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NOTETAKER CHECKLIST FORM

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Name: Ori Katz **Email/Phone:** ORIKATZ.OK@gmail.com

Speaker's Name: Alain Brizard

Talk Title: Symplectic gyrokinetic Vlasov-Maxwell theory

Date: 10/08/18 **Time:** 2:00 am / (pm) (circle one)

Please summarize the lecture in 5 or fewer sentences: Brizard considers a general form of electromagnetic gyrokinetic Vlasov-Maxwell theory in which the gyrocenter symplectic structure contains electric & magnetic perturbations that are necessary to cause the first-order gyrocenter polarization to vanish. The gyrocenter Hamilton equations satisfy the Liouville property exactly with a time-dependent gyrocenter Jacobian. He shows that the new symplectic gyrokinetic equations retain all first-order polarization & magnetization effects.

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Symplectic gyrokinetic Vlasov-Maxwell theory

Alain J. Brizard
Department of Physics, Saint Michael's College

Hamiltonian systems, from topology to applications through analysis I

MSRI, Berkeley (California) October 8-12, 2018

Outline

• The Vlasov-Maxwell equations offer a Hamiltonian description of the self-consistent interactions between charged particles (mass m and charge e) and an electromagnetic field

$$\left.\begin{array}{r} \nabla\cdot\mathbf{B} = 0 \\ \\ \nabla\times\mathbf{E} + c^{-1}\partial\mathbf{B}/\partial t = 0 \end{array}\right\} \rightarrow \left\{\begin{array}{lcl} \mathbf{E} & = & -\,\nabla\Phi \;-\; c^{-1}\partial\mathbf{A}/\partial t \\ \\ \mathbf{B} & = & \nabla\times\mathbf{A} \end{array}\right.$$

• In this talk, I present the variational (Lagrangian) principles for exact and reduced (guiding-center and gyrocenter) Vlasov-Maxwell equations.

• In a generic noncanonical Lagrangian formulation, both the symplectic (Poisson-bracket) structure and the Hamiltonian depend on variational fields $(\Phi, \mathbf{A}; \mathbf{E}, \mathbf{B})$.

Avertissement

The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure.

Joseph Louis de Lagrange
Mécanique Analytique (1788)

Review of Lagrangian & Hamiltonian Dynamics

- Lagrangian dynamics in extended phase space $(\mathbf{x}, \mathbf{p}; w, t)$

$$L = \left(\frac{e}{c}\mathbf{A} + \mathbf{p}\right)\cdot\dot{\mathbf{x}} - w\,\dot{t} - \left(\frac{|\mathbf{p}|^2}{2m} + e\,\Phi - w\right)$$

◦ Extended Hamiltonian $\mathcal{H} = (|\mathbf{p}|^2/2m + e\,\Phi) - w = H - w \equiv 0$

◦ Euler-Lagrange equations $(\dot{t} = 1)$

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{p}/m \\
\dot{\mathbf{p}} &= -e\left(\nabla\Phi + \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) + \frac{e}{c}\left(\nabla\mathbf{A}\cdot\dot{\mathbf{x}} - \dot{\mathbf{x}}\cdot\nabla\mathbf{A}\right) \\
&\equiv e\,\mathbf{E} + \frac{e}{c}\dot{\mathbf{x}}\times\mathbf{B} \\
\dot{w} &= e\frac{\partial\Phi}{\partial t} - \frac{e}{c}\dot{\mathbf{x}}\cdot\frac{\partial\mathbf{A}}{\partial t}
\end{aligned}$$

- Hamiltonian dynamics in extended phase space $(\mathbf{x}, \mathbf{p}; w, t)$

◦ Lagrange bracket $\omega_{\alpha\beta} \;\rightarrow\;$ Poisson bracket $J^{\alpha\beta} = \{z^\alpha,\; z^\beta\}$

$$\gamma = \left(\frac{e}{c}\mathbf{A} + \mathbf{p}\right) \cdot \mathrm{d}\mathbf{x} - w\,\mathrm{d}t \;\rightarrow\; \omega = \mathrm{d}\gamma = \frac{1}{2}\,\omega_{\alpha\beta}\,\mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta \;\rightarrow\; J \equiv \omega^{-1}$$

◦ Extended (noncanonical) Poisson bracket

$$\begin{aligned}\{f,\; g\} \;&=\; \left(\nabla f \cdot \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \nabla g\right) - \left(\frac{\partial f}{\partial w}\,\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\,\frac{\partial g}{\partial w}\right) \\ &\quad + \frac{e}{c}\mathbf{B} \cdot \left(\frac{\partial f}{\partial \mathbf{p}} \times \frac{\partial g}{\partial \mathbf{p}}\right) + \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} \cdot \left(\frac{\partial f}{\partial \mathbf{p}}\,\frac{\partial g}{\partial w} - \frac{\partial g}{\partial \mathbf{p}}\,\frac{\partial f}{\partial w}\right)\end{aligned}$$

◦ Hamilton equations $\rightarrow$ Liouville Theorem (Jacobian $\mathcal{J} = 1$)

$$\dot{z}^\alpha \;=\; \left\{z^\alpha,\; \mathcal{H}\right\} = J^{\alpha\beta}\,\frac{\partial \mathcal{H}}{\partial z^\beta} \;\rightarrow\; \frac{1}{\mathcal{J}}\,\frac{\partial}{\partial z^\alpha}\left(\mathcal{J}\,\dot{z}^\alpha\right) \;=\; 0$$

Vlasov-Maxwell Equations

- Vlasov equation for Vlasov distribution $f(\mathbf{x}, \mathbf{p}; t)$

$$0 \;=\; \frac{\partial f}{\partial t} \;+\; \dot{z}^{\alpha}\,\frac{\partial f}{\partial z^{\alpha}} \;=\; \frac{1}{\mathcal{J}}\left[\frac{\partial}{\partial t}(\mathcal{J}\,f) \;+\; \frac{\partial}{\partial z^{\alpha}}\left(\mathcal{J}\,f\,\dot{z}^{\alpha}\right)\right]$$

◦ Vlasov equation in extended phase space

$$\mathcal{F}(\mathbf{x}, \mathbf{p}; t, w) \;\equiv\; f(\mathbf{x}, \mathbf{p}; t)\,\delta\left(w \;-\; H(\mathbf{x}, \mathbf{p}; t)\right) \;\rightarrow\; \left\{\mathcal{F},\, \mathcal{H}\right\} \;=\; 0$$

- Maxwell equations (with particle sources)

$$\nabla\cdot\mathbf{E}(\mathbf{x}, t) \;=\; 4\pi\,\varrho = 4\pi\sum e\int_{\mathbf{p}} f(\mathbf{x}, \mathbf{p}; t)$$

$$\nabla\times\mathbf{B}(\mathbf{x}, t) - c^{-1}\partial\mathbf{E}(\mathbf{x}, t)/\partial t \;=\; \frac{4\pi}{c}\,\mathbf{J} = \frac{4\pi}{c}\sum e\int_{\mathbf{p}} \dot{\mathbf{x}}\,f(\mathbf{x}, \mathbf{p}; t)$$

◦ Source-free Maxwell equations

$$\nabla\cdot\mathbf{B}(\mathbf{x}, t) \;=\; 0 \;=\; \nabla\times\mathbf{E}(\mathbf{x}, t) + c^{-1}\partial\mathbf{B}(\mathbf{x}, t)/\partial t$$

Variational Principle for Vlasov-Maxwell Equations

- Vlasov-Maxwell Lagrangian density [Brizard (2000)]

$$\mathcal{L} \;\equiv\; \frac{1}{8\pi}\left(|\mathbf{E}|^2 \;-\; |\mathbf{B}|^2\right) \;-\; \sum \int_{\mathbf{p},w} \mathcal{F}\,\mathcal{H}$$

- ◦ Hamiltonian variation: $\delta\mathcal{H} \;=\; e\,\delta\Phi$

- ◦ Constrained Variation for $\mathcal{F}$: Canonical part + Symplectic part

$$\delta\mathcal{F} \;\equiv\; \{\delta\mathcal{S},\; \mathcal{F}\} \;+\; \frac{e}{c}\,\delta\mathbf{A}\cdot\{\mathbf{x},\; \mathcal{F}\}$$

- ◦ Constrained Variation for $(\mathbf{E}, \mathbf{B})$:

$$\delta\mathbf{E} \;=\; -\nabla\delta\Phi - c^{-1}\partial\delta\mathbf{A}/\partial t \quad \text{and} \quad \delta\mathbf{B} \;=\; \nabla\times\delta\mathbf{A}$$

- Variational principle $\int \delta\mathcal{L}\, d^3x\, dt \;=\; 0$

$$\delta\mathcal{L} \;=\; -\sum \int_{\mathbf{p},w} \left[\delta\mathcal{S}\, \left\{ \mathcal{F},\, \mathcal{H} \right\} \;+\; \mathcal{F} \left(e\,\delta\Phi \;-\; \frac{e}{c}\delta\mathbf{A}\cdot\frac{\partial\mathcal{H}}{\partial\mathbf{p}} \right) \right]$$
$$+\; \frac{\delta\Phi}{4\pi}\,(\nabla\cdot\mathbf{E}) \;+\; \frac{\delta\mathbf{A}}{4\pi}\cdot \left(\frac{1}{c}\,\frac{\partial\mathbf{E}}{\partial t} \;-\; \nabla\times\mathbf{B} \right)$$
$$+\; \left(\partial\delta\mathcal{N}/\partial t \;+\; \nabla\cdot\delta\mathbf{N} \right) \;\rightarrow\; \textbf{Noether equation}$$

