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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: ORI KATZ **Email/Phone:** ORI KATZ.OK@gmail.com

Speaker's Name: Joshua Burby

Talk Title: Dark Plasma

Date: 10 / 11 / 18 **Time:** 2:00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Burby argues for existence of plasmas that are "dark" in the sense that their coherent emission is extremely weak. Dark plasma motions are identified with a slow manifold in the Vlasov-Maxwell phase space. In the case where collisions are extremely rare, Burby gives a complete description of the Hamiltonian formulation of dynamics on the slow manifold.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - → **Computer Presentations**: Obtain a copy of their presentation
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Dark plasma

J. W. Burby and G. Miloshevich
(MSRI)

October 11th, 2018
Hamiltonian systems, from topology to applications through analysis: I
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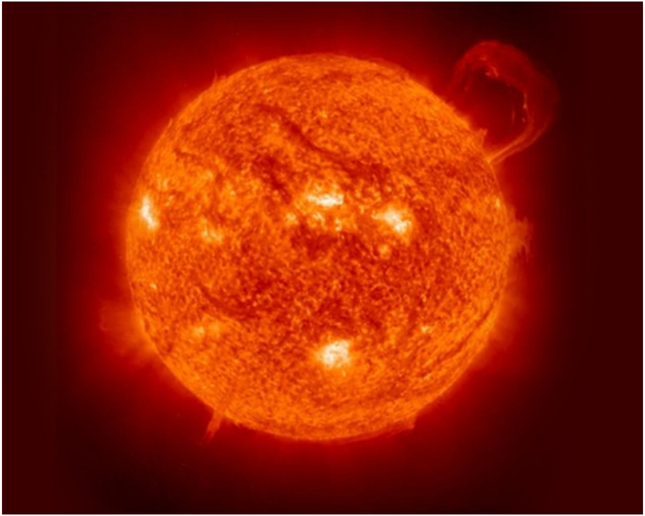
Plasmas are ionized gases.



Plasmas are ionized gases.

Weakly-coupled plasmas are especially common:

Weakly-coupled plasmas satisfy

$$\Lambda_p = n\lambda_D^3 \gg 1,$$

where the *Debye length* is

$$\Lambda_D = \left(\frac{T}{4\pi e^2 n}\right)^{1/2}.$$

Most plasmas emit light.

Mean-field radiation:

As $1/\Lambda_p \to 0$, plasma behaves as continuous medium $\Rightarrow$ *mean-field radiation* driven by

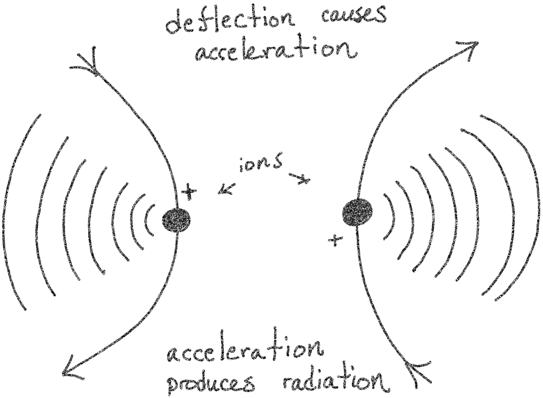
$$\boldsymbol{J}(\boldsymbol{x}, t) = \sum_{\sigma} en(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t)$$
$$\rho(\boldsymbol{x}, t) = \sum_{\sigma} en(\boldsymbol{x}, t),$$

where $n, \boldsymbol{u}$ are local number density and fluid velocity.

Most plasmas emit light.

Bremsstrahlung radiation:

Strongest discrete particle effect contributing to radiation is *Bremsstrahlung radiation*.



But some don't!

The purpose of this talk

(1) Argue that there are dark plasmas

(2) Describe basic features of dark plasmas

Outline for rest of talk

Part I. Why are there dark plasmas?

- collisionless plasma dynamics as fast-slow system
- dark plasma as motion on the slow manifold

Part II. Properties of dark plasma

- low-order truncations of slow dynamics
- dark plasma as a Hamiltonian system

Part I: Why are there dark plasmas?

Bremsstrahlung is weak in rarefied, hot plasmas.

Collision frequency in weakly-coupled plasmas:

Two-body collisions occur at a rate

$$\nu = \frac{\ln \Lambda_p}{\Lambda_p}\omega_p \propto \frac{n}{T^{3/2}},$$

where the plasma frequency is

$$\omega_p^2 = \frac{4\pi e^2 n}{m}$$

But are there collisionless plasmas without mean-field radiation?

Fundamental collisionless plasma model: Vlasov-Maxwell system

$$\partial_t f + \nabla \cdot (\boldsymbol{v} f) + \nabla_{\boldsymbol{p}} \cdot (e[\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{B}]f) = 0$$

$$c^{-1}\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}$$

$$c^{-1}\partial_t \boldsymbol{E} = \nabla \times \boldsymbol{B} - 4\pi c^{-1}(\boldsymbol{J} - \boldsymbol{J}_H)$$

But are there collisionless plasmas without mean-field radiation?

Fundamental collisionless plasma model: Vlasov-Maxwell system

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{m\sqrt{1+|\boldsymbol{p}|^2/(mc)^2}} = \frac{\boldsymbol{p}}{\gamma m}$$

$$\boldsymbol{J} = \sum_\sigma e \int \boldsymbol{v} f \, d^3\boldsymbol{p}$$

$$\rho = \sum_\sigma e \int f \, d^3\boldsymbol{p}$$

But are there collisionless plasmas without mean-field radiation?

