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NOTETAKER CHECKLIST FORM

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Name: ORI KATZ **Email/Phone:** ORIKATZ.OK@gmail.com

Speaker's Name: Renato Calleja

Talk Title: Whiskered KAM tori of Conformally Symplectic Systems

Date: 10 / 12 / 18 **Time:** 11 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: The lecture discusses existence of whiskered tori in a family of conformally symplectic maps depending on a parameter. Main results: Given an approximately invariant embedding of a torus for a parameter value μ_0, there is an invariant embedding & invariant splittings for new parameters. Then study domains of analiticity in perturbations of Hamiltonian systems with dissipation.

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Whiskered KAM Tori of Conformally Symplectic Systems

Renato Calleja

IIMAS–UNAM

Hamiltonian systems, from topology to applications through analysis I
MSRI

October 12^{th}, 2018

Joint work with A. Celletti, R. de la Llave, A. P. Bustamante

Outline

Conformally symplectic systems

Existence of whiskered tori

Small dissipation limit

Existence of Lindstedt Series and their convergence

Numerical evidence supporting our conjecture

KAM Theory for dissipative systems

Dissipation $\Longrightarrow$ many orbits to have the same asymptotic behaviour $\Longrightarrow$ less asyptotic behaviours $\Longrightarrow$ adjust parameters

- A KAM theory with adjustment of parameters was developed in remarkable and pioneering papers: (Moser '67, Broer-Huitema-Takens-Braakama '90, Broer-Huitema-Sevryuk '96, Simó, ...)
- Conformally symplectic systems are a a subset of dissipative systems and appear in physics and economics
- In C-Celletti-Llave '13, we developed a KAM throry for conformally symplectic systems with a different parameter count that the more general cases of KAM theory

Conformally symplectic flow

Let Ω be a symplectic form such that

$$\Omega_x(u, v) = (u, J(x)v)$$

and X a vector field such that there exists a constant $\eta \in \mathbb{R}$ such that

$$\mathcal{L}_X \Omega = \eta \Omega.$$

The time t flow f_t satisfies that

$$(f_t)^* \Omega = \exp(\eta t) \Omega.$$

Conformally symplectic mappings

Let Ω be a symplectic form such that

$$\Omega_x(u, v) = (u, J(x)v).$$

A conformally symplectic map $f : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ is

$$f^*\Omega = \lambda\Omega$$

for $\lambda \in \mathbb{R}$.

Conformally symplectic systems transport a symplectic form into a multiple of itself

- Any Hamiltonian system with friction proportional to the velocity
- Celestial Mechanics Tidal torques (e.g. Biasco, Chierchia, Celletti, Laskar, Correia).
- Hamiltonian chains with energy dissipation (e.g. Wayne, Eckmann, Cuneo)
- Aubry-Mather Theory (e.g. Sorrentino, Maró)
- Euler-Lagrange equations of exponentially dicounted systems (e.g. Bensoussan, Davini, Fathi, Iturriaga, Zavidovique, Siconolfi)
- Gaussian thermostats Non-equilibrium Statistical Mechanics (e.g. Wojtowski and Liverani).
- Nosé-Hoover model (more Statistical Mechanics)

Dissipative standard map

An example of a conformally symplectic map

The family f_μ given by

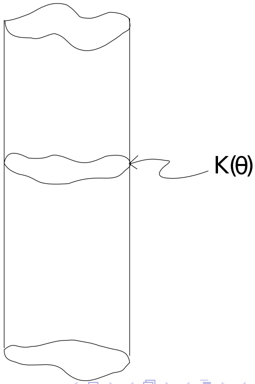
$$y_{n+1} = \lambda y_n + \mu + \varepsilon\, V'(x_n)$$
$$x_{n+1} = x_n + y_{n+1}$$

- $|Df_\mu| = \lambda$
- $\lambda = 1$, $\mu = 0$ -standard map (symplectic).
- $0 < \lambda < 1$ dissipative map.
- $\lambda > 1$ expanding map.

Parameterization of an invariant circle

Quasi-periodic solutions are orbits of the form

$$(q_n, p_n) = K(n\omega), \quad \omega \in \mathbb{R} \setminus \mathbb{Q}$$



In such a case, we have

$$f_\mu \circ K(\theta) = K(\theta + \omega).$$

We will assume

$$K(\theta + 1) = K(\theta) + (1, 0)$$

“non-contractible circles”.