- Vlasov-Maxwell equations ($\int_{\mathbf{p},w} \mathcal{F} = \int_{\mathbf{p}} f$)

$$\delta\mathcal{S} \;\rightarrow\; \left\{ \mathcal{F},\, \mathcal{H} \right\} \;=\; 0$$
$$\delta\Phi \;\rightarrow\; \nabla\cdot\mathbf{E} - 4\pi \sum e \int_{\mathbf{p}} f \;=\; 0$$
$$\delta\mathbf{A} \;\rightarrow\; \frac{1}{c}\,\frac{\partial\mathbf{E}}{\partial t} - \nabla\times\mathbf{B} + 4\pi \sum \frac{e}{c} \int_{\mathbf{p}} f\,\dot{\mathbf{x}} \;=\; 0$$

- Noether equation

$$\delta\mathcal{L} = \frac{\partial}{\partial t}\left(\sum\int_{\mathbf{p},w}\delta\mathcal{S}\,\mathcal{F} - \delta\mathbf{A}\cdot\frac{\mathbf{E}}{4\pi c}\right) + \nabla\cdot\left[\sum\int_{\mathbf{p},w}\delta\mathcal{S}\,\mathcal{F}\,\dot{\mathbf{x}} - \frac{1}{4\pi}\left(\delta\Phi\,\mathbf{E} + \delta\mathbf{A}\times\mathbf{B}\right)\right]$$

  ◦ Momentum-Energy conservation $\leftrightarrow$ Space-Time symmetry

$$\begin{aligned} \delta\mathcal{S} &= \mathbf{P}\cdot\delta\mathbf{x} - w\,\delta t \\ \delta\Phi &= \mathbf{E}\cdot\delta\mathbf{x} - c^{-1}\partial\delta\chi/\partial t \\ \delta\mathbf{A} &= \mathbf{E}\,c\delta t + \delta\mathbf{x}\times\mathbf{B} + \nabla\delta\chi \end{aligned}$$

  ◦ Gauge variation: $\delta\chi \equiv \Phi\,c\delta t - \mathbf{A}\cdot\delta\mathbf{x}$

  ◦ Lagrangian variation ($\mathcal{L}_{\rm Vlasov} = -\int_w \mathcal{F}\,\mathcal{H} = 0$)

$$\delta\mathcal{L} \equiv -\frac{\partial}{\partial t}\left(\delta t\,\mathcal{L}_{\rm Maxwell}\right) - \nabla\cdot\left(\delta\mathbf{x}\,\mathcal{L}_{\rm Maxwell}\right)$$

◦ Gauge-variation cancellations

$$\delta\mathcal{S} + \frac{e}{c}\delta\chi = \left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right)\cdot\delta\mathbf{x} - \left(w - e\,\Phi\right)\,\delta t$$
$$= \mathbf{p}\cdot\delta\mathbf{x} - \left(|\mathbf{p}|^2/2m\right)\,\delta t \equiv \mathbf{p}\cdot\delta\mathbf{x} - K\,\delta t$$

◦ Noether Theorem

$$\delta\mathcal{L} = \frac{\partial}{\partial t}\left[\sum\int_{\mathbf{p}} f\left(\mathbf{p}\cdot\delta\mathbf{x} - K\,\delta t\right) - \left(\mathbf{E}\,c\delta t + \delta\mathbf{x}\times\mathbf{B}\right)\cdot\frac{\mathbf{E}}{4\pi c}\right]$$
$$+ \nabla\cdot\left[\sum\int_{\mathbf{p}} f\left(\mathbf{p}\cdot\delta\mathbf{x} - K\,\delta t\right)\dot{\mathbf{x}} - \left(\frac{\mathbf{E}\times\mathbf{B}}{4\pi}\right)c\,\delta t\right.$$
$$\left. - \frac{1}{4\pi}\left(\mathbf{E}\,\mathbf{E}\cdot\delta\mathbf{x} + (\delta\mathbf{x}\times\mathbf{B})\times\mathbf{B}\right)\right]$$

- Energy-momentum conservation laws

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla\cdot\mathbf{S} \;=\; 0 \;=\; \frac{\partial \mathbb{P}}{\partial t} \;+\; \nabla\cdot\mathsf{T}$$

◦ Vlasov-Maxwell energy density and energy-density flux

$$\begin{aligned} \mathcal{E} &= \sum\int_{\mathbf{p}} f\; K \;+\; \left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)/8\pi \\ \mathbf{S} &= \sum\int_{\mathbf{p}} f\;\dot{\mathbf{x}}\;K \;+\; c\,(\mathbf{E}\times\mathbf{B})/4\pi \end{aligned}$$

◦ Vlasov-Maxwell momentum density and stress tensor

$$\begin{aligned} \mathbb{P} &= \sum\int_{\mathbf{p}} f\;\mathbf{p} \;+\; (\mathbf{E}\times\mathbf{B})/4\pi c \\ \mathsf{T} &= \sum\int_{\mathbf{p}} f\;m\,\dot{\mathbf{x}}\dot{\mathbf{x}} \;+\; \frac{\mathbf{I}}{8\pi}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) \;-\; \frac{1}{4\pi}\Big(\mathbf{E}\,\mathbf{E} \;+\; \mathbf{B}\,\mathbf{B}\Big) \end{aligned}$$

◦ Symmetry $T^{ij} = T^{ji} \Rightarrow$ Conservation of angular momentum

Guiding-center Vlasov-Maxwell Theory

- Guiding-center Lagrangian one-form [Brizard & Tronci (2016)]

$$\gamma_{\rm gc} = \left(\frac{e}{c}\,\mathbf{A} + p_{\|}\,\widehat{\mathsf{b}}\right)\cdot d\mathbf{X} + J\,d\theta \;-\; w\,dt \equiv \frac{e}{c}\mathbf{A}^{*}\cdot d\mathbf{X} + J\,d\theta \;-\; w\,dt$$

- ◦ Asymptotic decoupling of fast gyromotion (J, θ) from reduced guiding-center motion $(\mathbf{X}, p_{\|})$.

- ◦ Guiding-center Poisson bracket ($\mathbf{B}^{*} = \nabla\times\mathbf{A}^{*},\;\; B_{\|}^{*} = \widehat{\mathsf{b}}\cdot\mathbf{B}^{*}$)

$$\begin{aligned}\{\mathcal{F},\,\mathcal{G}\}_{\rm gc} \;\;&=\;\; \left(\frac{\partial\mathcal{F}}{\partial\theta}\frac{\partial\mathcal{G}}{\partial J} - \frac{\partial\mathcal{F}}{\partial J}\frac{\partial\mathcal{G}}{\partial\theta}\right) + \frac{\mathbf{B}^{*}}{B_{\|}^{*}}\cdot\left(\nabla^{*}\mathcal{F}\,\frac{\partial\mathcal{G}}{\partial \mathsf{p}_{\|}} - \frac{\partial\mathcal{F}}{\partial \mathsf{p}_{\|}}\,\nabla^{*}\mathcal{G}\right)\\ &\quad - \frac{c\widehat{\mathsf{b}}}{eB_{\|}^{*}}\cdot\nabla^{*}\mathcal{F}\times\nabla^{*}\mathcal{G} \;+\; \left(\frac{\partial\mathcal{F}}{\partial w}\;\frac{\partial\mathcal{G}}{\partial t} \;-\; \frac{\partial\mathcal{F}}{\partial t}\;\frac{\partial\mathcal{G}}{\partial w}\right)\end{aligned}$$

where $\nabla^{*} \equiv \nabla - [(e/c)\,\partial\mathbf{A}^{*}/\partial t]\;\partial/\partial w$.