Fundamental collisionless plasma model: Vlasov-Maxwell system

- $Q \approx S^1 \times S^1 \times S^1$: spatial domain
- $\boldsymbol{E} \in \Omega^1_T \oplus \Omega^1_L$: vector field on Q in $H(\mathrm{curl})$ with $\boldsymbol{E}_H = \int_Q \boldsymbol{E}\, d^3\boldsymbol{x} / \int_Q d^3\boldsymbol{x} = 0$
- $\boldsymbol{B} \in \Omega^2_T$: vector field in $\mathrm{curl}(H(\mathrm{curl}))$
- $f \in L^1(T^*Q)$: integrable function on $T^*Q \approx Q \times \mathbb{R}^3$
- c: speed of light in vacuum

But are there collisionless plasmas without mean-field radiation?

Hodge decomposition of $H(\mathrm{curl})$:

There is an L^2-orthogonal direct-sum decomposition

$$H(\mathrm{curl}) = \Omega^1_T \oplus \Omega^1_L \oplus \Omega^1_H$$
$$\boldsymbol{u} \in H(\mathrm{curl}) = \boldsymbol{u}_T + \boldsymbol{u}_L + \boldsymbol{u}_H$$

where

$$\Omega^1_T = \mathrm{curl}^\dagger(H(\mathrm{div}))$$
$$\Omega^1_L = \mathrm{grad}(H^1)$$
$$\Omega^1_H = \ker(\Delta)$$

There are!

Here's why:

The argument comes from dynamical systems theory.

1. For small c^{-1}, Vlasov-Maxwell is a fast-slow system

   $\Rightarrow$ There is a slow manifold in VM phase space

2. Dynamics on slow manifold does not contain light waves

   $\Rightarrow$ Dynamics on slow manifold is dark!

Rudiments of fast-slow systems theory

What is a fast-slow system?

Definition: (Fast-slow dynamical system)

Let X, Y be Banach spaces and $\epsilon \ll 1$. A fast-slow dynamical system is an ODE on $X \times Y$ of the form

$$\begin{aligned} \epsilon \dot{y} &= f(x, y) \\ \dot{x} &= g(x, y) \end{aligned}$$

with $D_y f(x, y) : Y \to Y$ an isomorphism when $(x, y) \in f^{-1}(\{0\})$.

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Convention:

y is called the fast variable, x the slow variable.

Invariant manifolds parameterized by slow variables satisfy a non-linear PDE

Lemma 1: (invariance equation)

Suppose a fast-slow dynamical system admits an invariant manifold S_ϵ of the form

$$S_\epsilon = \{(x, y) \in X \times Y \mid y = y_\epsilon^*(x)\}$$

for some smooth map $y_\epsilon^* : X \to Y$. Then the *invariance equation*

$$\epsilon D y_\epsilon^*(x)[g(x, y_\epsilon^*(x))] = f(x, y_\epsilon^*(x)),$$

holds for each $x \in X$.

Slow manifolds are formal power series solutions of invariance equation

Definition: (slaving function)

A *slaving function* $y_\epsilon^* = y_0^* + \epsilon y_1^* + \epsilon^2 y_2^* + \dots$ is a formal power series solution of the invariance equation.

Slow manifolds are formal power series solutions of invariance equation

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Definition: (slow manifold)

Let y_ϵ^* be a slaving function. An n'th-order *slow manifold* is a submanifold $S_\epsilon^{(n)} \subset X \times Y$ of the form

$$S_\epsilon^{(n)} = \{(x, y) \mid y = (y_0^* + \dots + \epsilon^n y_n)(x)\}.$$

Slow manifolds are formal power series solutions of invariance equation

Definition: (slaving function)

A *slaving function* $y^*_\epsilon = y^*_0 + \epsilon y^*_1 + \epsilon^2 y^*_2 + \dots$ is a formal power series solution of the invariance equation.

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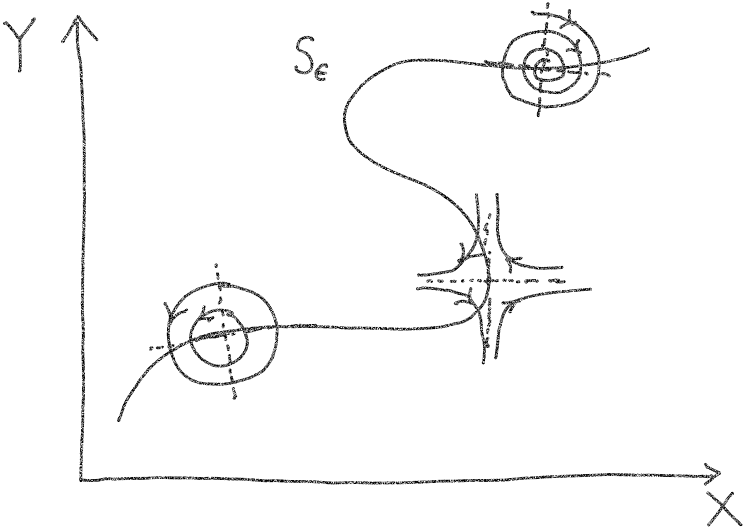
$$S^{(n)}_\epsilon = \{(x, y) \mid y = (y^*_0 + \cdots + \epsilon^n y_n)(x)\}.$$

Definition: (slow system)

Given a fast-slow system and a slaving function y^*_ϵ, the associated *slow system* is the formal power series $\dot{x}^*_\epsilon = g(x, y^*_\epsilon(x))$.

