2.4

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NOTETAKER CHECKLIST FORM

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Name: ORI KATZ Email/Phone: ORIKATZ.OK@gmail.com

Speaker's Name: Emanuele Tassi

Talk Title: Hamiltonian reduced fluid models for non-dissipative plasmas

Date: 10/9/18 Time: 2:00 am /(pm) (circle one)

Please summarize the lecture in 5 or fewer sentences: The Hamiltonian structure of reduced fluid models for plasmas is not always easy to identify. Tassi presented a procedure to derive Hamiltonian reduced fluid models from parent gyrokinetic models. Shown explicitly for drift-kinetic models, but amenable to further extensions including parallel magnetic fluctuations, finite Larmor radius effects & equilibrium temperature anisotropy.

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Hamiltonian reduced fluid models for non-dissipative plasmas

E. Tassi

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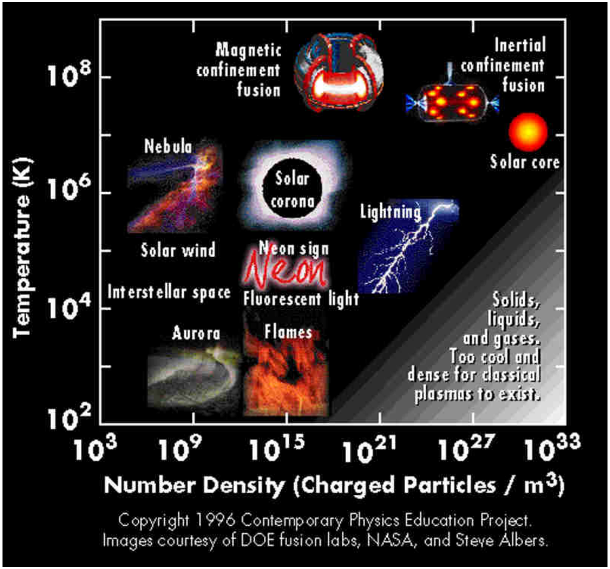
1. Plasmas

- Ionized gas quasi-neutral on sufficiently large scales
  ($n_e \approx n_{ions}$ for $L \gg \lambda_D$)
- Naturally present in various astrophysical environments
- Created in laboratory (e.g. nuclear fusion experiments)



2. Modelling plasma dynamics

- Very large number of particles (10^{20} per m^3 in tokamak fusion devices) interacting with electromagnetic fields

  $\Rightarrow$ describing dynamics of all particles not feasible

  $\Rightarrow$ Continuum models are often adopted:

- Kinetic description: distribution function for each species $f(\mathbf{x}, \mathbf{v}, t)$ coupled with electromagnetic fields

  (e.g. Vlasov-Maxwell, Vlasov-Poisson)

- Multifluid description: particle species treated as fluids interacting with electromagnetic fields via moments (density $n(\mathbf{x}, t)$, velocity $u(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t), \cdots$)

- Magnetohydrodynamics (MHD): Single conducting fluid

- A multitude of other reduced kinetic/fluid models obtained from the previous ones by asymptotic expansions, truncations, averagings,...

3. Reduced fluid models

- Parent model: ideal MHD (no dissipation)

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - (\nabla \times \mathbf{B}) \times \mathbf{B} = 0,$$
$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0,$$
$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0$$

  - $\mathbf{B}(\mathbf{x}, t)$ magnetic field, $\mathbf{v}(\mathbf{x}, t)$ velocity field p pressure
  - $\mathbf{B}, \mathbf{v} : \mathcal{D} \times \mathbb{R} \to \mathbb{R}^3$ where $\mathbf{x} = (x, y, z) \in \mathcal{D} \subseteq \mathbb{R}^3$, $t \geq 0$
    - $p : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ (p determined via $\nabla \cdot \mathbf{v} = 0$)

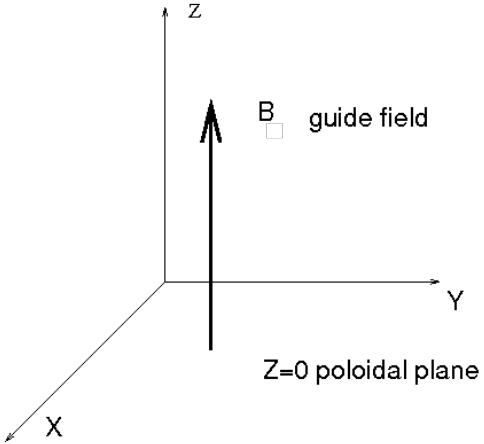
- Fusion and astrophysical plasmas phenomena (turbulence, reconnection, instabilities,..)