- Reduced guiding-center equations of motion

$$\left(\frac{d_{\rm gc}\mathbf{X}}{dt},\;\frac{d_{\rm gc}p_{\|}}{dt}\right) \;=\; \left(\frac{p_{\|}}{m}\,\frac{\mathbf{B}^{*}}{B_{\|}^{*}} \;+\; \mathbf{E}^{*}\times\frac{c\,\widehat{\sf b}}{B_{\|}^{*}},\; e\,\mathbf{E}^{*}\cdot\frac{\mathbf{B}^{*}}{B_{\|}^{*}}\right)$$

- Guiding-center Lagrangian $\Lambda_{\rm gc} = e\,\mathbf{A}^{*}\cdot\dot{\mathbf{X}} - (e\Phi^{*} + p_{\|}^{2}/2)$

$$\left.\begin{array}{l} e\,\Phi^{*}(\mathbf{z},t;\mu) \equiv e\,\Phi + \mu\,B \\ e\,\mathbf{A}^{*}(\mathbf{z},t;\mu) \equiv e\,\mathbf{A} + c\,p_{\|}\,\widehat{\sf b} \end{array}\right\} \rightarrow \left\{\begin{array}{l} \mathbf{E}^{*} \equiv -\nabla\Phi^{*} - c^{-1}\partial\mathbf{A}^{*}/\partial t \\ \mathbf{B}^{*} \equiv \nabla\times\mathbf{A}^{*} \end{array}\right.$$

- Guiding-center Jacobian $\rightarrow$ Guiding-center Liouville Theorem

$$\mathcal{J}_{\rm gc} \equiv (e/c)B_{\|}^{*} \;\rightarrow\; \frac{\partial}{\partial z^{a}}\left(\mathcal{J}_{\rm gc}\,\frac{d_{\rm gc}Z^{a}}{dt}\right) = -\,\frac{\partial\mathcal{J}_{\rm gc}}{\partial t}$$

Guiding-center Polarization and Magnetization

- Guiding-center Polarization and Magnetization (Pfirsch 1984 & Kaufman 1986)

  ◦ Guiding-center electric-dipole moment

$$\boldsymbol{\pi}_{\rm gc} \;=\; \frac{e\,\widehat{\sf b}}{\Omega}\times\frac{d_{\rm gc}\mathbf{X}}{dt} \;=\; -\,\frac{e}{m\,\Omega^{2}}\;\left(\mu\,\nabla_{\perp}B \;+\; \frac{p_{\|}^{2}}{m}\,\widehat{\sf b}\cdot\nabla\widehat{\sf b}\right)$$

  ◦ Intrinsic (Hamiltonian) guiding-center magnetic-dipole moment

$$\boldsymbol{\mu}_{\rm gc} \;=\; -\,\mu\,\widehat{\sf b} \;\equiv\; -\,\mu\,\frac{\partial B}{\partial\mathbf{B}}$$

  ◦ Moving electric-dipole (symplectic) contribution

$$\boldsymbol{\pi}_{\rm gc}\times\frac{p_{\|}}{mc}\,\widehat{\sf b} \;=\; \frac{p_{\|}}{B}\left(\frac{d_{\rm gc}\mathbf{X}}{dt}\right)_{\perp} \;\equiv\; p_{\|}\,\frac{\partial\widehat{\sf b}}{\partial\mathbf{B}}\cdot\frac{d_{\rm gc}\mathbf{X}}{dt}$$

- Guiding-center Vlasov-Maxwell Equations

◦ Guiding-center Vlasov equation $f_\mu(\mathbf{X}, p_\|, t)$

$$\frac{\partial f_\mu}{\partial t} = -\frac{d_{\rm gc}\mathbf{X}}{dt}\cdot\nabla f_\mu - \frac{d_{\rm gc}p_\|}{dt}\frac{\partial f_\mu}{\partial p_\|}$$

◦ Divergence form (+ Liouville Theorem): $F_\mu \equiv \mathcal{J}_{\rm gc}\, f_\mu$

$$\frac{\partial F_\mu}{\partial t} = -\nabla\cdot\left(F_\mu\,\frac{d_{\rm gc}\mathbf{X}}{dt}\right) - \frac{\partial}{\partial p_\|}\left(F_\mu\,\frac{d_{\rm gc}p_\|}{dt}\right)$$

◦ Source-free guiding-center Maxwell equations

$$\nabla\cdot\mathbf{B}^* = 0 = \nabla\cdot\mathbf{B} \quad\text{and}\quad \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0 = \nabla\times\mathbf{E}^* + \frac{1}{c}\frac{\partial\mathbf{B}^*}{\partial t}$$

◦ Guiding-center Maxwell equations

$$\nabla\cdot\mathbf{E} = 4\pi\,\varrho_{\rm gc} \;\;\text{and}\;\; c\,\nabla\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = 4\pi\left(\mathbf{J}_{\rm gc} \,+\, c\,\nabla\times\mathbb{M}_{\rm gc}\right)$$

◦ Guiding-center charge and current densities ($\Sigma^{\mu} \equiv \sum\int d\mu$)

$$(\varrho_{\rm gc},\;\mathbf{J}_{\rm gc}) \;\equiv\; \Sigma^{\mu}\int\left(e,\; e\,\frac{d_{\rm gc}\mathbf{X}}{dt}\right)\;F_{\mu}\,dp_{\|}$$

◦ Guiding-center magnetization (with moving electric-dipole)

$$\mathbb{M}_{\rm gc} \;\equiv\; \Sigma^{\mu}\int\frac{\partial\Lambda_{\rm gc}}{\partial\mathbf{B}}\;F_{\mu}\,dp_{\|}$$

where

$$\frac{\partial\Lambda_{\rm gc}}{\partial\mathbf{B}} \;=\; \frac{e}{c}\frac{\partial\mathbf{A}^{*}}{\partial\mathbf{B}}\cdot\dot{\mathbf{X}} - e\,\frac{\partial\Phi^{*}}{\partial\mathbf{B}} \;=\; \left(\widehat{\mathsf{b}}\times\dot{\mathbf{X}}\right)\;\times\;\frac{p_{\|}\,\widehat{\mathsf{b}}}{B} - \mu\,\widehat{\mathsf{b}}$$

$$\equiv\;\; \boldsymbol{\pi}_{\rm gc}\times\frac{p_{\|}\,\widehat{\mathsf{b}}}{mc} \;+\; \boldsymbol{\mu}_{\rm gc}$$

Guiding-center Vlasov-Maxwell Variational Principle

- Guiding-center Vlasov-Maxwell Lagrangian density (Brizard 2000, Brizard & Tronci 2016)

$$\mathcal{L}_{\rm gc} \;=\; \frac{1}{8\pi}\left(|\mathbf{E}|^2 \;-\; |\mathbf{B}|^2\right) \;-\; \sum^{\mu} \int \mathcal{F}_\mu\, \mathcal{H}\, dp_\|\, dw$$

◦ Extended Eulerian fields $\mathcal{F}_\mu$ and $\mathcal{H}$, with $Z = (\mathbf{X}, p_\|; w, t)$

$$\begin{aligned} \mathcal{H} &\equiv \left(e\,\Phi^* + {\rm p}_\|^2/2m\right) \;-\; w \;=\; H_{\rm gc} \;-\; w \\ \mathcal{F}_\mu &\equiv F_\mu(\mathbf{X}, {\rm p}_\|, t)\, \delta(w - H_{\rm gc}) \end{aligned}$$

- Eulerian field variations

  - Guiding-center Hamiltonian variation

$$\delta\mathcal{H} \;\equiv\; e\,\delta\Phi^{*} \;=\; e\,\delta\Phi \;+\; \mu\,\widehat{\mathsf{b}}\cdot\delta\mathbf{B}$$

  - Guiding-center Vlasov variation ($\overline{\mathcal{F}}_{\mu} \equiv \mathcal{F}_{\mu}/B_{\|}^{*}$):

$$\delta\mathcal{F}_{\mu} = \text{Jacobian part} + \text{canonical part} + \text{symplectic part}$$

$$\delta\mathcal{F}_{\mu} \;\equiv\; \overline{\mathcal{F}}_{\mu}\,\delta B_{\|}^{*} + B_{\|}^{*}\left(\left\{\delta\mathcal{S},\,\overline{\mathcal{F}}_{\mu}\right\}_{\rm gc} + \frac{e}{c}\,\delta\mathbf{A}^{*}\cdot\left\{\mathbf{X},\,\overline{\mathcal{F}}_{\mu}\right\}_{\rm gc}\right)$$

  - Guiding-center symplectic & Jacobian variations

$$\frac{e}{c}\,\delta\mathbf{A}^{*} \;=\; \frac{e}{c}\,\delta\mathbf{A} \;+\; \mathsf{p}_{\|}\,\delta\mathbf{B}\cdot\frac{\partial\widehat{\mathsf{b}}}{\partial\mathbf{B}}$$

$$\delta B_{\|}^{*} \;=\; \delta\mathbf{B}^{*}\cdot\widehat{\mathsf{b}} \;+\; \left(\delta\mathbf{B}\cdot\frac{\partial\widehat{\mathsf{b}}}{\partial\mathbf{B}}\right)\cdot\mathbf{B}^{*}$$

- Guiding-center Lagrangian variation

$$\begin{aligned} \delta\mathcal{L}_{\rm gc} \;\; &\equiv \;\; -\sum^{\mu}\int B_{\|}^{*}\,\delta\mathcal{S}\,\left\{\overline{\mathcal{F}}_{\mu},\,\mathcal{H}\right\}_{\rm gc}\,dp_{\|}\,dw \\ &+\frac{\delta\Phi}{4\pi}\left(\nabla\cdot\mathbf{E}-4\pi\,\varrho_{\rm gc}\right)+\frac{\delta\mathbf{A}}{4\pi}\cdot\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}-\nabla\times\mathbb{H}_{\rm gc}+\frac{4\pi}{c}\,\mathbf{J}_{\rm gc}\right) \\ &+\frac{\partial\delta\mathcal{N}_{\rm gc}}{\partial t}+\nabla\cdot\delta\mathbf{N}_{\rm gc}\;\;\rightarrow\;\;\text{(Noether equation)} \end{aligned}$$

