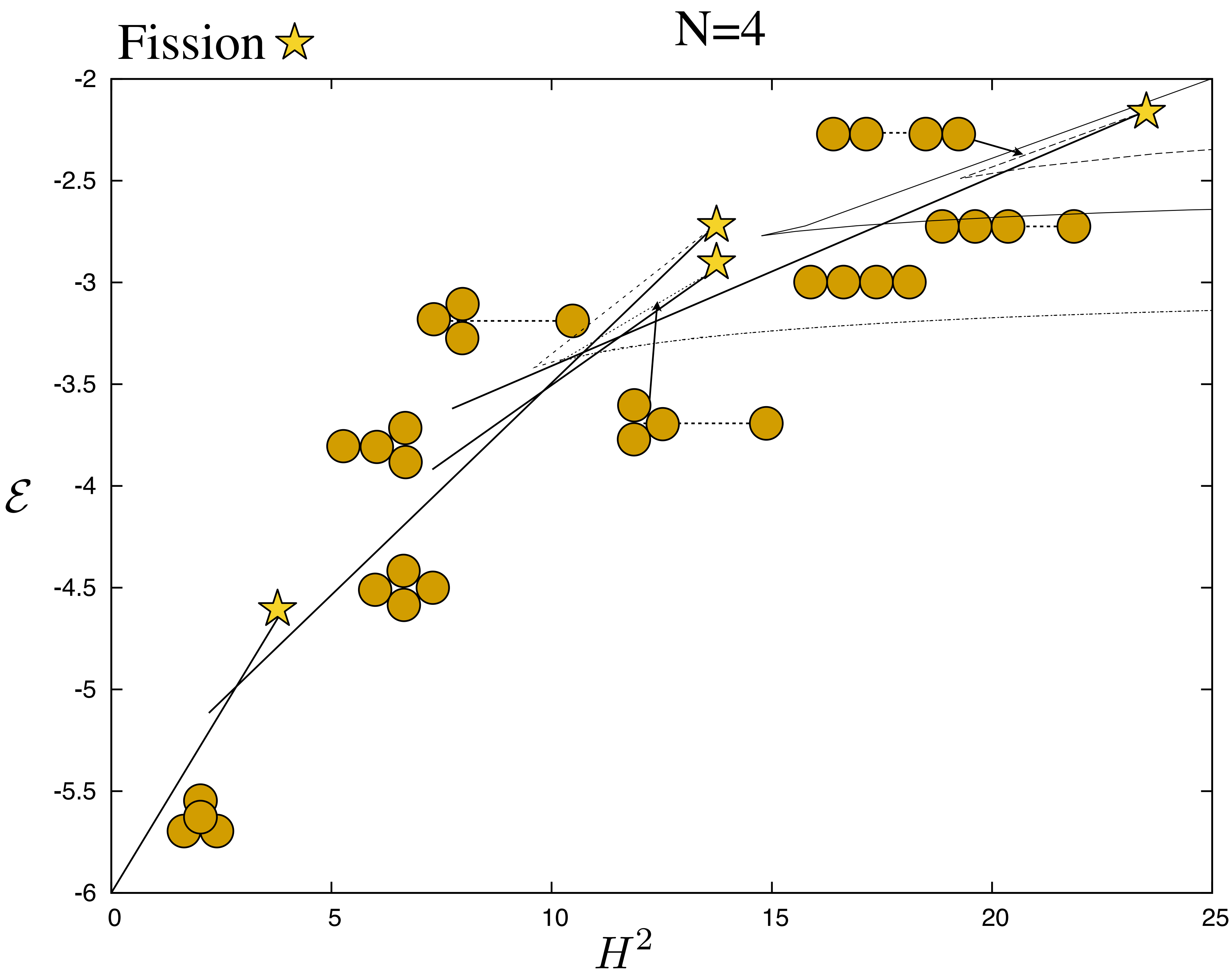


Energy Limits for N = 5



D.J

Fission ★



For all four bodies to disperse requires E > 0

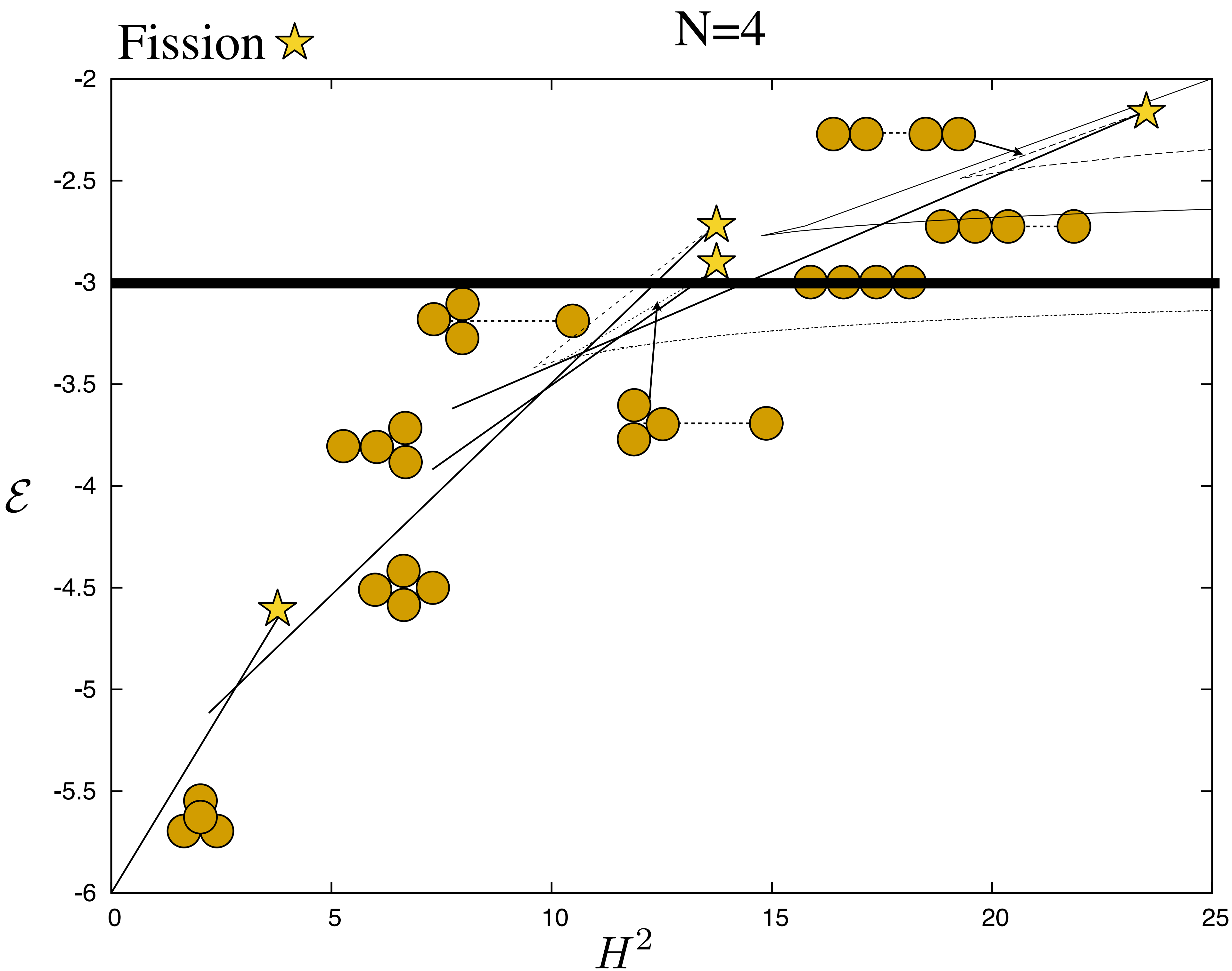
If E > -2 then two equal mass bodies can escape each other

If E > -3, a single body can escape

Fission can lead to a single body escaping

More complex for non-equal mass bodies

Fission ☆



For all four bodies to disperse requires E > 0

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Fission can lead to a single body escaping