- Simplified fluid models are often desirable (numerics and analytics)

- Obtained from more general fluid models via reduction in number of dynamical variables

4. Low-frequency reduced fluid models for plasmas in strong guide fields

- Strong constant and uniform component of magnetic field in one direction : $B_x/B \sim B_y/B \sim \epsilon \ll 1$
- E.g. toroidal magnetic field in tokamaks, mean magnetic field in solar wind



- Low frequency: $\omega/\omega_{ci} \sim \epsilon$ where $\omega_{ci} \equiv eB/(m_i c)$
- Strong anisotropy: $\partial_z/\partial_x \sim \partial_z/\partial_y \sim \epsilon$
- Small pressure : $p \sim \epsilon^2$
- Negligible parallel velocity : $v_z \approx 0$

5. Example : reduced MHD (RMHD) (Strauss (1976, 1977))

$$
\begin{aligned}
&\frac{\partial A}{\partial t} + [\phi, A] + \frac{\partial \phi}{\partial z} = 0, \\
&\frac{\partial \nabla_\perp^2 \phi}{\partial t} + [\phi, \nabla_\perp^2 \phi] - [A, \nabla_\perp^2 A] + \frac{\partial \nabla_\perp^2 A}{\partial z} = 0
\end{aligned}
$$

- $\mathbf{B}(\mathbf{x}, t) = \nabla \times (A(\mathbf{x}, t)\hat{z}) + B\hat{z}$ magnetic field

- $\mathbf{v}(\mathbf{x}, t) = \hat{z} \times \nabla \phi(\mathbf{x}, t)$ velocity field

  where $A, \phi : \mathcal{D} \times \mathbb{R} \to \mathbb{R}, \qquad B \in \mathbb{R}, \quad B \geq 0$

  $\nabla_\perp f \equiv \partial_x f\, \hat{x} + \partial_y f\, \hat{y}, \qquad \nabla_\perp^2 f \equiv \partial_{xx} f + \partial_{yy} f, \qquad [f, g] \equiv \partial_x f\, \partial_y g - \partial_y f\, \partial_x g$

- Less accurate but also less demanding, analytically and numerically, than parent general models

- $B \gg |\nabla_\perp A| \to$ Reduction from $B_x, B_y, B_z, v_x, v_y, v_z$ of MHD down to A, ϕ

6. Further examples of low-frequency reduced fluid models (not exhaustive)

- Electrostatic drift waves (Hasegawa and Mima, 1978)
- Nonlinear tokamak dynamics (Hazeltine, Kotschenreuther and Morrison (1985), Hazeltine and Meiss (1985), Hazeltine, Hsu and Morrison (1986,1987))
- Ion temperature gradient instability (Kim, Horton and Hamaguchi (1989))
- Magnetic reconnection (Schep, Pegoraro and Kuvshinov (1994))
- Electromagnetic drift waves (Camargo, Biskamp and Scott (1996))
- Interchange turbulence (Dagnelund and Pavlenko (2005))
- Gyrofluid turbulence (Brizard (1992), Dorland and Hammett (1993), Snyder and Hammett (2001), Scott (2010))

7. Hamiltonian structure

- Important : models should be Hamiltonian when dissipation is neglected
- $(\chi_1(\mathbf{z},t),\cdots,\chi_N(\mathbf{z},t)) \in U$ space of field variables (e.g. density, electromagnetic fields, distribution functions,..)
  $\mathbf{z}$ set of coordinates (space and velocity, in general)
- $\mathcal{F} = \{F : U \to \mathbb{R}\}$ set of observables
- Hamiltonian model if

$$\frac{\partial F}{\partial t} = \{F, H\}, \qquad \forall F \in \mathcal{F}$$

- $H \in \mathcal{F}$ Hamiltonian observable (total energy)
- $\{\,,\,\}$ Poisson bracket:
- Reduced fluid models typically formulated in terms of noncanonical variables $\Rightarrow$ noncanonical Poisson brackets
- No pairs of canonically conjugate variables globally on phase space
- Symplectic leaves foliate phase space
- Casimir invariants $C \in \mathcal{F}$: $\{C, F\} = 0, \qquad \forall F \in \mathcal{F}$