Slow manifolds are formal power series solutions of invariance equation



There is at most one slaving function

Theorem 2:

If $y_\epsilon^* = y_0^* + \epsilon y_1^* + \epsilon^2 y_2^* + \dots$ is a slaving function, then the coefficients y_k^* are unique. Moreover, they may be explicitly computed using:

$$
\begin{aligned}
0 &= f(x, y_0^*(x)) \\
y_1^*(x) &= (D_y f(x, y_0^*(x)))^{-1} \Big[D y_0^*(x)[g(x, y_0^*(x))] \Big] \\
y_2^*(x) &= (D_y f(x, y_0^*(x)))^{-1} \Big[D y_1^*(x)[g(x, y_0^*(x))] \\
&+ D y_0^*(x) \Big[D_y g(x, y_0^*(x))[y_1^*(x)] \Big] - \frac{1}{2} D_y^2 f(x, y_0^*(x))[y_1^*(x), y_1^*(x)] \Big] \\
y_3^*(x) &= (D_y f(x, y_0^*(x)))^{-1} [\dots] \\
&\dots
\end{aligned}
$$

Vlasov-Maxwell
as a fast-slow system

Small parameter is $\epsilon = c^{-1}$.

Scaled Vlasov-Maxwell system

$$\partial_t f + \nabla \cdot (\boldsymbol{v} f) + \nabla_{\boldsymbol{p}} \cdot (e[\boldsymbol{E} + c^{-1} \boldsymbol{v} \times \boldsymbol{B}] f) = 0$$

$$c^{-1} \partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}$$

$$c^{-1} \partial_t \boldsymbol{E} = \nabla \times \boldsymbol{B} - 4\pi c^{-1} (\boldsymbol{J} - \boldsymbol{J}_H)$$

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$$\epsilon \partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}$$
$$\epsilon \partial_t \boldsymbol{E} = \nabla \times \boldsymbol{B} - 4\pi\epsilon(\boldsymbol{J}_\epsilon - (\boldsymbol{J}_\epsilon)_H)$$

Slow variables are $\boldsymbol{E}_L$ and f.

Definition: (slow variables for VM)

The space of slow variables is

$$X = L^1(T^*Q) \times \Omega^1_L \ni (f, \boldsymbol{E}_L)$$

Fast variables are E_T and B.

Definition: (fast variables for VM)

The space of fast variables is

$$Y = \Omega_T^1 \times \Omega_T^2 \ni (\boldsymbol{E}_T, \boldsymbol{B})$$

Weakly-relativistic VM is a fast-slow system on $X \times Y$.

VM as a fast-slow system

Fast variable evolution equations ($\epsilon \dot{y} = f(x, y)$):

$$\epsilon \partial_t \boldsymbol{E}_T = \nabla \times \boldsymbol{B} - 4\pi\epsilon(\boldsymbol{J}_\epsilon)_T$$
$$\epsilon \partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}_T$$

Slow variable evolution equations ($\dot{x} = g(x, y)$):

$$\partial_t f = -\nabla \cdot (\boldsymbol{v}_\epsilon f) - \nabla_{\boldsymbol{p}} \cdot (e[\boldsymbol{E} + \epsilon \boldsymbol{v}_\epsilon \times \boldsymbol{B}] f)$$
$$\partial_t \boldsymbol{E}_L = -4\pi(\boldsymbol{J}_\epsilon)_L$$

Weakly-relativistic VM is a fast-slow system on $X \times Y$.

Lemma:

Set

$$\begin{aligned} \delta y &= (\delta \boldsymbol{E}_T, \delta \boldsymbol{B}) \in \Omega^1_T \times \Omega^2_T \\ C &= \mathrm{curl} \\ G &= (\Delta \mid \Omega^1_T \oplus \Omega^1_L)^{-1}. \end{aligned}$$

We have

$$D_y f_0(x,y)[\delta y] = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix} \begin{pmatrix} \delta \boldsymbol{E}_T \\ \delta \boldsymbol{B} \end{pmatrix}$$

$$(D_y f_0(x,y))^{-1}[\delta y] = \begin{pmatrix} 0 & GC \\ -GC & 0 \end{pmatrix} \begin{pmatrix} \delta \boldsymbol{E}_T \\ \delta \boldsymbol{B} \end{pmatrix}$$

In fast time $\tau = t/\epsilon$, leading-order dynamics is light-wave propagation.

Leading-order dynamics in fast time:

Fast variable evolution equations:

$$\partial_\tau \boldsymbol{E}_T = \nabla \times \boldsymbol{B}$$
$$\partial_\tau \boldsymbol{B} = -\nabla \times \boldsymbol{E}_T$$

Slow variable evolution equations:

$$\partial_\tau f = 0$$
$$\partial_\tau \boldsymbol{E}_L = 0$$

In fast time $\tau = t/\epsilon$, leading-order dynamics is light-wave propagation.

Leading-order dynamics in fast time:

Fast variable evolution equations:

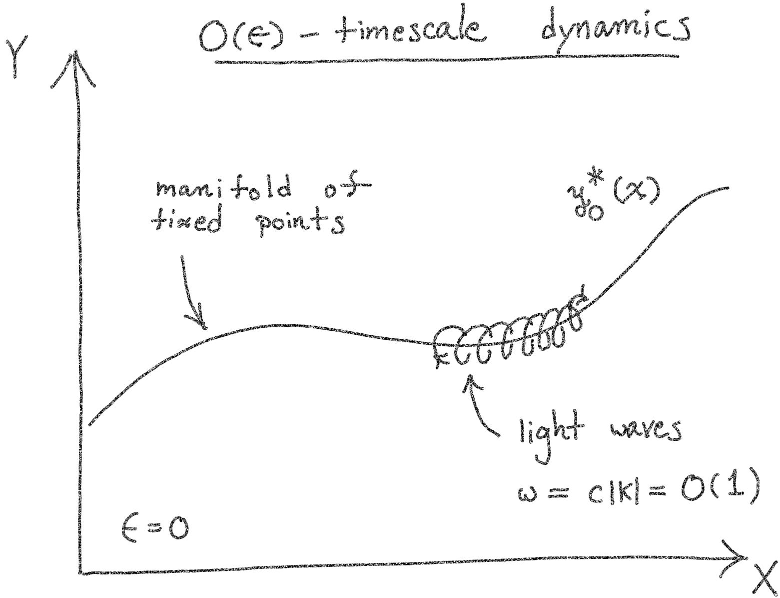
$$\partial_\tau \boldsymbol{E}_T = \nabla \times \boldsymbol{B}$$
$$\partial_\tau \boldsymbol{B} = -\nabla \times \boldsymbol{E}_T$$