  - Guiding-center magnetic H-field: $\mathbb{H}_{\rm gc}\equiv\mathbf{B}\;-\;4\pi\,\mathbb{M}_{\rm gc}$

  - Guiding-center Noether equation

$$\begin{aligned} \delta\mathcal{N}_{\rm gc} \;\; &\equiv \;\; \sum^{\mu}\int\delta\mathcal{S}\,\mathcal{F}_{\mu}\,dp_{\|}\,dw\;-\;\frac{\mathbf{E}\cdot\delta\mathbf{A}}{4\pi\,c} \\ \delta\mathbf{N}_{\rm gc} \;\; &\equiv \;\; \sum^{\mu}\int\delta\mathcal{S}\,\mathcal{F}_{\mu}\,\frac{d_{\rm gc}\mathbf{X}}{dt}\,dp_{\|}\,dw-\frac{1}{4\pi}\left(\delta\Phi\,\mathbf{E}\;+\;\delta\mathbf{A}\times\mathbb{H}_{\rm gc}\right) \end{aligned}$$

Guiding-center Conservation Laws

- Guiding-center energy conservation law

$$\frac{\partial \mathcal{E}_{\rm gc}}{\partial t} + \nabla\cdot\mathbf{S}_{\rm gc} = 0$$

- ∘ Guiding-center energy density ($K_{\rm gc} \equiv \mu B + p_{\|}^{2}/2m$)

$$\mathcal{E}_{\rm gc} \equiv \sum^{\mu}\int F_{\mu}\,K_{\rm gc}\,d\mathrm{p}_{\|} + \frac{1}{8\pi}\left(|\mathbf{E}|^{2} + |\mathbf{B}|^{2}\right)$$

- ∘ Guiding-center energy-density flux

$$\mathbf{S}_{\rm gc} \equiv \sum^{\mu}\int F_{\mu}\,K_{\rm gc}\,\frac{d_{\rm gc}\mathbf{X}}{dt}\,d\mathrm{p}_{\|} + \frac{c}{4\pi}\,\mathbf{E}\times\mathbb{H}_{\rm gc}$$

- Guiding-center momentum conservation law

$$\frac{\partial \mathbb{P}_{\rm gc}}{\partial t} \;+\; \nabla\cdot\mathsf{T}_{\rm gc} \;=\; 0$$

- Guiding-center momentum density

$$\mathbb{P}_{\rm gc} \;\equiv\; \sum^{\mu}\int \mathrm{p}_{\|}\,\widehat{\mathrm{b}}\,F_{\mu}\,d\mathrm{p}_{\|} \;+\; \frac{\mathbf{E}\times\mathbf{B}}{4\pi\,c}$$

- **Symmetric** guiding-center stress tensor $\mathsf{T}_{\rm gc} \equiv \mathsf{T}_{\rm M} + \mathsf{T}_{\rm gcV}$

$$\mathsf{T}_{\rm M} \;\equiv\; \left(|\mathbf{E}|^{2} \;+\; |\mathbf{B}|^{2}\right)\frac{\mathbf{I}}{8\pi} - \frac{1}{4\pi}\left(\mathbf{EE} \;+\; \mathbf{BB}\right)$$

$$\mathsf{T}_{\rm gcV} \;\equiv\; \mathsf{P}_{\rm CGL} + \sum^{\mu}\int\left(\dot{\mathbf{X}}_{\perp}\,\mathrm{p}_{\|}\,\widehat{\mathrm{b}} + \mathrm{p}_{\|}\,\widehat{\mathrm{b}}\,\dot{\mathbf{X}}_{\perp}\right)F_{\mu}\,d\mathrm{p}_{\|}$$

- CGL pressure tensor: $\dot{\mathbf{X}}\,p_{\|}\,\widehat{\mathrm{b}} = (p_{\|}^{2}/m)\,\widehat{\mathrm{b}}\,\widehat{\mathrm{b}} + \dot{\mathbf{X}}_{\perp}\,p_{\|}\,\widehat{\mathrm{b}}$

$$\mathsf{P}_{\rm CGL} \equiv \sum^{\mu}\int\left[\frac{\mathrm{p}_{\|}^{2}}{m}\,\widehat{\mathrm{b}}\,\widehat{\mathrm{b}} + \;\mu\,B\;\left(\mathbf{I} - \widehat{\mathrm{bb}}\right)\right]\;F_{\mu}\,d\mathrm{p}_{\|}$$

Symplectic Gyrokinetic Maxwell-Vlasov Theory

- Perturbed guiding-center dynamics

$$
\begin{aligned}
\Gamma_{\rm gc} &= \Gamma_{0\rm gc} + \epsilon\,\frac{e}{c}\,\mathbf{A}_{1\rm gc}\cdot\left(\mathrm{d}\mathbf{X} + \mathrm{d}\boldsymbol{\rho}_{\rm gc}\right) \\
\mathcal{H}_{\rm gc} &= \mathcal{H}_{0\rm gc} + \epsilon\, e\,\Phi_{1\rm gc}
\end{aligned}
$$

◦ Finite-Larmor-radius effects: $\mathsf{T}_{\rm gc}^{-1}\mathbf{x} = \mathbf{X} + \boldsymbol{\rho}_{\rm gc}$

$$
\left(\mathsf{T}_{\rm gc}^{-1}\Phi_1,\; \mathsf{T}_{\rm gc}^{-1}\mathbf{A}_1\right) \equiv \left(\Phi_{1\rm gc},\; \mathbf{A}_{1\rm gc}\right)
$$

$\Rightarrow$ Electromagnetic perturbations destroy the adiabatic invariance of the guiding-center gyroaction J: $d_{\rm gc}J/dt = \mathcal{O}(\epsilon)$

- Gyrocenter transformation $(\mathbf{X}, p_\|, J, \theta, w, t) \rightarrow (\overline{\mathbf{X}}, \overline{p}_\|, \overline{J}, \overline{\theta}, \overline{w}, t)$

$$
\begin{aligned}
\Gamma_{\rm gy} &\equiv \left(\frac{e}{c}\,\mathbf{A}_0 + \overline{p}_\|\,\widehat{\mathsf{b}}_0 + \boldsymbol{\Pi}_{\rm gy}\right)\cdot\mathrm{d}\overline{\mathbf{X}} + \overline{J}\,\mathrm{d}\overline{\theta} - \overline{w}\,\mathrm{d}t \\
\mathcal{H}_{\rm gy} &\equiv \overline{p}_\|^2/2m + \overline{J}\,\Omega + e\,\Psi_{\rm gy} - \overline{w} \equiv H_{\rm gy} - \overline{w}
\end{aligned}
$$

◦ Gyrocenter Jacobian

$$\mathcal{J}_{\rm gy} \;=\; \frac{e}{c}\,\mathsf{b}^{*}\cdot\mathbf{B}^{*} \;=\; \frac{e}{c}\,B_0 \;+\; \mathsf{b}^{*}\cdot\nabla\times\left(p_{\|}\,\widehat{\mathsf{b}}_0 + \boldsymbol{\Pi}_{\rm gy}\right)$$

where

$$\mathsf{b}^{*} \;\equiv\; \widehat{\mathsf{b}}_0 \;+\; \partial\boldsymbol{\Pi}_{\rm gy}/\partial p_{\|} \;\rightarrow\; \widehat{\mathsf{b}}_0\cdot\mathsf{b}^{*} \;=\; 1$$

$$\mathbf{B}^{*} \;\equiv\; \mathbf{B}_0 \;+\; (c/e)\,\nabla\times(p_{\|}\,\widehat{\mathsf{b}}_0 + \boldsymbol{\Pi}_{\rm gy})$$

◦ Gyrocenter Jacobian is time-dependent:

$$\frac{\partial\mathcal{J}_{\rm gy}}{\partial t} \;=\; \frac{\partial\mathsf{b}^{*}}{\partial t}\cdot\frac{e}{c}\,\mathbf{B}^{*} \;+\; \mathsf{b}^{*}\cdot\nabla\times\frac{\partial\boldsymbol{\Pi}_{\rm gy}}{\partial t}$$