8. Ex. Hamiltonian structure for RMHD (Morrison and Hazeltine (1984))

$$\frac{\partial A}{\partial t} + [\phi, A] + \frac{\partial \phi}{\partial z} = 0,$$
$$\frac{\partial \nabla_\perp^2 \phi}{\partial t} + [\phi, \nabla_\perp^2 \phi] - [A, \nabla_\perp^2 A] + \frac{\partial \nabla_\perp^2 A}{\partial z} = 0$$

- Field variables $\chi_1(\mathbf{x}, t) = A(\mathbf{x}, t), \quad \chi_2(\mathbf{x}, t) = \omega(\mathbf{x}, t) \equiv \nabla_\perp^2 \phi(\mathbf{x}, t)$

  $\mathcal{D} = \{(x, y, z) : -L_x \le x \le L_x, \quad -L_y \le y \le L_y \quad -L_z \le z \le L_z\}$,
  $L_x, L_y, L_z > 0$ periodic boundary conditions

- Hamiltonian
$$H(A, \omega) = \frac{1}{2} \int d^3x \, \left(|\nabla_\perp \phi|^2 + |\nabla_\perp A|^2 \right)$$
  kinetic energy + magnetic energy

- Poisson bracket
$$\{F, G\} = \int d^3x \, \left(\omega \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right] + A \left(\left[\frac{\delta F}{\delta A}, \frac{\delta G}{\delta \omega} \right] + \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta A} \right] \right) \right.$$
$$\left. + \frac{\delta F}{\delta A} \frac{\partial}{\partial z} \frac{\delta G}{\delta \omega} + \frac{\delta F}{\delta \omega} \frac{\partial}{\partial z} \frac{\delta G}{\delta A} \right)$$

9. Ex. Hamiltonian structure for RMHD - II

- Casimir invariants:

$$C_1 = \int d^3x\, A, \qquad C_2 = \int d^3x\, \omega \tag{1}$$

In 2D ($\partial_z = 0$):

$$C_1 = \int d^2x\, \mathcal{C}_1(A), \qquad C_2 = \int d^2x\, \omega\, \mathcal{C}_2(A) \tag{2}$$

with $\mathcal{C}_{1,2}$ arbitrary functions $\Rightarrow$ infinite families of invariants

  - Dual of the Lie algebra of a semidirect product

(see, e.g. Marsden and Ratiu (2002))

- Hamiltonian functional suggested by energy conservation
- Jacobi identity verified with the help of Lemma in Morrison (1982) (second order functional derivatives do not contribute)

10. Hamiltonian structure - II

- Guarantees energy conservation and no fake dissipation
- Helps in finding further conservation laws (Casimir invariants)
- Energy-Casimir method for stability (see, e.g. Fjortoft (1950), Kruskal and Oberman (1958), Bernstein (1958), Arnol'd (1966), Holm, Marsden, Ratiu and Weinstein (1985), Morrison (1998))
- Structure-preserving numerical algorithms (e.g. review in Morrison (2017))

However,

- Derivation of such models often does not take into account structures
- Hamiltonian structure might be identified a posteriori :
  might not be easy and still unknown for some models
- Otherwise devise derivation guaranteeing Hamiltonian structures
  (e.g. Poisson reductions, Dirac brackets,..)

11. Reduced fluid models from gyrokinetics

- Reduced fluid models from taking moments of "δf" gyrokinetic equations ("gyrofluid") (Brizard (1992))
- Low-frequency phenomena $\omega/\omega_{ci} \sim \epsilon \ll 1 \rightarrow$ gyrocenter dynamics

  Remove dependence on gyration angle

  Preserve magnetic moment $\mu_s = m_s v_\perp^2/(2B) + \mathcal{O}(\epsilon), \quad s = e, i,$

  m_s particle mass (see, e.g. Brizard and Hahm (2007))

  $v_\parallel \equiv v_z, \quad v_\perp \equiv \sqrt{v_x^2 + v_y^2}$ velocity coordinates

FIG. 4. Exact and reduced single-particle orbits in a magnetic field.