Slow variable evolution equations:

$$\partial_\tau f = 0$$
$$\partial_\tau \boldsymbol{E}_L = 0$$

Vacuum light wave propagation!

Therefore dynamics on slow manifold is dimly lit.



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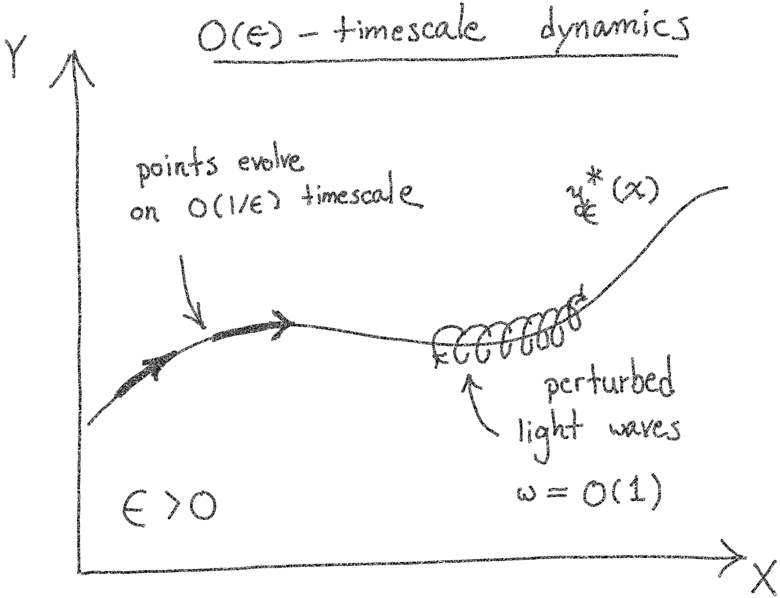


Part II: Properties of dark plasma

I will describe two features of dark plasma dynamics

(1) Low-order approximations of dark plasma dynamics

(2) Dark plasma dynamics is Hamiltonian

(1) Low-order approximations of dark plasma dynamics

(2) Dark plasma dynamics is Hamiltonian

Leading-order dark motions governed by Vlasov-Poisson

Theorem: (leading-order slaving function)

The 0'th-order coefficient in the slaving function $y^*_\epsilon = (\boldsymbol{E}^*_{T\epsilon}, \boldsymbol{B}^*_\epsilon)$ is given by

$$\boldsymbol{E}^*_{T0}(x) = 0$$
$$\boldsymbol{B}^*_0(x) = 0$$

where $x = (f, \boldsymbol{E}_L) \in L^1(T^*Q) \times \Omega^1_L$.

Leading-order dark motions governed by Vlasov-Poisson

Theorem: (leading-order slow system)

The leading-order coefficient in the slow system $\dot{x}_\epsilon^* = (\dot{f}_\epsilon^*, \dot{\boldsymbol{E}}_{L\epsilon}^*)$ is given by

$$\dot{f}_0^*(x) = -\nabla \cdot (\boldsymbol{v}_0 f) - \underbrace{\nabla_{\boldsymbol{p}} \cdot (e\boldsymbol{E}_L f)}_{\text{Coulomb force}} \tag{1}$$

$$\dot{\boldsymbol{E}}_{L0}^*(x) = -4\pi \Pi_L \sum_\sigma e \int \boldsymbol{v}_0 f \, d^3\boldsymbol{p} \tag{2}$$

where $\boldsymbol{v}_0 = \boldsymbol{p}/m$ and $\Pi_L : H(\text{curl}) \to \Omega_L^1$ is the L^2-orthogonal projection.

Leading-order dark motions governed by Vlasov-Poisson

Connection with the Poisson equation:

By leading-order Vlasov equation (1)

$$\partial_t \rho + \nabla \cdot \boldsymbol{J}_0 = 0.$$

Therefore the divergence of leading-order Ampére equation (2) implies

$$\begin{aligned} & \partial_t \left(\nabla \cdot \boldsymbol{E}_L - 4\pi\rho \right) = 0 \\ \Leftrightarrow \quad & \nabla \cdot \boldsymbol{E}_L = 4\pi(\rho - \rho_H) + 4\pi\rho_{\mathrm{ext}}, \end{aligned}$$

where ρ_{ext} is arbitrary time-independent function with $(\rho_{\mathrm{ext}})_H = 0$.

First-order slaving function captures magnetostatic fields

Theorem: (first-order slaving function)

The 1'st-order coefficient in the slaving function $y_\epsilon^* = (\boldsymbol{E}_{T\epsilon}^*, \boldsymbol{B}_\epsilon^*)$ is given by

$$\boldsymbol{E}_{T1}^*(x) = 0$$

$$\boldsymbol{B}_1^*(x) = -4\pi G C \sum_\sigma e \int \boldsymbol{v}_0 f \, d^3\boldsymbol{p}$$

where $x = (f, \boldsymbol{E}_L) \in L^1(T^*Q) \times \Omega_L^1$.