Gyrocenter Hamilton Equations

- Gyrocenter Poisson bracket (omit gyrocenter bar unless needed)

$$
\begin{aligned}
\{\mathcal{F},\, \mathcal{G}\}_{\rm gy} \;&=\; \frac{\partial \mathcal{F}}{\partial w}\frac{\partial^* \mathcal{G}}{\partial t} - \frac{\partial^* \mathcal{F}}{\partial t}\frac{\partial \mathcal{G}}{\partial w} \;+\; \frac{\partial \mathcal{F}}{\partial \theta}\frac{\partial \mathcal{G}}{\partial J} - \frac{\partial \mathcal{F}}{\partial J}\frac{\partial \mathcal{G}}{\partial \theta} \\
&+ \frac{e\mathbf{B}^*}{c\mathcal{J}_{\rm gy}} \cdot \left(\nabla^* \mathcal{F}\frac{\partial \mathcal{G}}{\partial p_\|} - \frac{\partial \mathcal{F}}{\partial p_\|}\nabla^* \mathcal{G} \right) \\
&- \frac{\mathsf{b}^*}{\mathcal{J}_{\rm gy}} \cdot \nabla^* \mathcal{F} \times \nabla^* \mathcal{G}
\end{aligned}
$$

where the modified space-time operators are

$$
\begin{aligned}
\frac{\partial^*}{\partial t} \;&\equiv\; \frac{\partial}{\partial t} + \frac{\mathsf{b}^*}{\mathcal{J}_{\rm gy}} \cdot \left(\frac{\partial \boldsymbol{\Pi}_{\rm gy}}{\partial t} \times \frac{\partial \boldsymbol{\Pi}_{\rm gy}}{\partial J} \right) \frac{\partial}{\partial \theta}, \\
\nabla^* \;&\equiv\; \nabla - \frac{\partial \boldsymbol{\Pi}_{\rm gy}}{\partial J}\,\frac{\partial}{\partial \theta} - \frac{\partial \boldsymbol{\Pi}_{\rm gy}}{\partial t}\,\frac{\partial}{\partial w}.
\end{aligned}
$$

- Reduced gyrocenter Hamilton equations

$$\dot{\mathbf{X}} = \frac{\mathsf{b}^*}{\mathcal{J}_{\rm gy}} \times \left(\nabla H_{\rm gy} + \frac{\partial \mathbf{\Pi}_{\rm gy}}{\partial t} \right) + \frac{e\mathbf{B}^*}{c\mathcal{J}_{\rm gy}} \frac{\partial H_{\rm gy}}{\partial p_\|}$$

$$\dot{p}_\| = -\frac{e\mathbf{B}^*}{c\mathcal{J}_{\rm gy}} \cdot \left(\nabla H_{\rm gy} + \frac{\partial \mathbf{\Pi}_{\rm gy}}{\partial t} \right)$$

  - Gyrocenter gyromotion equations

$$\dot{J} = -\frac{\partial H_{\rm gy}}{\partial \theta} \equiv 0 \quad \text{and} \quad \dot{\theta} = \frac{\partial H_{\rm gy}}{\partial J} - \dot{\mathbf{X}} \cdot \frac{\partial \mathbf{\Pi}_{\rm gy}}{\partial J}$$

  - Gyrocenter Liouville Theorem:

$$\frac{\partial \mathcal{J}_{\rm gy}}{\partial t} + \nabla \cdot \left(\mathcal{J}_{\rm gy}\, \dot{\mathbf{X}} \right) + \frac{\partial}{\partial p_\|} \left(\mathcal{J}_{\rm gy}\, \dot{p}_\| \right) = 0$$

Gyrocenter near-identity phase-space transformation

- The gyrocenter symplectic one-form and Hamiltonian

$$\Gamma_{\rm gy} \;\equiv\; \mathsf{T}_{\rm gy}^{-1}\Gamma_{\rm gc} \;+\; {\sf d}S \quad \text{and} \quad \mathcal{H}_{\rm gy} \;\equiv\; \mathsf{T}_{\rm gy}^{-1}\mathcal{H}_{\rm gc}$$

are derived by a near-identity gyrocenter transformation

$$\overline{\mathcal{Z}}^{a} \;=\; \mathcal{Z}^{a} + \epsilon\,\mathcal{G}_1^{a} + \epsilon^2\left(\mathcal{G}_2^{a} + \frac{1}{2}\mathcal{G}_1^{b}\frac{\partial \mathcal{G}_1^{a}}{\partial \mathcal{Z}^{b}}\right) + \cdots$$

  - The gyrocenter push-forward operator

$$\mathsf{T}_{\rm gy}^{-1} \equiv \cdots \exp(-\epsilon^2\mathcal{L}_2)\,\exp(-\epsilon\mathcal{L}_1)$$

where the nth-order Lie derivative $\mathcal{L}_n$ is generated by $\mathcal{G}_n$, and the gauge function $S \equiv \epsilon\,S_1 + \epsilon^2 S_2 + \cdots$ represents the canonical part of the phase-space transformation.

- First-order analysis

◦ First-order symplectic equation

$$\mathbf{\Pi}_{1\mathrm{gy}} \cdot \frac{\partial \mathbf{X}}{\partial \mathcal{Z}^b} \;=\; \frac{e}{c}\,\mathbf{A}_{1\mathrm{gc}} \cdot \frac{\partial(\mathbf{X} + \boldsymbol{\rho}_{\mathrm{gc}})}{\partial \mathcal{Z}^b} \;-\; \mathcal{G}_1^a\,\omega_{0ab} \;+\; \frac{\partial S_1}{\partial \mathcal{Z}^b}$$

from which we obtain the first-order components

$$\mathcal{G}_1^a = \{S_1, \mathcal{Z}^a\}_0 + \frac{e}{c}\,\mathbf{A}_{1\mathrm{gc}} \cdot \left\{\mathbf{X} + \boldsymbol{\rho}_{\mathrm{gc}}, \mathcal{Z}^a\right\}_0 \;-\; \mathbf{\Pi}_{1\mathrm{gy}} \cdot \{\mathbf{X}, \mathcal{Z}^a\}_0$$

where $\{\;,\;\}_0$ unperturbed (guiding-center) Poisson bracket.

◦ First-order Hamiltonian equation

$$e\,\Psi_{1\mathrm{gy}} = e\,\left(\Phi_{1\mathrm{gc}} - \frac{\mathbf{v}_0}{c} \cdot \mathbf{A}_{1\mathrm{gc}}\right) \;+\; \mathbf{\Pi}_{1\mathrm{gy}} \cdot \dot{\mathbf{X}}_0 \;-\; \{S_1, \mathcal{H}_0\}_0$$

where $\mathbf{v}_0 = (p_\| / m)\,\widehat{\mathsf{b}}_0 + \Omega\,\partial \boldsymbol{\rho}_0 / \partial\theta$ and $\dot{\mathbf{X}}_0 = (p_\| / m)\,\widehat{\mathsf{b}}_0$.

◦ Since we want $\Psi_{1\mathrm{gy}}$ to be gyroangle-independent, we define

$$e\,\Psi_{1\mathrm{gy}} \;\equiv\; e\,\left\langle \Phi_{1\mathrm{gc}} - \frac{\mathbf{v}_0}{c}\cdot\mathbf{A}_{1\mathrm{gc}}\right\rangle \;+\; \mathbf{\Pi}_{1\mathrm{gy}}\cdot\dot{\mathbf{X}}_0$$

where $\langle\cdots\rangle$ denotes gyroangle averaging.

◦ First-order gauge function S_1 (here, $\langle S_1\rangle = 0$):

$$\{S_1, \mathcal{H}_0\}_0 \;\equiv\; e\,\left(\Phi_{1\mathrm{gc}} - \frac{\mathbf{v}_0}{c}\cdot\mathbf{A}_{1\mathrm{gc}}\right) - e\,\left\langle \Phi_{1\mathrm{gc}} - \frac{\mathbf{v}_0}{c}\cdot\mathbf{A}_{1\mathrm{gc}}\right\rangle \;\equiv\; e\,\widetilde{\psi}_{1\mathrm{gc}}$$

◦ Gyrocenter gyroaction is independent of the choice of $\mathbf{\Pi}_{1\mathrm{gy}}$:

$$G_1^J \;=\; \frac{\partial S_1}{\partial\theta} \;+\; \frac{e}{c}\,\mathbf{A}_{1\perp\mathrm{gc}}\cdot\frac{\partial\boldsymbol{\rho}_{\mathrm{gc}}}{\partial\theta}$$

◦ We now introduce a new choice for the gyrocenter symplectic momentum $\mathbf{\Pi}_{1\mathrm{gy}}$ based on the first-order gyrocenter polarization displacement

Gyrocenter Polarization Displacement

- Calculate the first-order gyrocenter displacement

$$\boldsymbol{\rho}_{1\rm gy} \;\equiv\; -\,\mathcal{G}_1 \cdot \mathsf{d}(\mathbf{X} + \boldsymbol{\rho}_0) \;=\; \{\mathbf{X} + \boldsymbol{\rho}_0,\; S_1\}_0 \;+\; \boldsymbol{\Pi}_{1\rm gy} \times \frac{c\widehat{\mathsf{b}}_0}{eB_0}$$

  - Gyroangle-averaged first-order gyrocenter displacement

$$\begin{aligned}\langle\boldsymbol{\rho}_{1\rm gy}\rangle \;&=\; \Big\langle \{\boldsymbol{\rho}_0,\; S_1\}_0 \Big\rangle \;+\; \boldsymbol{\Pi}_{1\rm gy} \times \frac{c\widehat{\mathsf{b}}_0}{eB_0} \\ &\simeq\; \frac{e\widehat{\mathsf{b}}_0}{\Omega} \times \left[\frac{e}{mc}\langle\mathbf{A}_{1\perp\rm gc}\rangle + \frac{c\widehat{\mathsf{b}}_0}{B_0} \times \nabla\langle\psi_{1\rm gc}\rangle \right] \;+\; \boldsymbol{\Pi}_{1\rm gy} \times \frac{c\widehat{\mathsf{b}}_0}{eB_0}\end{aligned}$$

where

$$e\,\langle\psi_{1\rm gc}\rangle \;=\; e\left(\langle\Phi_{1\rm gc}\rangle - \frac{p_\|}{mc}\,\langle A_{1\|\rm gc}\rangle\right) \;+\; \mu\,\langle\langle B_{1\|\rm gc}\rangle\rangle$$