(Haro, Canadell, Figueras, Luque, Mondelo)

Dissipative Froeschlé Map

Another example in four dimensions
The family $f_{(\mu_1,\mu_2)}$ given by

$$\begin{aligned}
y_{n+1}^{(1)} &= \lambda y_n^{(1)} + \mu_1 + \varepsilon_1\ V'(x_n^{(1)}) + \varepsilon_3 W'(x_n^{(1)}, x_n^{(2)}) \\
x_{n+1}^{(1)} &= x_n^{(1)} + y_{n+1}^{(1)} \\
y_{n+1}^{(2)} &= \lambda y_n^{(2)} + \mu_2 + \varepsilon_2\ V'(x_n^{(2)}) + \varepsilon_3 W'(x_n^{(1)}, x_n^{(2)}) \\
x_{n+1}^{(2)} &= x_n^{(2)} + y_{n+1}^{(2)}
\end{aligned}$$

is conformally symplectic

$$f^*\Omega = \lambda\Omega$$

Parameterization of an invariant torus

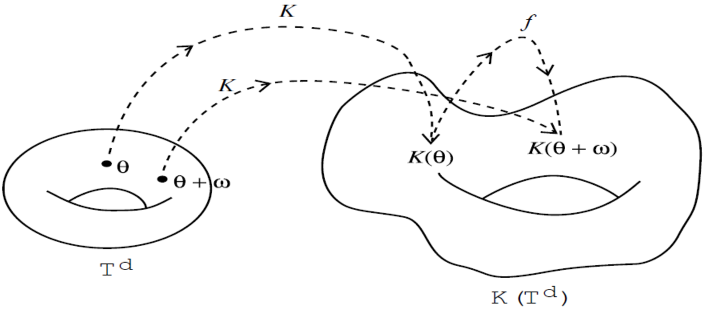
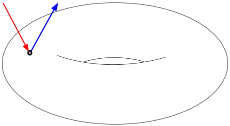


Figure: The invariance equation $f \circ K(\theta) = K(\theta + \omega)$.

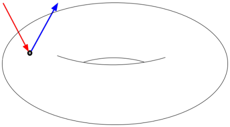
Whiskered tori

- Motion is conjugated to a rotation and many hyperbolic directions (exponentially contracting in the future or in the past under linearized evolution)
- Instability for nearly integrable systems (Arnold ’63, ’64, ...)
- Considered many times in the literature (Graff ’72, Zehnder ’76, Jorba-Villanueva ’97, Li-Yi ’05)



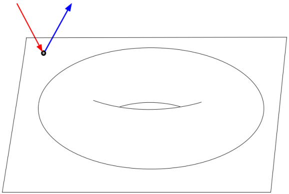
Whiskered tori in dissipative systems

- Dissipation $\implies$ many orbits to have the same asymptotic behaviour $\implies$ less asymptotic behaviours $\implies$ one cannot choose initial conditions to obtain a long term behaviour (adjust parameters)
- Conformally symplectic is a particular case of more general methods like (Moser '67, Broer-Huitema-Takens-Braaksma '90, Broer-Hitema-Sevryuk '96, Canadell-Haro '17)
- Parameterization method is closer to Llave-Fontich-Sire '09



Whiskered tori in conformally symplectic systems

- The dimension of hyperbolic directions is as large as possible given that f_μ is conformally symplectic
- Conformally symplectic allows to obtain more results that could be false in the general setting
- Small dissipation is a very singular perturbation
- In center bundle, exponential rates straddle the conformally symplectic constant



Spaces of analytic functions

For any $\rho > 0$, we denote by $\mathbb{T}^d_\rho$ the set

$$\mathbb{T}^d_\rho = \{z = x + iy \in \mathbb{C}^d/\mathbb{Z}^d : \ x \in \mathbb{T}^d, \ |y_j| \le \rho \ , \ \ j = 1, ..., d\} \ .$$

Given $\rho > 0$, we denote by $\mathcal{A}_\rho$ the set of functions which are analytic in Int$(\mathbb{T}^d_\rho)$ and extend continuously to the boundary.

- A norm in $\mathcal{A}_\rho$ is

$$\|f\|_{\mathcal{A}_\rho} = \sup_{z \in \mathbb{T}^d_\rho} |f(z)|$$

- These spaces are standard in KAM theory

We fix a Diophantine frequency ω

$$|\omega \cdot k - n| \geq \nu |k|^{-\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, n \in \mathbb{N}.$$

The invariance equation for $K_a \in \mathcal{A}_\rho$, $\mu_a \in \Lambda$

$$Err(\theta) = f_{\mu_a} \circ K_a(\theta) - K_a(\theta + \omega)$$

A solution is K_e, μ_e so that $\quad f_{\mu_e} \circ K_e(\theta) = K_e(\theta + \omega)$.

Cocycles

$$\Gamma^j \equiv Df_\mu \circ K \circ T_{j\omega} \times Df_\mu \circ K \circ T_{(j-1)\omega} \times \cdots \times Df_\mu \circ K \ ,$$

which are quasi-periodic cocycles of the form

$$\Gamma^j = \gamma_\theta \circ T_{j\omega} \times \cdots \times \gamma_\theta$$

with $\gamma_\theta = Df_\mu \circ K(\theta)$. The cocycle above satisfies the property: $\Gamma^{j+m} = \Gamma^j \circ T_{m\omega}\, \Gamma^m$.