First-order slaving function captures magnetostatic fields

Theorem: (first-order slow system)

The 1'st-order coefficient in the slow system $\dot{x}^*_\epsilon = (\dot{f}^*_\epsilon, \dot{\boldsymbol{E}}^*_{L\epsilon})$ is given by

$$\dot{f}^*_1(x) = 0 \tag{3}$$

$$\dot{\boldsymbol{E}}^*_{L0}(x) = 0. \tag{4}$$

First-order slaving function captures magnetostatic fields

Connection with Biot-Savart:

The leading-order contribution to the magnetic field $\boldsymbol{B}^*_\epsilon = \epsilon \boldsymbol{B}^*_1 + O(\epsilon^2)$ satisfies the *magnetostatic equation*:

$$\nabla \times \boldsymbol{B}^*_1 = 4\pi (\boldsymbol{J}_0)_T.$$

If our spatial domain was $Q = \mathbb{R}^3$ instead of $Q = T^3$, then $\boldsymbol{B}^*_1$ would be given by the Biot-Savart law:

$$\boldsymbol{B}^*_1(\boldsymbol{r}) = \int_{\mathbb{R}^3} \frac{(\boldsymbol{J}_0)_T(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\, d^3\boldsymbol{r}'$$

Second-order slaving leads to a piezoelectric field.

Theorem: (second-order slaving function)

The 2'nd-order coefficient of the slaving function $y_\epsilon^* = (\boldsymbol{E}_{T\epsilon}^*, \boldsymbol{B}_\epsilon^*)$ is given by

$$\boldsymbol{E}_{T2}^*(x) = G\Pi_T\omega_p^2\boldsymbol{E}_L - \underbrace{4\pi G\Pi_T\sum_\sigma e\nabla\cdot\mathbb{T}_0}_{\text{piezoelectric field}}$$

$$\boldsymbol{B}_2^*(x) = 0$$

where the stress tensor $\mathbb{T}_0$ is given by

$$\mathbb{T}_0 = \int \boldsymbol{v}_0\boldsymbol{v}_0\, f\, d^3\boldsymbol{p}.$$

Second-order slaving leads to a piezoelectric field.

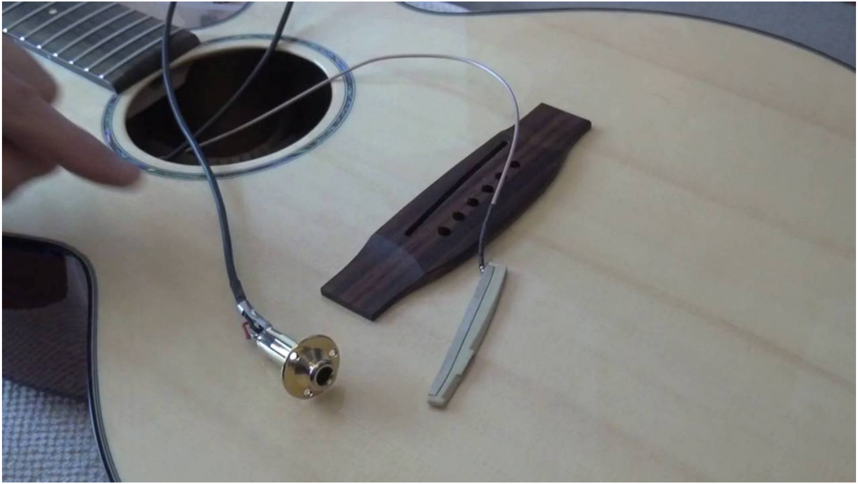


Figure: Piezoelectric acoustic guitar pickup

Second-order slow dynamics gives Darwin's correction to Coulomb force

Theorem: (second-order slow system)

The 2'nd-order coefficient in the slow system $\dot{x}^*_\epsilon = (\dot{f}^*_\epsilon, \dot{\boldsymbol{E}}^*_{L\epsilon})$ is given by

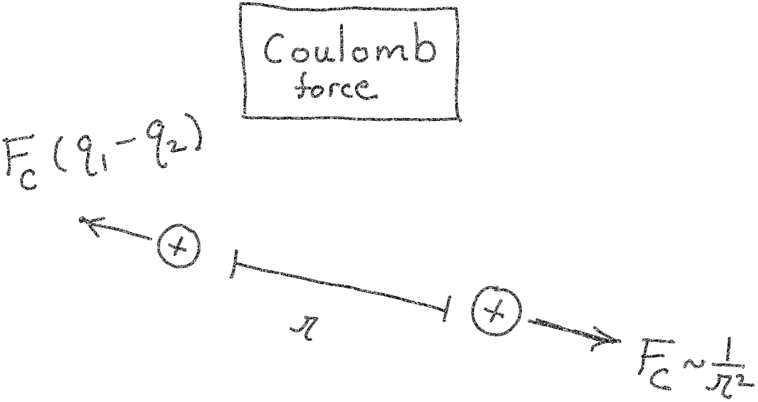
$$\dot{f}^*_2(x) = -\nabla \cdot (\boldsymbol{v}_2 f) - \underbrace{\nabla_{\boldsymbol{p}} \cdot (e[\boldsymbol{E}^*_{T2} + \boldsymbol{v}_0 \times \boldsymbol{B}^*_1] f)}_{\text{Darwin's correction to Coulomb}} \tag{5}$$

$$\dot{\boldsymbol{E}}^*_{L2}(x) = -4\pi \sum_\sigma e \int \boldsymbol{v}_2 \, f \, d^3\boldsymbol{p}. \tag{6}$$

where

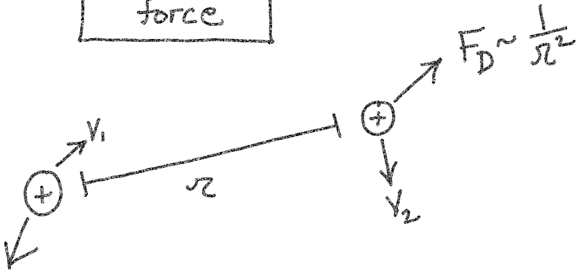
$$\boldsymbol{v}_2 = -\frac{1}{2}\frac{|\boldsymbol{p}|^2}{m^2}\frac{\boldsymbol{p}}{m}$$