- Zero gyrocenter polarization displacement $\langle \boldsymbol{\rho}_{1\mathrm{gy}} \rangle = 0$:
  $\Rightarrow$ Gyrocenter symplectic momentum

$$\boldsymbol{\Pi}_{1\mathrm{gy}} = \frac{e}{c}\langle \mathbf{A}_{1\mathrm{gc}} \rangle + \frac{p_{\|}}{B_0}\langle \mathbf{B}_{1\perp\mathrm{gc}} \rangle + \frac{\widehat{\mathrm{b}}_0}{\Omega} \times \left(e\,\nabla \langle \Phi_{1\mathrm{gc}} \rangle \;+\; \mu\,\nabla \langle\langle B_{1\|\mathrm{gc}} \rangle\rangle \right)$$

- Gyrocenter polarization reappears in the gyrocenter Jacobian

$$\begin{aligned}
\mathcal{J}_{\mathrm{gy}} &= \mathrm{b}^* \cdot \frac{e}{c}\mathbf{B}^* \\
&= \left(\widehat{\mathrm{b}}_0 + \epsilon \frac{\langle \mathbf{B}_{1\perp\mathrm{gc}} \rangle}{B_0} \right) \cdot \left[\frac{e}{c}\,\mathbf{B}_0 + \nabla \times \left(p_{\|}\,\widehat{\mathrm{b}}_0 \;+\; \epsilon\,\boldsymbol{\Pi}_{1\mathrm{gy}} \right) \right] \\
&\simeq \mathcal{J}_{0\mathrm{gc}} \;+\; \epsilon\,\mathcal{J}_{0\mathrm{gc}}\,\nabla \cdot \left(\boldsymbol{\Pi}_{1\mathrm{gy}} \times \frac{c\widehat{\mathrm{b}}_0}{eB_0} \right)
\end{aligned}$$

- First-order gyrocenter Hamiltonian

$$e\,\Psi_{1\mathrm{gy}} \;=\; e\,\langle \Phi_{1\mathrm{gc}} \rangle \;+\; \mu\,\langle\langle B_{1\|\mathrm{gc}} \rangle\rangle$$

Symplectic Gyrokinetic Variational Principle

- Symplectic gyrokinetic Vlasov-Maxwell Lagrangian density

$$\mathcal{L}_{\rm gy} \equiv -\int_P \mathcal{F}_{\rm gy}\,\mathcal{H}_{\rm gy} \;+\; \frac{1}{8\pi}\left(|\mathbf{E}|^2 \;-\; |\mathbf{B}|^2\right)$$

where summation over particle species is implicitly assumed and $\int_P$ denotes a four-momentum integration involving $(p_\|, J, \theta, w)$.

- ∘ Gyrocenter extended Vlasov density $\mathcal{F}_{\rm gy} \;\equiv\; \mathcal{J}_{\rm gy}\,\mathcal{F}$

- ∘ Gyrocenter extended Hamiltonian

$$\mathcal{H}_{\rm gy} \;=\; \left(p_\|^2/2m + \mu\,B_0\right) \;+\; \epsilon\left(e\,\langle\Phi_{1\rm gc}\rangle \;+\; \mu\,\langle\langle B_{1\|\rm gc}\rangle\rangle\right) \;-\; w$$

- Variation of the gyrokinetic Lagrangian density

$$\delta\mathcal{L}_{\rm gy} \;=\; -\int_P \left(\delta\mathcal{F}_{\rm gy}\,\mathcal{H}_{\rm gy} + \mathcal{F}_{\rm gy}\,\delta\mathcal{H}_{\rm gy}\right) \;+\; \frac{1}{4\pi}\left(\delta\mathbf{E}\cdot\mathbf{E} \;-\; \delta\mathbf{B}\cdot\mathbf{B}\right)$$

- Electromagnetic variations

$$\begin{aligned}\delta\mathbf{E}\cdot\mathbf{E} - \delta\mathbf{B}\cdot\mathbf{B} \;&=\; \epsilon\,\delta\mathbf{A}_1\cdot\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} - \nabla\times\mathbf{B}\right) \;+\; \epsilon\,\delta\Phi_1\,(\nabla\cdot\mathbf{E}) \\ &\quad -\frac{\partial}{\partial t}\left(\frac{\epsilon}{c}\,\delta\mathbf{A}_1\cdot\mathbf{E}\right) - \nabla\cdot\left(\epsilon\,\delta\Phi_1\,\mathbf{E} + \epsilon\,\delta\mathbf{A}_1\times\mathbf{B}\right).\end{aligned}$$

- Variation of extended gyrocenter Vlasov density:

$$\begin{aligned}\delta\mathcal{F}_{\rm gy} \;&=\; \delta\mathcal{J}_{\rm gy}\,\mathcal{F} \;+\; \mathcal{J}_{\rm gy}\,\delta\mathcal{F} \\ &=\; \mathcal{F}\left(\frac{\partial\delta\mathbf{\Pi}_{\rm gy}}{\partial p_\|}\cdot\frac{e}{c}\mathbf{B}^* + \mathrm{b}^*\cdot\nabla\times\delta\mathbf{\Pi}_{\rm gy}\right) \\ &\quad + \mathcal{J}_{\rm gy}\left(\{\delta\mathcal{S},\;\mathcal{F}\}_{\rm gy} \;+\; \delta\mathbf{\Pi}_{\rm gy}\cdot\{\mathbf{X},\;\mathcal{F}\}_{\rm gy}\right)\end{aligned}$$

◦ Identity I (integration by parts)

$$
\begin{aligned}
-\int_P \delta(\mathcal{F}_{\rm gy}\,\mathcal{H}_{\rm gy}) \;\;=\;\; & -\int_P \mathcal{J}_{\rm gy}\{\mathcal{F},\,\mathcal{H}_{\rm gy}\}_{\rm gy}\;\delta\mathcal{S} \\
& +\int_P \mathcal{F}_{\rm gy}\;\left(\delta\boldsymbol{\Pi}_{\rm gy}\cdot\dot{\mathbf{X}} - e\,\delta\Psi_{\rm gy}\right) \\
& +\frac{\partial}{\partial t}\left(\int_P \mathcal{F}_{\rm gy}\,\delta\mathcal{S}\right) \;+\; \nabla\cdot\left(\int_P \dot{\mathbf{X}}\,\mathcal{F}_{\rm gy}\,\delta\mathcal{S}\right)
\end{aligned}
$$

where $\dot{\mathbf{X}} \equiv \{\mathbf{X}, \mathcal{H}_{\rm gy}\}_{\rm gy}$ denotes the full gyrocenter velocity.

◦ Identity II (keep lowest order in $B_{1\|{\rm gc}}$)

$$
\begin{aligned}
\delta\boldsymbol{\Pi}_{1{\rm gy}}\cdot\dot{\mathbf{X}} - e\,\delta\Psi_{1{\rm gy}} \;\;=\;\; & -e\,\langle\delta\Phi_{1{\rm gc}}\rangle + \frac{e}{c}\,\langle\delta\mathbf{A}_{1{\rm gc}}\rangle\cdot\dot{\mathbf{X}} - \mu\,\widehat{\mathsf{b}}_0\cdot\langle\langle\delta\mathbf{B}_{1{\rm gc}}\rangle\rangle \\
& + \left(\langle\delta\mathbf{E}_{1{\rm gc}}\rangle + \frac{p_\|\,\widehat{\mathsf{b}}_0}{mc}\times\langle\delta\mathbf{B}_{1{\rm gc}}\rangle\right)\cdot\frac{e\widehat{\mathsf{b}}_0}{\Omega}\times\dot{\mathbf{X}}
\end{aligned}
$$

- Identity III

$$\int_P \mathcal{F}_{\rm gy} \left(\delta\mathbf{\Pi}_{\rm gy}\cdot\dot{\mathbf{X}} - e\,\delta\Psi_{\rm gy}\right) \;\equiv\; -\,\epsilon\,\delta\Phi_1\;\varrho_{\rm gy} \;+\; \epsilon\,\delta\mathbf{A}_1\cdot\mathbf{J}_{\rm gy}/c \;+\;\epsilon\,\delta\mathbf{E}_1\cdot\mathbb{P}_{\rm gy} \;+\; \epsilon\,\delta\mathbf{B}_1\cdot\mathbb{M}_{\rm gy}$$

- Gyrocenter charge and current densities

$$\varrho_{\rm gy} \;\equiv\; \int_P \mathcal{F}_{\rm gy} \left(e\,\frac{\delta\langle\Phi_{1\rm gc}\rangle}{\delta\Phi_1}\right)$$

$$\mathbf{J}_{\rm gy} \;\equiv\; \int_P \mathcal{F}_{\rm gy} \left(\frac{\delta\langle\mathbf{A}_{1\rm gc}\rangle}{\delta\mathbf{A}_1}\right)\cdot e\,\dot{\mathbf{X}}$$