Exponential trichotomy

The cocycle admits an exponential trichotomy when we can find a decomposition

$$\mathbb{R}^n = E^s_\theta \oplus E^c_\theta \oplus E^u_\theta \,, \qquad \theta \in \mathbb{T}^d \,,$$

rates of decay $\lambda_- < \lambda_c^- \leq \lambda_c^+ < \lambda_+$, $\lambda_- < 1 < \lambda_+$ and a constant $C_0 > 0$, such that

$$\begin{aligned}
v \in E^s_\theta &\iff |\Gamma^j(\theta) v| \leq C_0 \lambda_-^j |v|, \quad j \geq 0 \\
v \in E^u_\theta &\iff |\Gamma^j(\theta) v| \leq C_0 \lambda_+^j |v|, \quad j \leq 0 \\
v \in E^c_\theta &\iff \begin{array}{l} |\Gamma^j(\theta) v| \leq C_0 (\lambda_c^-)^j |v|, \quad j \geq 0 \\ |\Gamma^j(\theta) v| \leq C_0 (\lambda_c^+)^j |v|, \quad j \leq 0 \,. \end{array}
\end{aligned}$$

Approximately invariant splittings

Given a splitting $E^s_\theta \oplus E^u_\theta \oplus E^c_\theta$ and a cocycle γ_θ, we define

$$\gamma_\theta^{\sigma,\sigma'} = \Pi^\sigma_{\theta+\omega}\gamma_\theta\Pi^{\sigma'}_\theta \ ,$$

so that we conclude that the splitting is invariant under the cocycle if and only if

$$\gamma_\theta^{\sigma,\sigma'} \equiv 0 \ , \qquad \sigma \neq \sigma' \ .$$

$$\gamma_\theta = Df_\mu \circ K(\theta) = \begin{pmatrix} \gamma_\theta^{s,s} & 0 & 0 \\ 0 & \gamma_\theta^{c,c} & 0 \\ 0 & 0 & \gamma_\theta^{u,u} \end{pmatrix}$$

$$\tilde{E}^{\sigma}_{\theta} = \{ v \in \mathbb{R}^n, v = x + A^{\sigma}_{\theta} x \,|\, x \in E^{\sigma}_{\theta} \} \ .$$

Lemma

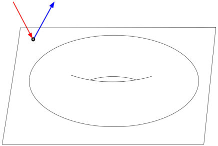
Fix an analytic reference splitting on $\mathbb{T}^d_\rho$ and let $\mathcal{U}$ be a sufficiently small neighborhood of this splitting, so that A^{σ}_{θ} as with $\|A^{\sigma}_{\theta}\|_\rho < M_1$.
Let E be an analytic splitting in the neighborhood $\mathcal{U}$.
Let γ be an analytic cocycle over a rotation defined on $\mathbb{T}^d_\rho$ with $\|\gamma\|_\rho < M_2$.
Assume that E is approximately invariant under γ, and that γ is approximately hyperbolic for the reference splitting.
Then, there is a locally unique splitting $\tilde{E}$ invariant under γ close to E and the splitting $\tilde{E}$ satisfies a trichotomy.
The constants can be chosen uniformly depending only on M_1, M_2.

Theorem

- ω *Diophantine,* $d \leq n$
- $f_\mu : \mathcal{M} \to \mathcal{M}$, $\mu \in \mathbb{R}^d$, *be a family of real analytic, conformally symplectic mappings*
- $0 < \lambda < 1$

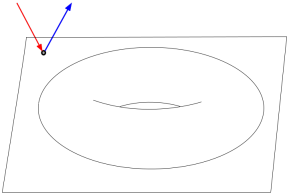
$(H1)$ *Appproximate solution:*
(K_0, μ_0) *with* $K_0 : \mathbb{T}^d \to \mathcal{M}$, $K_0 \in \mathcal{A}_\rho$, *and* $\mu_0 \in \mathbb{R}^d$

$$\|f_{\mu_0} \circ K_0 - K_0 \circ T_\omega\|_{\mathcal{A}_\rho} \leq \mathcal{E}rr, \qquad \mathcal{E}rr > 0.$$



$(H2)$ Approximate splitting:
For $\theta \in \mathbb{T}^d_\rho$, $\exists$ splitting of the tangent space
-depending analytically on the angle
-bundles are approximately invariant under the cocycle

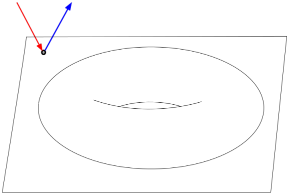
$$\gamma_\theta = Df_{\mu_0} \circ K_0(\theta)$$



($H3$) Spectral condition for the bundles (exponential trichotomy):
For $\theta \in \mathbb{T}^d_\rho$ the spaces in ($H2$) are approximately hyperbolic for γ_θ.
($H3'$) For the almost symplectic limit, we assume:

$$\lambda_- < \lambda\lambda_+ < \lambda_c^- \,, \qquad \lambda_c^- \leq \lambda \leq \lambda_c^+ \,.$$

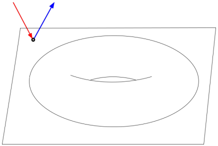
($H4$) The dimension of the center subspace is $2d$.