From Brizard and Hahm (2007).

- Gyrokinetic equations evolve distribution functions $\mathsf{f}_s(\mathbf{x}, v_\parallel, \mu_s, t)$

  $v_\parallel, \mu_s \in \mathbb{R}, \qquad \mu_s \geq 0, \qquad \mathbf{x} \in \mathcal{D}$ periodic boundary conditions

12. Hamiltonian closures

$$p_{mn_s}(\mathbf{x},t) \equiv (2\pi B/m_s) \int_0^{+\infty} d\mu_s \int_{-\infty}^{+\infty} dv_\parallel \, \mu_s^m v_\parallel^n \, \mathsf{f}_s(\mathbf{x}, v_\parallel, \mu_s, t),$$

- Gyrofluid moment of orders mn with $m, n = 0, 1, 2, \cdots$
- Gyrokinetic equations infinite hierarchy of evolution equations for p_{mn_s}
- Evolution equation for $p_{\bar{m}\bar{n}_s}$ depends in general on p_{mn_s} with $m > \bar{m}$ and $n > \bar{n}$
- Truncate hierarchy $\rightarrow$ closed reduced fluid model for finite number of moments
- Truncation such that :

Hamiltonian parent "δf" gyrokinetic model $\rightarrow$ Hamiltonian reduced fluid model

- Applied also to "full-f" drift-kinetic and Vlasov-Poisson systems (in collaboration with M. Perin, C. Chandre, P.J. Morrison)

13. Parent model with electron drift-kinetic equations

- Magnetic and electric field:

  $\mathbf{B}(\mathbf{x},t) = \nabla \times (A(\mathbf{x},t)\hat{z}) + B\hat{z}, \qquad \mathbf{E}(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t) - (1/c)\partial_t A(\mathbf{x},t)\hat{z}$

- For electrons (not most general case) :

  - Neglect Larmor radius effects: Gyrokinetics $\rightarrow$ Drift kinetics
  - Consider evolution of $g_e(\mathbf{x},v_\parallel,t) = \tilde{f}(\mathbf{x},v_\parallel,t) - ev_\parallel/(T_e c)\mathcal{F}_{eq}(v_\parallel)A(\mathbf{x},t)$
    where $\tilde{f}(\mathbf{x},v_\parallel,t) = (2\pi B/m_e)\int d\mu_e \, \mathsf{f}_e(\mathbf{x},v_\parallel,\mu_e,t)$
    ($\rightarrow$ moments with $n=0$)
    $\mathcal{F}_{eq}(v_\parallel) = n_0\left(\frac{m_e}{2\pi T_e}\right)^{1/2}\exp(-m_e v_\parallel^2/(2T_e))$ with $|\tilde{f}| \ll \mathcal{F}_{eq}$
  - Equilibrium Maxwellian with temperature T_e and density n_0

- For ions :

  - Assume isothermal gyrofluid description
    (only two moments involved)

14. Parent model with drift-kinetic electrons (from e.g. Scott (2010))

$$\frac{\partial g_e}{\partial t} + \frac{c}{B}[\phi - \frac{v_\|}{c}A, g_e] + v_\| \frac{\partial}{\partial z}\left(g_e - e\frac{\mathcal{F}_{eq}}{T_e}\left(\phi - \frac{v_\|}{c}A\right)\right) = 0, \tag{3}$$

$$\frac{\partial n_i}{\partial t} + \frac{c}{B}[\Gamma_0^{1/2}\phi, n_i] - \frac{n_0}{B}[\Gamma_0^{1/2}A, u_i] + n_0\frac{\partial u_i}{\partial z} = 0, \tag{4}$$

$$\frac{\partial D}{\partial t} + \frac{c}{B}\left[\Gamma_0^{1/2}\phi, D\right] - \frac{T_i}{m_i B}[\Gamma_0^{1/2}A, n_i] + \frac{T_i}{m_i}\frac{\partial n_i}{\partial z} + \frac{e n_0}{m_i}\frac{\partial \Gamma_0^{1/2}\phi}{\partial z} = 0, \tag{5}$$

$$\frac{e^2}{T_i} n_0 (1 - \Gamma_0)\phi = -e\Gamma_0^{1/2} n_i + e\int dv_\| \, g_e, \tag{6}$$

$$\frac{c}{4\pi}\nabla_\perp^2 A - \frac{e^2 n_0}{m_e c} A = -e n_0 \Gamma_0^{1/2} u_i + e\int dv_\| \, v_\| g_e \tag{7}$$