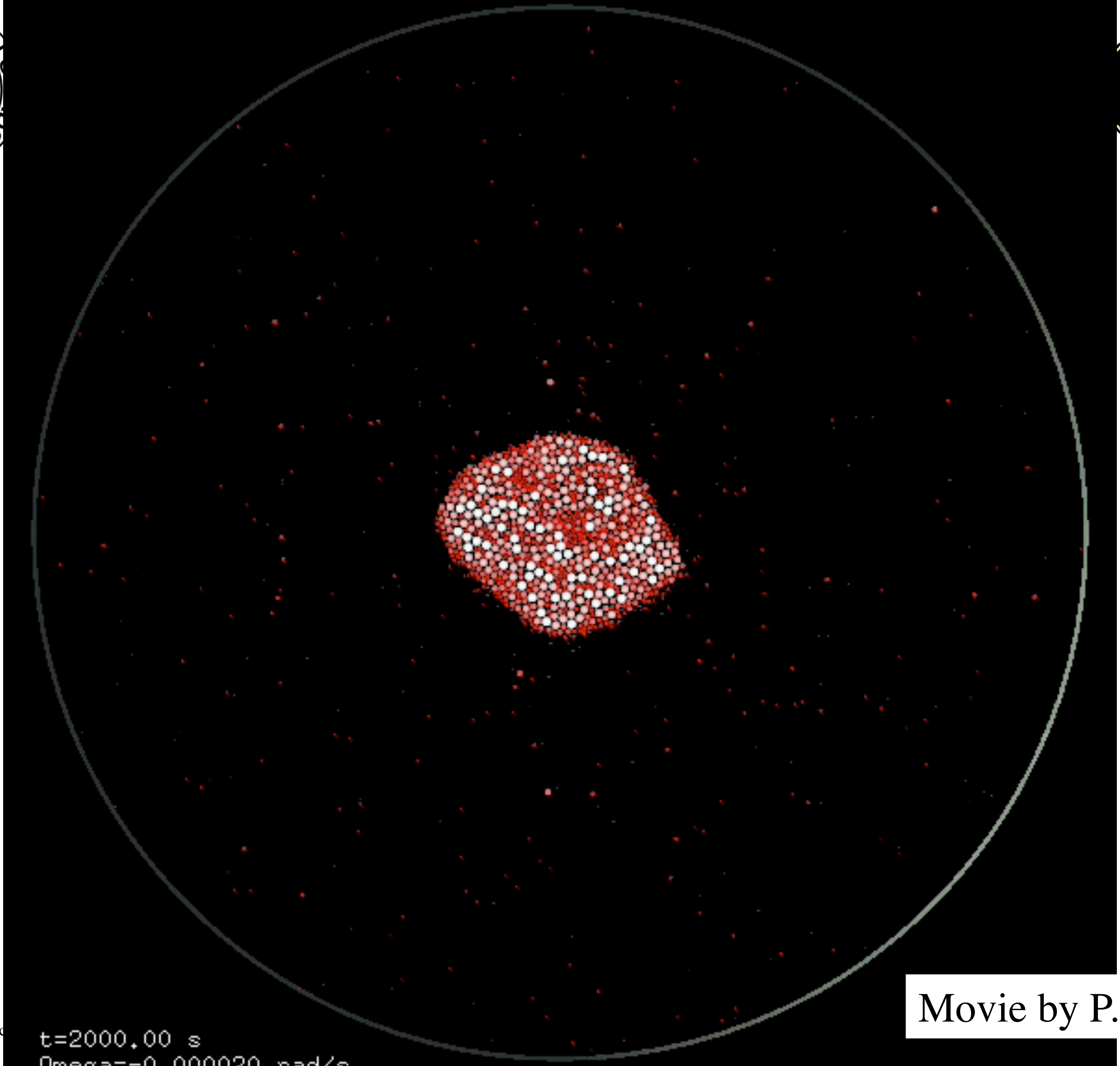
More complex for non-equal mass bodies

Summary

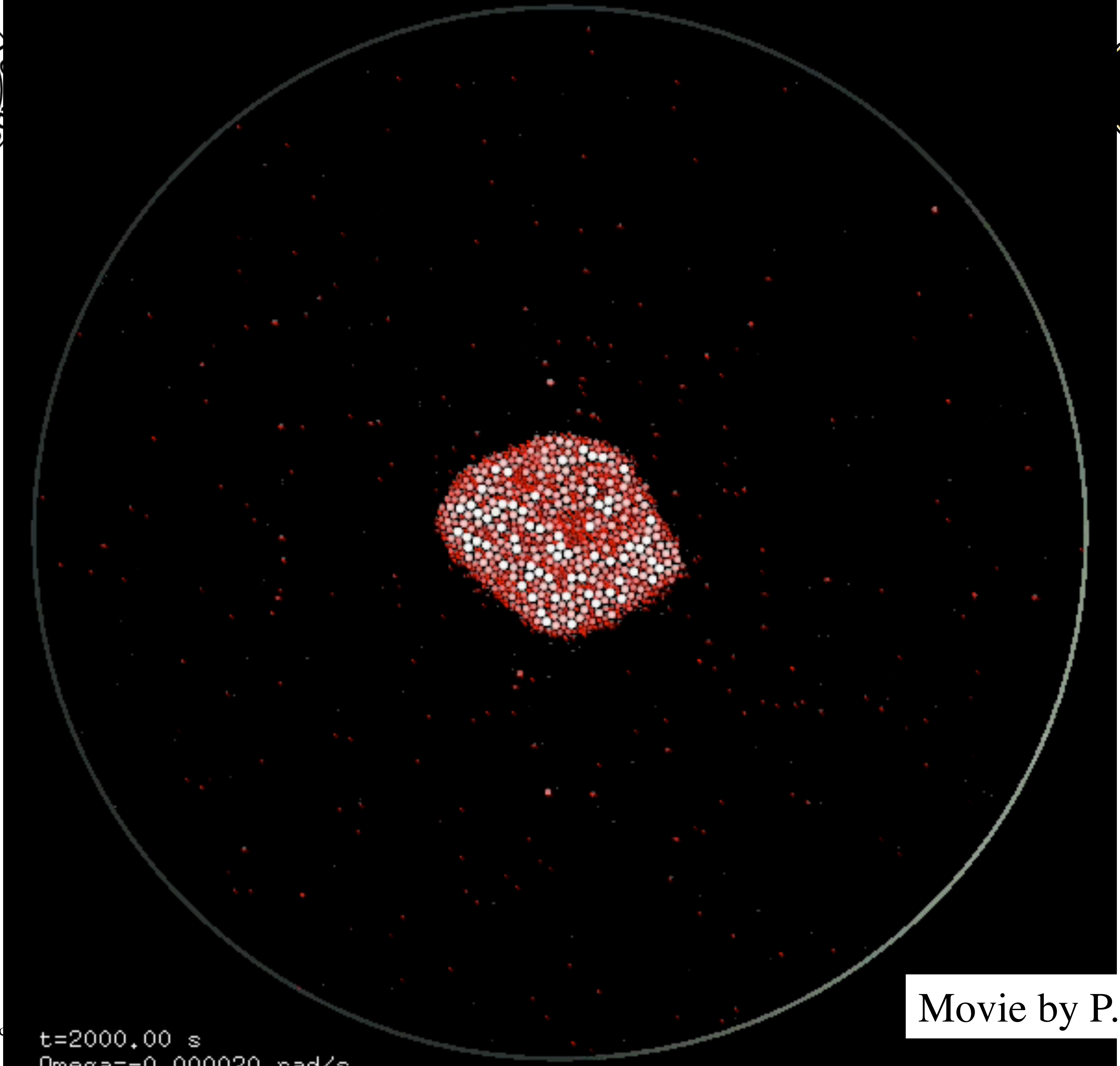
- Study of asteroids leads directly to study of minimum energy configurations in Celestial Mechanics in granular mechanics-types of situations
  – Only possible for bodies with finite density
- For finite density bodies, minimum energy and stable configurations are defined as a function of angular momentum by studying the amended potential:

$$\mathcal{E} = \frac{H^2}{2I_H} + \mathcal{U}$$

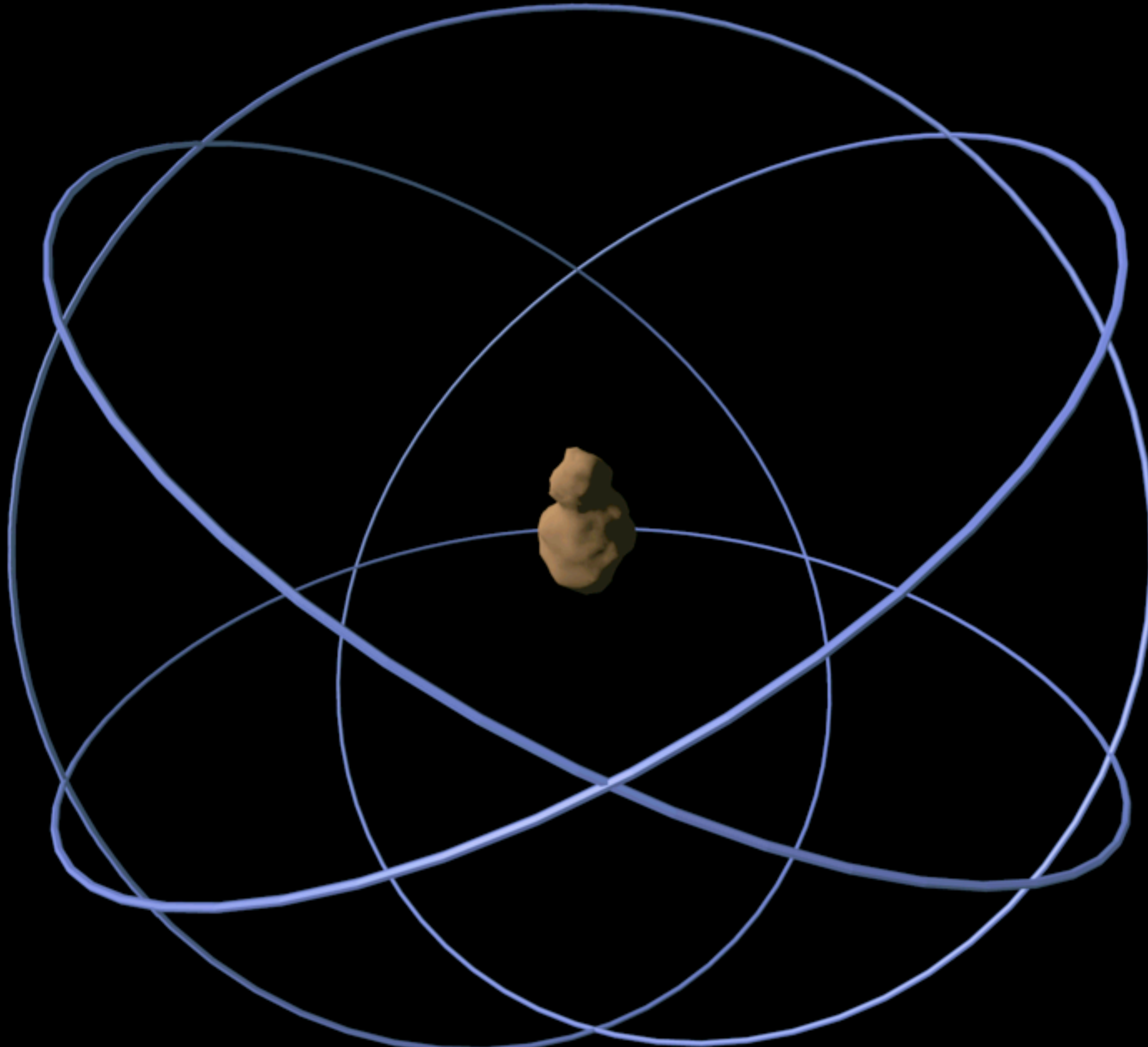
  – only a function of the internal, relative system configuration
  – Globally minimum energy configurations seem to be denumerable
- Simple few body systems can be fully explored
  – Need theories for polydisperse grains and *N >> 1*



Movie by P. Sanchez



Movie by P. Sanchez



Minimum Energy Configurations in the N-body Problem - Talk by Daniel Scheeres

Lecture notes (Ori S. Katz)