$(H5)$ Non–degeneracy:

$$\det\begin{pmatrix} \overline{S} & \overline{S(W_b)^0} + \overline{(M_a^{-1}\circ T_\omega D_\mu f_\mu \circ K_a)_1} \\ (\lambda - 1)\mathrm{Id} & \overline{(M_a^{-1}\circ T_\omega D_\mu f_\mu \circ K_a)_2} \end{pmatrix} \neq 0$$

$(W_b^c)^0$ is the solution of
$\lambda (W_b^c)^0 - (W_b^c)^0 \circ T_\omega = -((M_a^{-1}\circ T_\omega D_\mu f_\mu \circ K_a)_2^c)^0$,



Let $\sigma = \sigma(\tau)$ be an explicit number and assume that for some $0 < \delta < \rho$, we have

$$\begin{aligned}\mathcal{E}rr_h \leq & \mathcal{E}rr_h^*(\nu, \tau, C, \lambda_+, \lambda_-, \lambda_c^+, \lambda_c^-, \|\Pi_{K_0(\theta)}^{s/u/c}\|_{\mathcal{A}_\rho}, \\ & \|DK_0\|_{\mathcal{A}_\rho}, \|(DK_0^T DK_0)^{-1}\|_{\mathcal{A}_\rho}, |\mathcal{S}^{-1}|, \max_{j=0,1,2} \sup_{|\mu-\mu_0|\leq\eta} \|D^j f_\mu\|_{\mathcal{U}}), \\ \mathcal{E}rr \leq & \delta^{2\sigma} \mathcal{E}rr^*(\nu, \tau, C, \lambda_+, \lambda_-, \lambda_c^+, \lambda_c^-, \|\Pi_{K_0(\theta)}^{s/u/c}\|_{\mathcal{A}_\rho}, \\ & \|DK_0\|_{\mathcal{A}_\rho}, \|(DK_0^T DK_0)^{-1}\|_{\mathcal{A}_\rho}, |\mathcal{S}^{-1}|, \max_{j=0,1,2} \sup_{|\mu-\mu_0|\leq\eta} \|D^j f_\mu\|_{\mathcal{U}}),\end{aligned}$$

where an explicit expression for σ, $\mathcal{E}rr_h^*$, $\mathcal{E}rr^*$.

Then, there exists an exact solution (K_e, μ_e), such that

$$f_{\mu_e} \circ K_e - K_e \circ T_\omega = 0$$

with

$$\|K_e - K_0\|_{\mathcal{A}_{\rho-2\delta}} \le C\mathcal{E}rr\delta^{-\tau}, \qquad |\mu_e - \mu_0| \le C\mathcal{E}rr\delta^{-\tau} .$$

Furthermore, the invariant torus K_e is hyperbolic in the sense that there is an invariant splitting

$$\mathcal{T}_{K_e(\theta)}\mathcal{M} = E^s_{K_e(\theta)} \oplus E^c_{K_e(\theta)} \oplus E^u_{K_e(\theta)} ,$$

$$|\lambda_\pm - \widetilde{\lambda}_\pm| \le C(\mathcal{E}rr\delta^{-\sigma} + \mathcal{E}rr_h) , \quad |\lambda_c^\pm - \widetilde{\lambda}_c^\pm| \le C(\mathcal{E}rr\delta^{-\sigma} + \mathcal{E}rr_h) ,$$

$$|C_0 - \widetilde{C}_0| \le C(\mathcal{E}rr\delta^{-\sigma} + \mathcal{E}rr_h) .$$

Sketch of the proof

The centerpiece is,

$$f_{\mu'} \circ K'(\theta) - K' \circ T_\omega(\theta) = e'(\theta)$$

The Newton's equations for $K' = K + \Delta$, $\mu' = \mu + \Delta_\mu$

$$Df_\mu \circ K(\theta)\ \Delta(\theta) + D_\mu f_\mu \circ K(\theta)\ \Delta_\mu - \Delta(\theta + \omega) = -e(\theta)\ .$$

We project to stable/unstable/center subspaces, $\Delta^\xi(\theta) \equiv \Pi^\xi_{K(\theta+\omega)} \Delta(\theta)$, $e^\xi(\theta) \equiv \Pi^\xi_{K(\theta+\omega)} e(\theta)$ with $\xi = s, c, u$,

$$Df_\mu \circ K(\theta)\ \Delta^\xi(\theta) + \Pi_{K(\theta+\omega)^\xi} D_\mu f_\mu \circ K(\theta) \Delta_\mu - \Delta^\xi(\theta + \omega) = -e^\xi(\theta)\ ,$$

Center direction

$$\Delta^c = M\, W^c \,,$$

with

$$Df_\mu \circ K(\theta)\, M(\theta) = M(\theta+\omega) \begin{pmatrix} \mathrm{Id_d} & S(\theta) \\ 0 & \lambda \mathrm{Id_d} \end{pmatrix} + R(\theta)\,.$$