(3) electron "δf" drift-kinetic equation, (4) ion continuity equation
(5) ion parallel momentum equation
(6) quasi-neutrality : $\phi = \phi(n_i, g_e)$, (7) parallel Ampère's law : $A = A(D, g_e)$

$n_i(\mathbf{x}, t), u_i(\mathbf{x}, t)$ ion gyrocenter density and parallel velocity fluctuations

$D(\mathbf{x}, t) \equiv n_0(u_i(\mathbf{x}, t) + (e/(m_i c))\Gamma_0^{1/2} A(\mathbf{x}, t)) \propto$ ion gyrocenter parallel canonical momentum

$\Gamma_0(k_\perp^2 \rho_{thi}^2) \equiv I_0(k_\perp^2 \rho_{thi}^2)\exp(-k_\perp^2 \rho_{thi}^2)$ gyroaverage operator in Fourier space

$\rho_{thi} \equiv \sqrt{T_i/m_i}(eB/(m_i c))^{-1}$ ion thermal Larmor radius, e proton charge, $k_\perp \equiv \sqrt{k_x^2 + k_y^2}$

15. Hamiltonian of the parent model

- $\chi_1(\mathbf{x}, v_\parallel, t) = g_e(\mathbf{x}, v_\parallel, t), \qquad \chi_2(\mathbf{x}, t) = n_i(\mathbf{x}, t), \qquad \chi_3(\mathbf{x}, t) = D(\mathbf{x}, t)$

$$H_k(g_e, n_i, D) = \mathcal{H}_{ek}(g_e) + \mathcal{H}_i(n_i, D) + \mathcal{H}_{ck}(g_e, n_i, D) \qquad \text{with}$$

$$\mathcal{H}_{ek} = \frac{1}{2}\int d^3x dv_\parallel \, \frac{T_e}{\mathcal{F}_{eq}} g_e^2,$$

$$\mathcal{H}_i = \frac{1}{2}\int d^3x \left(\frac{T_i}{n_0} n_i^2 + \frac{m_i}{n_0} D^2\right),$$

$$\mathcal{H}_{ck} = -\frac{e}{2}\int d^3x \phi \left(\int dv_\parallel g_e - \Gamma_0^{1/2} n_i\right) + \frac{e}{2c}\int d^3x A \left(\int dv_\parallel \, v_\parallel g_e - \Gamma_0^{1/2} D\right)$$

$\mathcal{H}_{ek}$ electron free energy + magnetic contributions

$\mathcal{H}_i$ ion free energy + magnetic contributions

$\mathcal{H}_{ck}$ coupling term via electromagnetic fields

16. Poisson bracket of the parent model

$$\{F,G\}_k = \{F,G\}_{ek} + \{F,G\}_i \qquad \text{with}$$

$$\{F,G\}_{ek} = \int d^3x dv_\parallel \left(\frac{c}{eB} g_e [F_{g_e}, G_{g_e}] - v_\parallel \frac{\mathcal{F}_{eq}}{T_e} F_{g_e} \frac{\partial G_{g_e}}{\partial z} \right),$$

$$\{F,G\}_i = -\frac{c}{eB} \int d^3x \left(n_i [F_{n_i}, G_{n_i}] + D([F_{n_i}, G_D] + [F_D, G_{n_i}]) \right.$$
$$\left. + \frac{T_i n_0}{M} n_i [F_D, G_D] \right) - \frac{n_0}{M} \int d^3x \left(F_D \frac{\partial G_{n_i}}{\partial z} + F_{n_i} \frac{\partial G_D}{\partial z} \right)$$