- Gyrocenter polarization and magnetization

$$\mathbb{P}_{\rm gy} \;\equiv\; \int_P \frac{\delta\mathbf{\Pi}_{1\rm gy}}{\delta\mathbf{E}_1}\cdot\dot{\mathbf{X}}\;\mathcal{F}_{\rm gy}$$

$$\mathbb{M}_{\rm gy} \;\equiv\; \int_P \left(\frac{\delta\mathbf{\Pi}_{1\rm gy}}{\delta\mathbf{B}_1}\cdot\dot{\mathbf{X}} \;-\; \mu\,\widehat{\mathsf{b}}_0\cdot\frac{\partial\langle\mathbf{B}_{1\rm gc}\rangle}{\partial\mathbf{B}_1}\right)\;\mathcal{F}_{\rm gy}$$

- Variation of the gyrokinetic Lagrangian density

$$
\begin{aligned}
\delta\mathcal{L}_{\rm gy} \;\; = \;\; & -\int_P \mathcal{J}_{\rm gy}\{\mathcal{F},\, \mathcal{H}_{\rm gy}\}_{\rm gy}\,\delta\mathcal{S} \\
& + \frac{\epsilon\delta\Phi_1}{4\pi}\left(\nabla\cdot\mathbb{D}_{\rm gy} \;-\; 4\pi\;\varrho_{\rm gy}\right) \\
& + \frac{\epsilon\delta\mathbf{A}_1}{4\pi}\cdot\left(\frac{1}{c}\frac{\partial\mathbb{D}_{\rm gy}}{\partial t} \;-\; \nabla\times\mathbb{H}_{\rm gy} \;+\; \frac{4\pi}{c}\,\mathbf{J}_{\rm gy}\right) \\
& + \frac{\partial}{\partial t}\left(\int_P \mathcal{F}_{\rm gy}\,\delta\mathcal{S} \;-\; \frac{\epsilon}{4\pi c}\,\delta\mathbf{A}_1\cdot\mathbb{D}_{\rm gy}\right) \\
& + \nabla\cdot\left[\int_P \mathcal{F}_{\rm gy}\,\dot{\mathbf{X}}\,\delta\mathcal{S} \;-\; \frac{\epsilon}{4\pi}\left(\delta\Phi_1\,\mathbb{D}_{\rm gy} \;+\; \delta\mathbf{A}_1\times\mathbb{H}_{\rm gy}\right)\right]
\end{aligned}
$$

- ◦ Gyrocenter electromagnetic fields

$$
\begin{aligned}
\mathbb{D}_{\rm gy} & \equiv \mathbf{E} \;+\; 4\pi\;\mathbb{P}_{\rm gy} \\
\mathbb{H}_{\rm gy} & \equiv \mathbf{B} \;-\; 4\pi\;\mathbb{M}_{\rm gy}
\end{aligned}
$$

- Gyrocenter Vlasov equation: $\int \mathcal{J}_{\rm gy}\,\{\mathcal{F},\mathcal{H}_{\rm gy}\}_{\rm gy}\,dw \;=\; 0 \rightarrow$

$$\frac{\partial}{\partial t}\left(\mathcal{J}_{\rm gy}\,F\right) \;+\; \nabla\cdot\left(\mathcal{J}_{\rm gy}\,F\,\dot{\mathbf{X}}\right) \;+\; \frac{\partial}{\partial p_{\|}}\left(\mathcal{J}_{\rm gy}\,F\,\dot{p}_{\|}\right) \;=\; 0$$

- Gyrokinetic Maxwell equations

$$\begin{aligned}\nabla\cdot\mathbb{D}_{\rm gy} &= 4\pi\;\varrho_{\rm gy}\\ \nabla\times\mathbb{H}_{\rm gy} &= \frac{1}{c}\,\frac{\partial\mathbb{D}_{\rm gy}}{\partial t} \;+\; \frac{4\pi}{c}\,\mathbf{J}_{\rm gy}\end{aligned}$$

- Gyrokinetic Noether equation

$$\begin{aligned}\delta\mathcal{L}_{\rm gy} \;&=\; \frac{\partial}{\partial t}\left(\int_P \mathcal{F}_{\rm gy}\,\delta\mathcal{S} \;-\; \frac{\epsilon}{4\pi c}\,\delta\mathbf{A}_1\cdot\mathbb{D}_{\rm gy}\right)\\ &\quad+\;\nabla\cdot\left[\int_P \mathcal{F}_{\rm gy}\,\dot{\mathbf{X}}\,\delta\mathcal{S} \;-\; \frac{\epsilon}{4\pi}\left(\delta\Phi_1\,\mathbb{D}_{\rm gy} \;+\; \delta\mathbf{A}_1\times\mathbb{H}_{\rm gy}\right)\right]\end{aligned}$$

Note: Gyrokinetic polarization and magnetization effects obtained without 2^{nd}-order gyrocenter Hamiltonian

Symplectic Gyrokinetic Vlasov-Maxwell Equations

- Gyrokinetic Vlasov equation

$$
\begin{aligned}
0 &= \frac{\partial F}{\partial t} + \dot{\mathbf{X}} \cdot \nabla F + \dot{p}_{\|} \frac{\partial F}{\partial p_{\|}} \\
&= \frac{1}{\mathcal{J}_{\rm gy}} \left[\frac{\partial}{\partial t}(\mathcal{J}_{\rm gy}\, F) + \nabla \cdot \left(\mathcal{J}_{\rm gy}\, F\, \dot{\mathbf{X}} \right) + \frac{\partial}{\partial p_{\|}} \left(\mathcal{J}_{\rm gy}\, F\, \dot{p}_{\|} \right) \right]
\end{aligned}
$$

◦ Gyrocenter Hamiltonian

$$
H_{\rm gy} = p_{\|}^2/2m + \mu \left(B_0 + \epsilon \langle\langle B_{1\|{\rm gc}} \rangle\rangle \right) + \epsilon\, e\, \langle \Phi_{1{\rm gc}} \rangle
$$

◦ Gyrocenter symplectic momentum

$$
\mathbf{\Pi}_{1{\rm gy}} = \frac{e}{c} \langle \mathbf{A}_{1{\rm gc}} \rangle + \frac{p_{\|}}{B_0} \langle \mathbf{B}_{1\perp{\rm gc}} \rangle + \frac{e\widehat{\sf b}_0}{\Omega} \times \nabla \Psi_{1{\rm gy}}
$$

- Reduced gyrocenter Hamilton equations

$$\begin{aligned} \dot{\mathbf{X}} &= \frac{\mathsf{b}^*}{\mathcal{J}_{\rm gy}} \times \left(\nabla H_{\rm gy} + \frac{\partial \boldsymbol{\Pi}_{\rm gy}}{\partial t} \right) + \frac{e\mathbf{B}^*}{c\mathcal{J}_{\rm gy}} \frac{p_\|}{m} \\ \dot{p}_\| &= -\frac{e\mathbf{B}^*}{c\mathcal{J}_{\rm gy}} \cdot \left(\nabla H_{\rm gy} + \frac{\partial \boldsymbol{\Pi}_{\rm gy}}{\partial t} \right) \end{aligned}$$

- ◦ Gyrocenter symplectic magnetic field

$$\mathbf{B}^* = \left(\mathbf{B}_0 + \epsilon \left\langle \mathbf{B}_{1\rm gc} \right\rangle\right) + \frac{c}{e} \nabla \times \left(p_\| \, \mathsf{b}^* + \epsilon \, \frac{e\widehat{\mathsf{b}}_0}{\Omega} \times \nabla \Psi_{1\rm gy} \right)$$

- ◦ Gyrocenter drifts with polarization correction

$$\begin{aligned} \nabla H_{\rm gy} + \frac{\partial \boldsymbol{\Pi}_{\rm gy}}{\partial t} \;&\simeq\; \mu \nabla \left(B_0 + \epsilon \left\langle\left\langle B_{1\|\rm gc} \right\rangle\right\rangle \right) - \epsilon \, e \langle \mathbf{E}_{1\rm gc} \rangle \\ &\quad + \epsilon \, \frac{d_{0\rm gc} \langle \mathbf{E}_{1\perp\rm gc} \rangle}{dt} \times \frac{e\widehat{\mathsf{b}}_0}{\Omega} \end{aligned}$$

- Gyrokinetic Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi \left(\varrho_{\rm gy} - \nabla \cdot \mathbb{P}_{\rm gy} \right)$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \left(\mathbf{J}_{\rm gy} + \frac{\partial \mathbb{P}_{\rm gy}}{\partial t} + c\, \nabla \times \mathbb{M}_{\rm gy} \right)$$

  - Gyrocenter polarization

$$\mathbb{P}_{\rm gy} = \int \mathcal{J}_{\rm gy}\, F\, \langle \delta^3_{\rm gc} \rangle\, \frac{e\widehat{\mathsf{b}}_0}{\Omega} \times \dot{\mathbf{X}} \equiv \int \mathcal{J}_{\rm gy}\, F\, \boldsymbol{\pi}_{\rm gy}$$

  - Gyrocenter magnetization

$$\mathbb{M}_{\rm gy} = \int \mathcal{J}_{\rm gy}\, F \left(\boldsymbol{\pi}_{\rm gy} \times \frac{p_{\|}\widehat{\mathsf{b}}_0}{mc} - \mu\, \widehat{\mathsf{b}}_0 \right)$$

Symplectic gyrokinetic Vlasov-Maxwell theory - Talk by Alain Brizard

Lecture notes (Ori S. Katz)

October 11, 2018

Abstract

We consider a general form of electromagnetic gyrokinetic Vlasov-Maxwell theory in which the gyrocenter symplectic structure contains electric and magnetic perturbations that are necessary to cause the first-order gyrocenter polarization displacement to vanish. The gyrocenter Hamilton equations, which are expressed in terms of a gyrocenter Poisson bracket that contains electromagnetic perturbations and a gyrocenter Hamiltonian, satisfy the Liouville property exactly with a time-dependent gyrocenter Jacobian. The gyrokinetic Vlasov-Maxwell equations are derived from a variational principle, which also yields exact conservation laws through the Noether method. We show that the new symplectic gyrokinetic Vlasov-Maxwell equations retain all first-order polarization and magnetization effects without the need to consider second-order contributions in the gyrocenter Hamiltonian.