So the equations reduce to

$$\begin{pmatrix} \mathrm{Id_d} & S(\theta) \\ 0 & \lambda \mathrm{Id_d} \end{pmatrix} W^c(\theta) - W^c \circ T_\omega(\theta) = -\widetilde{e}^c(\theta) - \widetilde{A}^c(\theta)\Delta_\mu \,,$$

Hyperbolic directions

$$Df_\mu(K\circ T_{-\omega}(\theta'))\,\Delta^s(T_{-\omega}(\theta')) + \Pi^s_{K(\theta+\omega)} D_\mu f_\mu(K\circ T_{-\omega}(\theta'))\Delta_\mu \;-\; \Delta^s(\theta') = -\widetilde{e}^s(\theta')\,,$$

which can be solved for Δ^s in the form

$$\begin{aligned}\Delta^s(\theta') =& \widetilde{e}^s(\theta') + \sum_{k=1}^{\infty}\Big(Df_\mu(K\circ T_{-\omega}(\theta'))\times\cdots\times Df_\mu(K\circ T_{-k\omega}(\theta'))\Big)\,\widetilde{e}^s(T_{-k\omega}(\theta'))\\ &+\Pi^s_{K(\theta+\omega)} D_\mu f_\mu(K\circ T_{-\omega}(\theta'))\Delta_\mu\\ &+\sum_{k=1}^{\infty}\Big(Df_\mu(K\circ T_{-\omega}(\theta'))\times\ldots\times Df_\mu(K\circ T_{-k\omega}(\theta'))\,\Pi^s_{K(\theta+\omega)} D_\mu f_\mu(K\circ T_{-(k+1)\omega}(\theta')\Big)\,\Delta_\mu\,,\end{aligned}$$

The unstable space is similar,

$$\begin{aligned}\Delta^u(\theta) =& -\sum_{k=0}^{\infty}\Big((Df_\mu)^{-1}(K(\theta))\times\ldots\times(Df_\mu)^{-1}(K\circ T_{k\omega}(\theta))\Big)\,e^u(T_{k\omega}(\theta))\\ &-\sum_{k=0}^{\infty}\Big((Df_\mu)^{-1}(K(\theta))\times\ldots\times(Df_\mu)^{-1}(K\circ T_{k\omega}(\theta))\,\Pi^u_{K(\theta+\omega)} D_\mu f_\mu(K\circ T_{k\omega}(\theta)\Big)\Delta_\mu\,.\end{aligned}$$

Small dissipation

- The limit of small friction forces is natural in problems of Celestial Mechanics (Celletti-Chierchia, Correia-Laskar)
- In finance the limit corresponds to small inflation, final horizon taken to infinity
- In the context of KAM theory it lead to conjectures about the optimality of the domains on which the tori as functions of the small dissipative parameter are defined

We consider analytic families of mappings or flows with small paramerter ε, and an internal parameter μ, so that,

$$f_{\mu,\varepsilon}^{*}\Omega = \lambda(\varepsilon)\Omega, \qquad \lambda(0) = 1$$

and

$$L_{\mathcal{F}_{\mu,\varepsilon}}\Omega = \eta(\varepsilon)\Omega, \qquad \eta(0) = 0$$

We assume that the conformal factor is an analytic function of ε

$$\lambda(\varepsilon) = 1 + \alpha\varepsilon^a + O(|\varepsilon|^{a+1})$$

we are thinking that $\varepsilon \in \mathbb{C}$, since we want to talk about differentiability.

Diophantine frequency ω
$|\omega \cdot k - n| \geq \nu |k|^{-\tau}, \quad \forall k \in \mathbb{Z} \setminus \{0\}, n \in \mathbb{N}.$
We fix ω to be Diophantine by controlling the size of the constants ν,

$$\nu(\omega; \tau) \equiv sup_{k \in \mathbb{Z}^d \setminus \{0\}} |e^{2\pi i k \cdot \omega} - 1|^{-1} |k|^{-\tau} < \infty$$

and

$$\nu(\lambda(\varepsilon); \omega, \tau) \equiv sup_{k \in \mathbb{Z}^d \setminus \{0\}} |e^{2\pi i k \cdot \omega} - \lambda(\varepsilon)|^{-1} |k|^{-\tau} < \infty$$

Dissipative Froeschlé Map

The family $f_{(\mu_1,\mu_2)}$ given by

$$\begin{aligned} y^{(1)}_{n+1} &= (1-\varepsilon^3)y^{(1)}_n + \mu_1(\varepsilon) + \varepsilon_1\, V'(x^{(1)}_n) + \varepsilon_3 W'(x^{(1)}_n, x^{(2)}_n) \\ x^{(1)}_{n+1} &= x^{(1)}_n + y^{(1)}_{n+1} \\ y^{(2)}_{n+1} &= (1-\varepsilon^3)y^{(2)}_n + \mu_2(\varepsilon) + \varepsilon_2\, V'(x^{(2)}_n) + \varepsilon_3 W'(x^{(1)}_n, x^{(2)}_n) \\ x^{(2)}_{n+1} &= x^{(2)}_n + y^{(2)}_{n+1} \end{aligned}$$