October 11, 2018

Abstract

Celestial Mechanics systems have two fundamental conservation principles: conservation of momentum and conservation of (mechanical) energy. Of the two, conservation of momentum provides the most constraints on a general system, with three translational symmetries (which can be trivially removed) and three rotational symmetries. If no external force acts on the system, these quantities are always conserved independent of the internal interactions of the system. In contrast, conservation of energy involves assumptions on both the lack of exogenous forces and on the nature of internal interactions within the system. For this reason energy is often not conserved for "real" systems that involve internal interactions, such as tidal deformations or impacts, even though such systems conserve total momentum. Thus, mechanical energy generally decays through dissipation until the system has found a local or global minimum energy configuration that corresponds to its constant level of angular momentum. This observation motivates a fundamental question for celestial mechanics: What is the minimum energy configuration of a N-body system with a fixed level of angular momentum? It can be shown that this is an ill-defined question and has no answer for traditional point-mass celestial mechanics systems. If, instead, the system and problem are formulated accounting for finite density distributions this question becomes well posed and we can prove the existence of minimum energy configurations for all suitably formulated celestial mechanics systems (Scheeres, CMDA 113(3): 291-320, 2012). This small change also leads to fundamental changes in the nature and stability properties of relative equilibria and, ultimately, the dynamics of these systems. Finally, we can also show that this naturally leads to a "granular mechanics" extension of celestial mechanics, with fundamental links between this topic and the science of small solar system bodies.

1 Lecture notes

We're interested in the mechanics of small asteroids.

Stepping back, asteroids look like self-gravitating rocks. We can assume they are cohesionless collections of rocks resting on each other.

What is the expected configuration? How will they reconfigure themselves if the angular momentum changes with time, on account of, for example, sunlight inserting torque into the system? This would account for a very slow and gradual change of angular momentum.

These asteroids form into multi-component systems, for example binary asteroids.

Mathematical challenge - develop a realistic model for such systems. The approach is the N-body problem of gravitationally interacting rigid bodies. Must account for finite density, allowing for contact mechanics and minimum energy configurations.

The goal is to track the stable system configurations as a function of angular momentum to identify transitions, and put rigorous constraints on them.

Background: celestial mechanics of point mass bodies. If we force this system to dissipate energy, it will at some point violate some of its conservation constraints.

Celestial mechanics of finite density bodies - the system inherits the N-body conservational principles, but are subject to surface and tidal forcing, enabling a natural energy dissipation mechanism. This allows for resting equilibria to exist. It is important to treat this carefully to get a physical model as the momenta themselves deviate from initial values.

The Lagrangian can be expressed in terms of the potential energy $\mathcal{U}$, the kinetic energy $\mathcal{T}$ and the total angular momentum $\mathcal{H}$.

Then, the system can be reduced by transforming to a frame rotating with the total angular momentum. Due to AM conservation the angle itself is ignorable, and Routh reduction yields an amended potential, that can be generalized to account for non-holonomic interactions, for example if the bodies roll on each other.

Obtain the amended potential

$$\mathcal{E} = \frac{H^2}{2I_H} + \mathcal{U},$$

where I_H is the instantaneous moment of inertia of the entire system.

This is distinct from the usual amended potential $\mathcal{E}' = \frac{1}{2}\mathcal{H} \cdot I^{-1} \cdot \mathcal{H} + \mathcal{U}$, but produces the same results.

The full N-body problem - using the amended potential.

The total energy of the system is $E = T + \mathcal{E}$, so $\mathcal{E}(Q) \leq E$. The relative equilibria and their stability are analyzed through $\mathcal{E}$ as a function of a minimal coordinate set.

Differences between this and a point mass relative equilibrium: The moment of inertia I_H is different, as is the potential energy. The stability condition is not used generally in celestial mechanics because it is never satisfied for $N \geq 3$.

The finite-density relative equilibria - the equilibrium has two different forms, unconstrained and constrained. Thus, for the equilibrium to exist in the constrained case, the perturbation must be positive. In the finite-density case, given dissipation the system will always approach a relative equilibrium.

Finite density properties - the energy density is compact and bounded, therefore minimum energy configurations exist for any N. Basically, at one point the masses touch each other and we get to the bottom of the well.

Minimum energy configurations of the spherical full body problem - Gravitational potential remains the same as point masses, but we have to account for the moment of inertia for each sphere. Even for the two-body problem, this dramatically changes the structure of minimum energy configurations.

Thus, very different behaviors between the point mass case and the finite density case for the 2-body problem. In the point mass case, there is only one stationary orbit energy configuration. In the finite density case, if the angular momentum is too low, there are no orbits and the bodies rest on each other. For high enough angular momentum energy configurations - a bifurcation to 2 stationary orbit energy configurations per AM value.