Second-order slow dynamics gives Darwin's correction to Coulomb force



Second-order slow dynamics gives Darwin's correction to Coulomb force

Darwin force



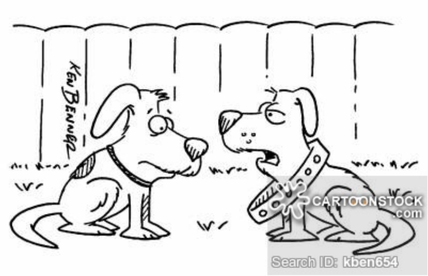
$F_D(q_1 - q_2, v_1, v_2)$

I will describe two features of dark plasma dynamics

(1) Low-order approximations of dark plasma dynamics

(2) Dark plasma dynamics is Hamiltonian

They receive hand-me-downs from their big brother

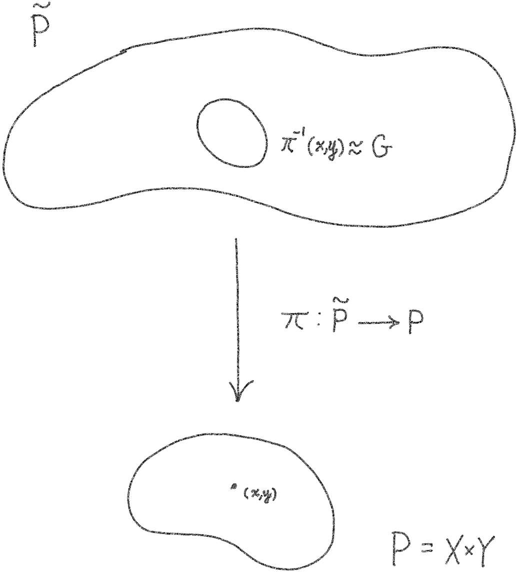


"I'm tired of wearing my brother's hand me downs."

Why is dark plasma dynamics Hamiltonian?

- $P = X \times Y$ is equal to $\tilde{P}/G$

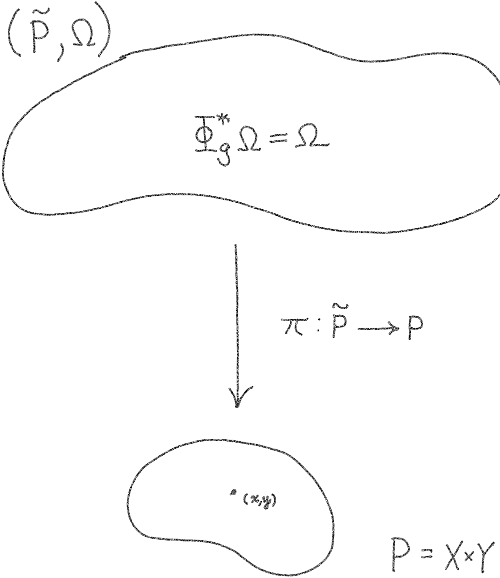
Why is dark plasma dynamics Hamiltonian?



Why is dark plasma dynamics Hamiltonian?

- $P = X \times Y$ is equal to $\tilde{P}/G$
- $\tilde{P}$ has a G-invariant symplectic form $\Omega = -\mathbf{d}\Theta$

Why is dark plasma dynamics Hamiltonian?



Why is dark plasma dynamics Hamiltonian?

- $P = X \times Y$ is equal to $\tilde{P}/G$
- $\tilde{P}$ has a G-invariant symplectic form $\Omega = -\mathbf{d}\Theta$
- VM dynamics on P lifts to G-invariant Hamiltonian dynamics on $\tilde{P}$

Why is dark plasma dynamics Hamiltonian?

$$\tilde{F}_t \circ \Phi_g = \Phi_g \circ \tilde{F}_t$$

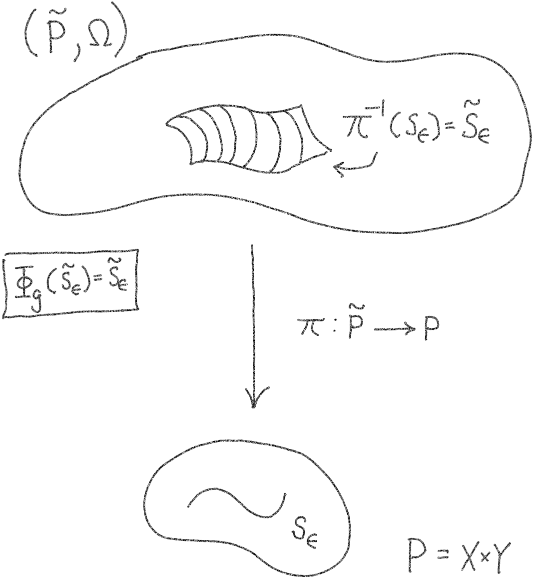
$$\tilde{F}_t^* \Omega = \Omega$$

$$\begin{tikzcd}
\tilde{P} \arrow[r, "\tilde{F}_t"] \arrow[d, "\pi"'] & \tilde{P} \arrow[d, "\pi"'] \\
P \arrow[r, "F_t"'] & P
\end{tikzcd}$$

Why is dark plasma dynamics Hamiltonian?