1 Lecture notes

Outline:

Vlasov-Maxwell equations. Specifically - variational principles for exact and reduced, by guiding center and gyrocenter, VM equations.

Lagrangian - not canonical transformations, so the Poisson brackets change.

Hamiltonian - transforming from the Lagrange bracket to the Poisson bracket. This procedure guarantees that the Poisson bracket will satisfy all the properties, specifically the Jacobi equality. From Hamilton's equations we get the Liouville theorem - Jacobian is 1.

Vlasov equation - characteristics are simple Hamilton's equations. It states that the distribution is a slaved function - whatever you do to the trajectory, the Vlasov distribution follows.

Vlasov equation in extended phase space - multiply physical Vlasov equation to a delta function.

We're going to recover the extended phase space version by variational principle.

Maxwell equation - divided into two groups - with and without particle sources.

Vlasov Maxwell equations- Brizard (2000) - Several versions exist. I present one derived in 2000 that exploits extended phase space version. EM Lagrangian $|E|^2 - |B|^2$ minus the integral over the momentum part.

Calculating the Eulerian variation, only the electric potential is varies, not the kinetic energy which is a function of the dummy variable.

The constrained variation has two parts - canonical and symplectic. The variation $\delta \boldsymbol{A}$ appears in the symplectic part.

The electric and magnetic field are not varied independently.

Rearranging terms, we obtain 4 terms, with δS, $\delta \phi$ and $\delta \boldsymbol{A}$. By rearranging the terms, we obtain an explicit time derivative, resulting in a Noether equation allowing a calculation of conserved quantities.

After the variational principle, the variation in the Lagrangian equals an explicit time derivative pluse space derivaticve.

Thus we obtain momentum-energy conservation, relating to space-time symmetry. There is also a gauge variation term, important because the δS is not gauge invariant.

The varition of the Lagrangian - we can impose that $\int FH = 0$ and the only variation left is the Maxwell one.

Comparing variations to zero, obtain the energy-momentum conservation laws, with the Vlasov-Maxwell energy density, energy density flux, momentum density and stress tensor.

So the variational principle is used to show that the Vlasov-Maxwell equations are subject to the variational method, and have explicit conserved quantities.

Guiding-center Lagrangian one-form - gyromotion has been transformed - J is the gyro action. wdt is the extended moment. Littlejohn - higher order contributions to the bracket.

Guiding center extended phase space Poisson bracket.

Here, $\boldsymbol{A}, \boldsymbol{B}$ are functions of time.

Deriving the reduced guiding center EOM, $\boldsymbol{E}^*$ can be written analogously to an electric field, with ϕ^* and $\boldsymbol{A}^*$ instead.

Guiding center Jacobian - non-canonical transformation so the Jacobian is no longer 1, it is an explicit function of time as it depends on B. This is in constrast to the standard guiding center case.

When moving to reduced picture, the rapid fluctuations shift from the Vlasov equations to the Maxwell equations.

Guiding center electric dipole moment - first order in magnetic field non-uniformity. Two contributions to magnetization - intrinsic (Hamiltonian), related to the derivative of the Hamiltonian w.r.t. B; and a symplectic contribution, taking into account only the perpendicular component of the guiding center velocity.

Guiding center Vlasov-Maxwell equations empoys the guiding center Vlasov equation as characteristics. Can be written either in standard or divergence form, using the Liouville theorem. Source-free guiding center Maxwell equations remains with starred fields.

Remaining guiding center Maxwell equations - current is the moment of the guiding center velocity. There is no guiding-center polarization here. However, the guiding center polarization enters into the symplectic part.

Guiding-center magnetization - two contributions.

Now, a variational form of these equations.

F_μ and H are the extended fields.

Varying the Hamiltonian, we have an additional term - the Hamiltonian contribution to magnetization. The varied field δF_μ will have three contributions, Jacobian, canonical and symplectic parts. Note that the $\delta \boldsymbol{A}^*$ has a part with $\delta \boldsymbol{A}$ and a part with $\delta \boldsymbol{B}$.

Eventually, obtain the guiding center Lagrangian variation, with a magnetic field and a Noether equation.

If there was magnetization, E would be replaced with a D.

Thus we obtain the guiding center conservation laws. Poynting flux now has $E \times H$.

The derived guiding center momentum density has only the parallel momentum, and $E \times B$.

The guiding center stress term has two contibutions, the Maxell and the guiding center Vlasov. They are symmetric.

In red - the guiding center magnetization from all contributions. If we wouldn't take all contributions into account the angular momentum would not be conserved and the tensor would not be symmetric.

Symplectic Gyrokinetic Maxwell-Vlasov equations:

Guiding center dynamics are perturbed by a magnetic field. Gyro radius - ρ_{gc}.

Thus the electromagnetic perturbations destroy the adiabatic invariance of the guiding center gyroaction.

Using new coordinates, the new structure have a contribution from the gyro perturbation. We have complete freedom in choosing what goes into the perturbations, and how much go into the symplectic vs. the Hamiltonian parts.

Gyrocenter Jacobian - dot product of parallel unit vector that is no longer parallel, with B^*.

The gyrocenter Jacobian is time dependent.

Since the gyrocenter momentum is time dependent, there are corrections to the coordinate and parallel momentum time derivatives. Liouville theorem is satisfied.

How to derive these new gyrocenter phase space coordinates?

Lie transform perturbation techniques, generated by the gyrocenter and the gauge function one form. The choice of S will not change the nature of the Poisson bracket, but it will appear in the definition of the Hamiltonian.

Generally, in order to obtain a first order Hamiltonian, we need to derive the second order Hamiltonian as well. Here the second order Hamiltonian is not needed.

First order analysis allows us to find the first-order Hamiltonian equation.

We choose S_1 to be gyroangle-independent, resulting in ψ_{1gy} to be gyroangle-independent. S_1 only depends on the gyroangle-independent part of the Hamiltonian.

While there are many representations, they all treat the gyroaction as a unique quantity, as the physics dictate. Indeed, the theory is quite explicit - gyroation does not depend on the choice made for the gyrocenter symplectic momentum ψ.

Gyrocenter polarization displacement - pushing forward the particle position into gyrocenter phase space - obtain two contributions, one due to S_1 and one due to Π_{1gy}. When calculating the gyrocenter average, obtain contributions from gyrocenter momentum, perpendicular complnent of A and the gradient of the average of ψ_{1gc}.

You could choose the gyrocenter polarization displacement to be zero, by choosing Π_{1gy} as written. We will see the impact this has on the equations.

We use this specific choice of the gyrocenter symplectic momentum Π_{1gy}. The physics of course do not change, just get shuffled somewhere else.

Thus the derived first order gyrocenter Hamiltonian has a very natural simple relationship.

Now, we will derive these equations from the variational principle. The Lagrangian density has the same form. Variational principle is derived by performing the variation. The variation of the extended gyrocenter Vlasov density again has three components.

Rearranging the variations, we can write the interactive portion in terms of functional derivatives. The magnetization $\mathbb{M}$ has an intrinsic and an electric dipole contribution.

Obtain - Gyrocenter Vlasov equation, gyrokinetic Maxwell equations and gyrokinetic Noether equation. This is derived straight from the variational principle without going through the calculation of the second order gyrocenter Hamiltonian.

Polarization charge density - $\mathbb{P}_{gy}$.

2 Questions

- This is important because leads to computational simplifications, are your final results used in gyrokinetic codes?

Gyrokinetic codes are used to simulate plasma systems. Reduced systems are useful for computational reasons. Very few computational codes took into account the second order gyrocenter Hamiltonian. This symplectic representation might provide an equally powerful set of equations without deriving the higher order effects.

- What is the principle by which the constraint variations of F and B were derived?

For the electro-magnetic fields, use that Gauss's law and Vlasov laws are conserved. For F, the phase space integral of δF must be zero.