Assuming the planar problem, constant density spheres, and normalizing the system so that the sum of radii and the sum of masses are 1, the configuration space can be described by a diagram - $m_1 + m_2 + m_3 = 1$ so every point in the diagram corresponds to a different configuration. We can restrict all cases to the lower half triangle in which $m_1 \geq m_2 \geq m_3$.

Planar problem - get an additional 23 planar equilibria to the classical five equilibria from the three body point mass problem - the Euler and Lagrage sets. The equilibria can be looked at as a function of unconstrained degrees of freedom - the new equilibria arise from the fact that as soon as we let two of these bodies touch each other, we reduce the number of degrees of freedom and change the structure and number of unconstrained degrees of freedom.

Overall - 28 equilibria for planar 3 body. None of them are stable for any value of AM, no matter what the masses and mass ratios are. But some are stable for certain sets of AM values and mass ratios. The five point-mass equilibria are still unstable.

Transitions from a stable relative equilibria to unstable relative equilibria - interesting because this relates to energy dissipation resulting in a perhaps stable state transitioning to an unstable equilibrium in a new energy regime in which additional stable equilibria exist.

Increasing AM can cause a transition from stable to unstable and vice versa, depending on configuration.

Eventually obtain detailed bifurcation charts. As the angular momentum goes to infinity, there are stable configurations.

Recalling initial trangle -the axis lines are $m_1 = m_2$, $m_2 = m_3$, $m_3 = m_1$. The green line marks the resting central configuration.

What happens at a bifurcation point where a resting configuration no longer exists? There is a stable relative equilibrium that still exists.

Movies - Right - rotating with center of mass. Can teach us about formation of asteroids.

Diagram - unstable and stable ER 132 sequence - always a minimum energy configuration. Along the small green line, stays stable.

Equal densities - if there are different densities the diagrams change a lot and there are more things that can be studied.

Can do the same analysis for all types of equilibria.

The bifurcation framework uses the energy-angular momentum $\mathcal{E} - \mathcal{H}^2$ diagram - $\mathcal{H}^2$ because then the lines of changing configurations are linear. We can use these diagrams to depict what happens to stability of different configurations. Transitioning between stable types the straight line breaks and transitions to a different line, and there is some excess energy that manifests as complex dynamics.

In the direction of reducing the angular momentum, when the system transitions from one stable type to another, the energy collapses down and the system dissipates energy through some mechanism, for example tidal forces..

So this can be used to trace out how the systems form around each other. Movie - at excess energy point.

Finite density 4-body problem - more tiresome.

They are all energetically unstable. ??

What happens after fission - after the systems break apart and start rotating around each other?

2-component rotational fission can be a smooth transition.

Free energy - total energy at which bodies split minus self potentials.

If disruption occurs and the bodies separate, the mutual potential goes to 0. Thus the sign of E_{free} determines whether or not the system can "catastrophically disrupt". So disruption vs. equilibrium depends on rate of energy dissipation.

This is used practically by looking at measurements of asteroids that were close to each other some time ago.

Different diagrams for different configurations.

Fission and escape for N bodies.

2 Questions

- If you have 2 bodies with more complicated shapes, how would you calculate the potential?

For 2 polyhedra, for example, there are no closed form formulas, but can be calculated up to any order, and since any shape can be approximated by polyhedra then any shape can be considered.

- Why are there minimum energy configurations for 2 point masses?

In the reduced case, fixing angular momentum limits the distance between the two point masses.

- From the last movie with the grains, how do we model that?

Realm of granular mechanics. Paul Sanchez is the expert in simulating granular materials. There are several different ways, the method in the movie is a soft sphere method - hits are accompanied by dissipation and spring energy - this is a very realistic model. However it is hard to get a good explanation of what happens in these large scale systems, and they take a long time to compute.

- In the many body case, do we see the phenomena and collective effects and could we come up with a mean field theory, like you would do in stellar dynamics - statistical approach?

These sorts of approaches could be possible. Could calculate which areas in phase space are more densely populated. One example - asteroids that integrated backwards come together to the same area, this is indicative of a many body configuration.

- Are there examples of triple asteroids in the solar system?

Lots of binary asteroids - 50 %. A much smaller - but still exist - 3 asteroids. A few 4- and 5-asteroid systems are known.