- $P = X \times Y$ is equal to $\tilde{P}/G$
- $\tilde{P}$ has a G-invariant symplectic form $\Omega = -\mathbf{d}\Theta$
- VM dynamics on P lifts to G-invariant Hamiltonian dynamics on $\tilde{P}$
- $S_\epsilon \subset P$ lifts to G-invariant invariant set $\tilde{S}_\epsilon \subset \tilde{P}$

Why is dark plasma dynamics Hamiltonian?



Why is dark plasma dynamics Hamiltonian?

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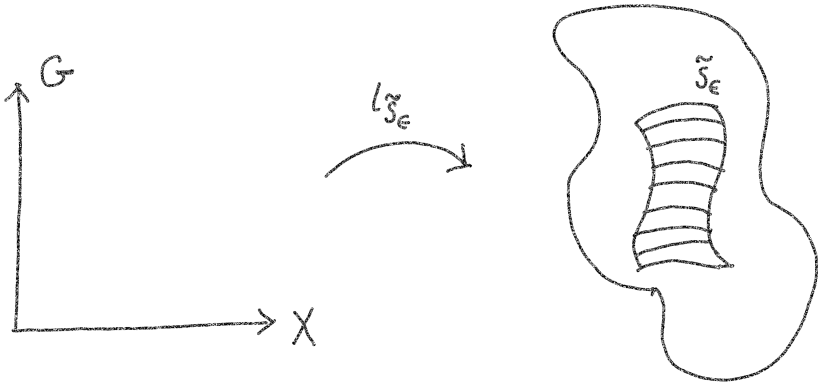
- $\tilde{P}$ has a G-invariant symplectic form $\Omega = -\mathbf{d}\Theta$
- VM dynamics on P lifts to G-invariant Hamiltonian dynamics on $\tilde{P}$
- $S_\epsilon \subset P$ lifts to G-invariant invariant set $\tilde{S}_\epsilon \subset \tilde{P}$
- $\tilde{S}_\epsilon$ is a symplectic submanifold in $(\tilde{P}, \Omega)$

Why is dark plasma dynamics Hamiltonian?

$(\tilde{S}_\epsilon, \iota_{\tilde{S}_\epsilon}^{*}\Omega)$ $(\tilde{P}, \Omega)$



$$T\tilde{P}|_{\tilde{S}_\epsilon} = T\tilde{S}_\epsilon \oplus T\tilde{S}_\epsilon^{\Omega}$$

Why is dark plasma dynamics Hamiltonian?

- $P = X \times Y$ is equal to $\tilde{P}/G$
- $\tilde{P}$ has a G-invariant symplectic form $\Omega = -\mathbf{d}\Theta$
- VM dynamics on P lifts to G-invariant Hamiltonian dynamics on $\tilde{P}$
- $S_\epsilon \subset P$ lifts to G-invariant invariant set $\tilde{S}_\epsilon \subset \tilde{P}$
- $\tilde{S}_\epsilon$ is a symplectic submanifold in $(\tilde{P}, \Omega)$

  $\Rightarrow$ $S_\epsilon = \tilde{S}_\epsilon/G$ is a Poisson manifold

$$
\begin{tikzcd}
\tilde{S}_\epsilon \arrow[d, "\pi"] & C^\infty(\tilde{S}_\epsilon)\times C^\infty(\tilde{S}_\epsilon) \arrow[r, "\{\cdot,\cdot\}_{\tilde{S}_\epsilon}"] & C^\infty(\tilde{S}_\epsilon) \\
S_\epsilon & C^\infty(S_\epsilon)\times C^\infty(S_\epsilon) \arrow[u, "\pi^*"] \arrow[r, "\{\cdot,\cdot\}_{S_\epsilon}"'] & C^\infty(S_\epsilon) \arrow[u, "\pi^*"']
\end{tikzcd}
$$

$$\{\pi^* C^\infty(S_\epsilon), \pi^* C^\infty(S_\epsilon)\}_{\tilde{S}_\epsilon} \subset \pi^* C^\infty(S_\epsilon)$$

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$$\begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{E}_T \\ \boldsymbol{E}_L \\ \boldsymbol{A}_T \\ \boldsymbol{A}_L \end{pmatrix} \cdot (\boldsymbol{\eta}, \psi) = \begin{pmatrix} \boldsymbol{g} \circ \boldsymbol{\eta} \\ \boldsymbol{E}_T \\ \boldsymbol{E}_L \\ \boldsymbol{A}_T \\ \boldsymbol{A}_L + \nabla \psi \end{pmatrix}$$

$$\text{Set } \delta Z_k = (\boldsymbol{\xi}_k, \delta \boldsymbol{E}_k, \delta \boldsymbol{A}_k) \in T\tilde{P}$$

What is the G-invariant symplectic form Ω?