is conformally symplectic

$$f^*\Omega = (1-\varepsilon^3)\Omega$$

Series expansions and geometry of the sets

$$K_\varepsilon^{[\leq N]} = \sum_{j=0}^{N} \varepsilon^j K_j, \quad \mu_\varepsilon^{[\leq N]} = \sum_{j=0}^{N} \varepsilon^j \mu_j$$

for any $N \in \mathbb{N}$
We define the sets

$$\mathcal{G}(A;\omega,\tau,N) = \{\varepsilon \in \mathbb{C} : \quad \nu(\lambda(\varepsilon);\omega,\tau)|\lambda(\varepsilon) - 1|^{N+1} \leq A\} \ .$$

and

$$\Lambda(A;\omega,\tau,N) = \{\lambda \in \mathbb{C} : \quad \nu(\lambda;\omega,\tau)|\lambda - 1|^{N+1} \leq A\} \ .$$

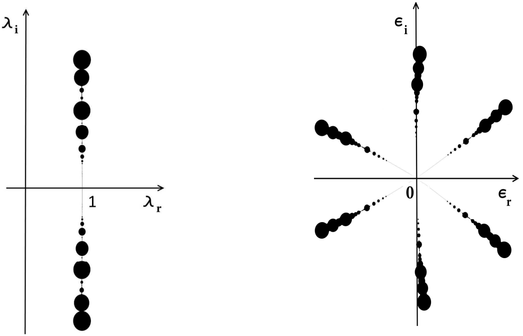


Figure: Sets Λ and $\mathcal{G}$

$$\lambda(\varepsilon) = 1 - \varepsilon^3$$

Existence of Series

Theorem

- ω *Diophantine* $\nu(\omega;\tau) < \infty$
- $K_0 \in \mathcal{A}_\rho$

$$f_{\mu_a,0} \circ K_0(\theta) = K_0(\theta + \omega)$$

- *Is a whiskered invariant torus with splitting* $T_{K(\theta)}\mathcal{M} = E^s_{K(\theta)} \oplus E^c_{K(\theta)} \oplus E^u_{K(\theta)}$, *rates, 2d dimensional center, and non-degeneracy condition as in the previous theorem*

A.1) We find a formal power series expansion $K_\varepsilon^{[N]} = \sum_{j=0}^N \varepsilon^j K_j$, $\mu_\varepsilon^{[\leq N]} = \sum_{j=0}^N \varepsilon^j \mu_j$ for any $N \in \mathbb{N}$ and $\rho > 0$, we have

$$\|f_{\mu_\varepsilon^{[\leq N]},\varepsilon} \circ K_\varepsilon^{[\leq N]} - K_\varepsilon^{[\leq N]} \circ T_\omega\|_{\rho'} \leq C_N|\varepsilon|^{N+1} \qquad (1)$$

for some $0 < \rho' < \rho$ and $C_N > 0$.

A.2) We can compute four formal power series expansions

$$A_\varepsilon^{\sigma,\infty} = \sum_{j=0}^{\infty} \varepsilon^j A_j^\sigma \,, \qquad A_j^\sigma(\theta) : E_0^\sigma(\theta) \to \mathcal{E}rr_0^{\hat{\sigma}}(\theta)\,, \qquad \sigma = s, \hat{s}, u, \hat{u}$$

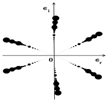
and the $A_j^\sigma \in \mathcal{A}_\rho$ in such a way that the operators for invariant dichotomies in the sense of power series.

A.3) We can find a set $\mathcal{G}_{r_0}$ of ε and functions

$$K : \mathcal{G}_{r_0} \to \mathcal{A}_{\rho'} \ , \quad \mu : \mathcal{G}_{r_0} \to \mathbb{C}^d$$

which are analytic in the interior of $\mathcal{G}_{r_0}$ and continous on the boundary of $\mathcal{G}_{r_0}$ and such that

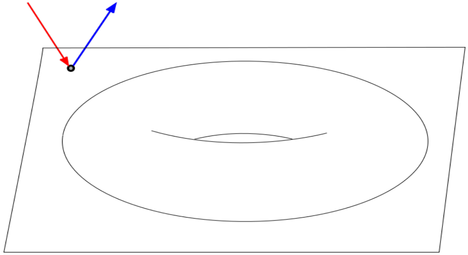
$$f_{\mu_\varepsilon,\varepsilon} \circ K_\varepsilon - K_\varepsilon \circ T_\omega = 0 \ . \tag{2}$$



At every order, the Lindstedt series have a stucture that can be solved by using the same geometric reduction