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$$+ \frac{1}{4\pi c} \int (\delta \boldsymbol{E}_1 \cdot \delta \boldsymbol{A}_2 - \delta \boldsymbol{E}_2 \cdot \delta \boldsymbol{A}_1)\, d^3\boldsymbol{x}$$

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$$\omega_B = -\mathbf{d}\theta_B$$
$$\theta_B = \boldsymbol{p} \cdot d\boldsymbol{x} + \tfrac{e}{c}\boldsymbol{A} \cdot d\boldsymbol{x}$$

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J_0 : canonical Poisson tensor on T^*Q

Summary

I. Specially-prepared collisionless plasmas don't emit light
II. Dark plasmas produce piezoelectric fields
III. Dark plasma dynamics is Hamiltonian

Dark Plasma - Talk by Joshua Burby

Lecture notes (Ori S. Katz)

October 15, 2018

Abstract

Fully ionized plasmas emit both incoherent light and coherent light. Incoherent emission occurs whenever plasma particles suffer Coulomb collisions, and is therefore ubiquitous. On the other hand, coherent emission involves macroscopic collections of particles moving in concert. This incoherent emission is well described by the Vlasov-Maxwell system of equations. In this talk I will argue for the existence of plasmas that are "dark" in the sense that their coherent emission is extremely weak. The dark plasma motions will be identified with a slow manifold in the Vlasov-Maxwell phase space. In the case where collisions are extremely rare, I will give a complete description of the Hamiltonian formulation of dynamics on the slow manifold. In the leading-order approximation, the dark motions are modeled by the Vlasov-Poisson system of equations. At the next order, which accounts for magnetostatic effects, the model also has a name: the Vlasov-Darwin system. Higher-order approximations account for non-radiative electromagnetic fields generated by collective acceleration of plasma particles. The dark motions may be modeled with any desired order of accuracy without sacrificing the problem's underlying Hamiltonian structure.

1 Lecture notes

Note: Not related to dark matter! Just describing plasma that doesn't emit light.

Weakly-coupled plasmas - many plasma particles in a sphere with radius equaling the Debye length.

Are there collision less plasmas without mean field radiation?

From the first equation, $f =$ distribution function of a single plasma particle in the 6D phase space.

The second equation is the Faraday equation, the 3rd is the Maxwell-Ampere equation.

$\mathbf{v}$ is related to $\mathbf{p}$ by the relativistic γ parameter.

$\mathbf{B}$ is the magnetic field.

Hodge decomposition: $H\,(curl)$ is the space of vector fields on Q.

Small c^{-1} - weakly relativistic regime. The slow manifold is a formally slow manifold.

Fast-slow systems definition: The assumption on the y derivative of f is technical, allowing us to make calculations.

Definition - slow manifold - while the definition makes sense for $n < \infty$, we think of this definition as relevant for $n \to \infty$.

S_ϵ is a slow invariant set. In general, there can be different types of behavior when perturbing around the slow manifold - inwards spiral, instability, oscillatory behavior.

Vlasov-Maxwell as a fast-slow system in the weakly relativistic regime. The small parameter is $\epsilon = c^{-1}$. This is formal, not a physical justification.

Space of fast variables - transverse electric field and magnetic field.

Is this a fast-slow system? We need to show the constraint of the y derivative of f - lemma. Therefore, this is formally a fast-slow system.

Claim - the slow manifold corresponds to motions without light waves. To show this, we look at the fast motions by rescaling time $\tau = t/\epsilon$. Thus, on very short time scales f and E_L (longitudinal electric field) are frozen in time, while E_T and B satisfy Maxwell's equations.

So for short time scales, obtain a manifold of fixed points for $\epsilon = 0$ and a slow evolution time-scale for $\epsilon > 0$. The light waves are a perturbation off of the slow manifold.

Is the invariant manifold attracting? No, it has normal ellipticity, and that's the reason why this development must be in a formal asymptotic series. The fact that the slow manifold does not attract is related to the underlying Hamiltonian structure.

Thus, dark plasma is related to motion along the slow manifold.

Is there a way to modify this to obtain quasi-magneto-plasma effects? Yes, we will talk about this soon.

Theorem - leading order slow system is related to a Vlasov-Poisson equation set.

Theorem - second-order slaving function - the first term of the transverse electric field is just an artifact. In the second term, $\mathbb{T}_0$ is the stress tensor, thus a piezoelectric field is created. The sum over σ is over charged species - electrons vs. ions, etc.

(Piezoelectric acoustic guitar pickup - a crystal with a piezoelectric property - by stressing the crystal an electric voltage is created.)

What is the Darwin force? Instantaneous force between two particles scaling like 1 over distance squared, but it is not directed along the line connecting the two particles. Going up to the next orders, we can get corrections to the Darwin force, and that's what we're working on now.

Why is dark plasma dynamics Hamiltonian? The preimage $\tilde{S}_\epsilon$ of the slow manifold S_ϵ on the large space $\tilde{P}$ with the symplectic form Ω is a symplectic submanifold. Therefore, by applying Poisson reduction we can see that $S_\epsilon = \tilde{S}_\epsilon / G$, which means that S_ϵ is a Poisson manifold.

What is the group G? Fix a 6-form $\tilde{f}_0 \in \Omega^6 \left(T^* Q_0\right)$ and perform the transformation.

What is the G-invariant symplectic form Ω on the large space $\tilde{P}$? Set the tangent vector δZ_k as a function of the tangent vectors to the electric vector field $\delta \mathbf{E}_k$ and the magnetic potential vector field $\delta \mathbf{A}_k$.

What is the Poisson bracket on S_ϵ? It can be calculated explicitly. Note that J_0 is the finite-dimensional canonical Poisson tensor on $T^* Q$.

2 Questions

- Can the dark plasma exist in nature?

How do you prepare a plasma in a particular configuration? It's typically hard experimentally, but if you have a device that resonates with electromagnetic waves you could perhaps suck radiation from plasma. Also, this theory requires motion on a torus.

- Usually, magnetic field is much larger than electric field, is there an alternate ordering scheme where the leading order is the magnetic term?

There is a way, a version of the same calculation with magnetic effects to leading order is possible.

- What about other waves in plasmas, in particular drift waves?

These other types of waves could be destabilized by the effects, with a possible result of bifurcations along the slow manifold. But the ordering would have to change in order to see this.