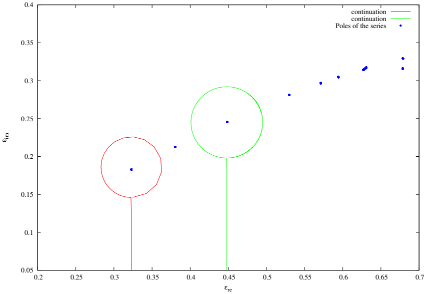
$$(Df_{\mu_0,0} \circ K_0)\ M_0 W_j - (M_0 W_j) \circ T_\omega + (D_\mu f_{\mu_0,0} \circ K_0)\mu_j$$

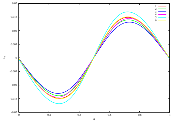
$$= F_j(K_0, \ldots, K_{j-1}, \mu_0, \ldots, \mu_{j-1})\ .$$

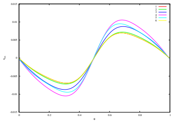


Trivial monodromy (C-Celletti-Llave '17)

We show that there is no monodromy of these continuations, either for the tori of for the stable manifolds.
The boundary of the domain is very thin, so that we can perform a unique analytic continuation of the invariant circles along closed circles enclosing the points of analyticity.







A dissipative standard map with vanishing dissipation

$$
\begin{aligned}
y_{n+1} &= (1 - \varepsilon^3) y_n + \mu_\varepsilon + \varepsilon \, V'(x_n) \\
x_{n+1} &= x_n + y_{n+1}
\end{aligned}
$$

is conformally symplectic with

$$
f^*\Omega = (1 - \varepsilon^3)\Omega
$$

Numerical evidence (joint with Adrián P. Bustamante)

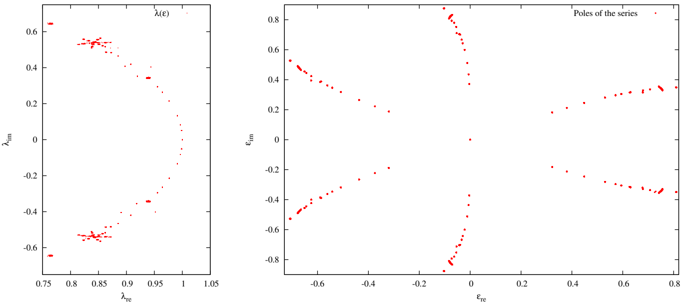
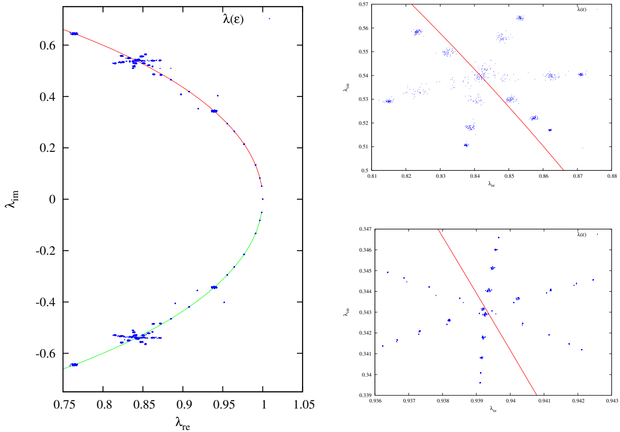


Figure: Sets Λ and $\mathcal{G}$ for the dissipative standard map

$$y_{n+1} = (1 - \varepsilon^3) y_n + \mu_\varepsilon + \varepsilon\, V'(x_n)$$
$$x_{n+1} = x_n + y_{n+1}$$



Sets Λ and $\mathcal{G}$ for the dissipative standard map

$$y_{n+1} = (1 - \varepsilon^3) y_n + \mu_\varepsilon + \varepsilon\, V'(x_n)$$

$$x_{n+1} = x_n + y_{n+1}$$

Gevrey Classes of functions

A function f is in a Gevrey class $\mathcal{G}^\sigma$ whenever there are constants C and R such that

$$|D^k f(\varepsilon)| \leq CR^k k^{\sigma k}$$

with ε in a compact set

Once we have the Lindstedt series expansion we can obtain numerical evidence that the funcitons we have are in a Gevrey class.

We evaluate

$$A_\rho(k) \equiv \frac{1}{k} \log \|u_k(\theta)\|_\rho,$$

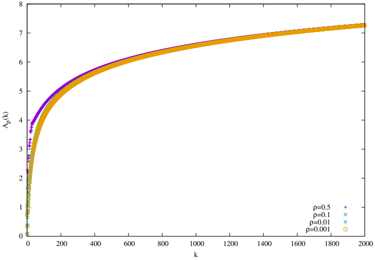


Figure: Analytic norms of the coefficients of the Linstedt expansion.

Summary

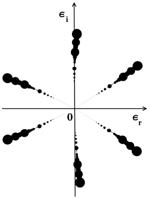
Two tools: KAM theory and Lindstedt series.

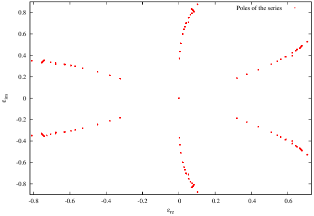
- Proof for existence of whiskered KAM tori of conformally symplectic systems
- The proof only requires the existence of an approximate solution
- The Lindstedt expansions are used to obtain expansions in complex domains
- We proof the existence of Lindstedt series to every order
- The two tools produce efficient algorithms and we use them to approximate the domains
- We conjecture that the domains we obtain are close to optimal
- The compuations support our conjectures

Thank you

- R.C., Celletti, A. and de la Llave, R., **Domains of analyticity of Lindstedt expansions of KAM tori in dissipative perturbations of Hamiltonian systems**, Nonlinearity 30, (2017) 3151-3202
- Bustamante, A. and R.C., **Computation of Domains of Analyticity for the dissipative standard map in the limit of small dissipation**, preprint: arXiv_1712.05476
- R.C., Celletti, A., and de la Llave, R., **KAM theory for conformally symplectic systems: Efficient algorithms and their validation**, J. Differential Equations 255 (2013)
- R.C. and Celletti, A., **Breakdown of invariant attractors for the dissipative standard map**, Chaos 20, 013121 (2010)

Thank you


ϵ_i
0
ϵ_r


Poles of the series
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
ϵ_{im}
ϵ_{re}

Whiskered KAM tori of conformally symplectic systems - Talk by Renato Calleja

Lecture notes (Ori S. Katz)

October 15, 2018

Abstract

Many physical problems are described by conformally symplectic systems. We study the existence of whiskered tori in a family f_μ of conformally symplectic maps depending on parameters μ. Whiskered tori are tori on which the motion is a rotation but having as many contracting/expanding directions as allowed by the preservation of the geometric structure. Our main result is formulated in an a-posteriori format. Given an approximately invariant embedding of the torus for a parameter value μ_0 with an approximately invariant splitting of the tangent space at the range of the embedding into stable/unstable/center bundles, there is an invariant embedding and invariant splittings for new parameters. Using the results of formal expansions as the starting point for the a-posteriori method, we study the domains of analiticity of parameterizations of whiskered tori in perturbations of Hamiltonian Systems with dissipation. The proofs of the results lead to efficient algorithms that are quite practical to implement.

Joint work with A. Celletti and R. de la Llave.

1 Lecture notes

Outline: Small dissipation limit - relevant for mechanics, celestial mechanics.

KAM theory for dissipative systems: Can explore phase space by varying parameters and looking at asymptotic behavior.

Conformally symplectic flow: Lie derivative of the symplectic flow $\mathcal{L}_X\Omega = \eta\Omega$, can assume for this talk that η is constant, although some of the stated results apply also in the non-constant case. If $\eta = 0$, that is the symplectic case obtained from Hamiltonian structure. If $\eta = 0$ the system is dissipating/expanding.

Conformally symplectic mappings: We will discuss maps, although everything here is relevant also for flows.

Dissipative standard map: The parameter μ inserts interesting behavior into the map.

Parameterization of an invariant torus: dynamics on the torus are conjugate to a rigid rotation by ω, so dynamics on perfect torus parameterize dynamics on invariant torus.

Whiskered tori: Motion on torus is conjugate to rigid rotation, but at the same time there are directions exponentially contracting in the future or the past.

Whiskered tori in conformally symplectic systems: Whiskered tori live in a center space which is a conformally symplectic system, with expanding or contracting directions.

Spaces of analytic functions: We want to have a KAM theorem for the existence of whiskered tori.

What do we need for KAM? Fix a Diophantine frequency ω. Need an invariance equation - start with an approximate solution K_a and an approximate parameter μ_a that make the error from the solution small. So we start with an approximately invariant torus, and correct it using the Newton method and other methods, until we obtain a solution K_e with the dynamics we were hoping to obtain (invariant torus).

The problem - there are also hyperbolic directions, so we have to look at cocycles Γ^j.

Exponential trichotomy: Decompose the phase space to stable E^S_θ, center E^C_θ and unstable E^U_θ.

Approximately invariant splittings: This is how we check our cocycle can be thus decomposed.

Theorem: Assumptions will be important to explore the small dissipation limit. (H5) non-degeneracy condition is reminiscent of the no-twist condition - when the parameter is moved, the rotation number changes as well.

Sketch of the proof: Start with the approximate solution, correct via a Newton-type method.

Center direction: It is possible to get rid of $S(\theta)$ by a diagonalization, but it turns out that in the non-dissipation limit where $\lambda \to 1$ this term is important, so it is a good idea to retain it.

Small dissipation: interesting in celestial mechanics because the dissipating forces (tidal) are very small and slow.

Dissipative Froeschle map: Why ϵ^3? So that the symplectic case 0 is crossed when ϵ is varied.

The black regions in the $\Lambda\mathcal{G}$ figure are the problematic regions, they are the boundaries. What happens to the functions for small ϵ around these boundaries? Trivial monodromy - when going around a boundary, you return to the same function. This was shown numerically for the dissipative standard map with vanishing dissipation. The poles of the series align on the boundaries. Looking at the λ plot (left plot) obtain balls with a type of self similar structure.