- subscripts on functionals indicate functional derivatives
- $\{\, ,\, \}_{ek}$ and $\{\, ,\, \}_i$ depend only on g_e and n_i, D respectively

  They contain functional derivatives only with respect to g_e and n_i, D respectively

  They are Poisson brackets of their own

  $\Rightarrow \{\, ,\, \}_k = \{\, ,\, \}_{ek} + \{\, ,\, \}_i$ is automatically a Poisson bracket

17. Hierarchy of electron fluid equation with respect to Hermite moments

- Convenient to take moments with respect to Hermite polynomials
  $H_0(v) = 1, \quad H_1(v) = v, \quad H_2(v) = v^2 - 1, \quad H_3(v) = v^3 - 3v, \cdots$
- Moments $g_n(\mathbf{x}, t) = \frac{1}{\sqrt{n!}} \int dv_\parallel \, H_n(v_\parallel / v_{te}) g_e(\mathbf{x}, v_\parallel, t), \quad$ where $v_{te} = \sqrt{T_e/m_e}$
- Physically meaningful ($g_0 \propto$ density, $g_1 \propto$ canonical momentum, $g_2 \propto$ parallel temperature, $g_3 \propto$ parallel heat flux fluctuations,...)

$$
\begin{aligned}
\frac{\partial g_0}{\partial t} &= -\frac{c}{B}[\phi, g_0] + \frac{v_{te}}{B}[A, g_1] - v_{te}\frac{\partial}{\partial z}\left(g_1 + \frac{v_{te}}{c}\frac{eA}{T_e}\right) \\
\frac{\partial g_1}{\partial t} &= -\frac{c}{B}[\phi, g_1] + \sqrt{2}\frac{v_{te}}{B}[A, g_2] + \frac{v_{te}}{B}[A, g_0] \\
&- v_{te}\frac{\partial}{\partial z}\left(\sqrt{2} g_2 + g_0 - \frac{e\phi}{T_e}\right), \\
&\vdots \\
\frac{\partial g_N}{\partial t} &= -\frac{c}{B}[\phi, g_N] + \sqrt{N+1}\frac{v_{ts}}{B}[A, g_{N+1}] + \sqrt{N}\frac{v_{te}}{B}[A, g_{N-1}] \\
&- \sqrt{N+1} v_{te}\frac{\partial g_{N+1}}{\partial z} - \sqrt{N} v_{te}\frac{\partial g_{N-1}}{\partial z},
\end{aligned}
$$

- Close hierarchy imposing $g_{N+1} = \alpha g_N, \quad \alpha \in \mathbb{R}$
  $\rightarrow$ Hamiltonian reduced fluid model for any N (Tassi (2015,2017))

18. Hamiltonian of reduced fluid models

- Hamiltonian of the resulting model:

  Replace $g_e(\mathbf{x}, v_\parallel, t) = \sum_{n=0}^{N} (1/\sqrt{n!}) g_n(\mathbf{x}, t) H_n\left(\frac{v_\parallel}{v_{te}}\right) \mathcal{F}_{eq}(v_\parallel)$

  into Hamiltonian of parent model $\rightarrow$

$$H(g_0, \cdots, g_N, n_i, D) = \mathcal{H}_e(g_0, \cdots, g_N) + \mathcal{H}_i(n_i, D) + \mathcal{H}_c(g_0, g_1, n_i, D) \quad (8)$$

$$\mathcal{H}_e = \frac{n_0 T_e}{2} \sum_{n=0}^{N} \int d^3x \, g_n^2,$$

$$\mathcal{H}_i = \frac{1}{2} \int d^3x \left(\frac{T_i}{n_0} n_i^2 + \frac{m_i}{n_0} D^2 \right),$$

$$\mathcal{H}_c = -\frac{e}{2} \int d^3x \phi \left(n_0 g_0 - \Gamma_0^{1/2} n_i \right) + \frac{e}{2c} \int d^3x A \left(n_0 v_{te} g_1 - \Gamma_0^{1/2} D \right)$$

- Functional (8) conserved by the reduced dynamics for any α
- H is a good candidate Hamiltonian

19. Poisson bracket for reduced fluid models

$$\frac{\partial g_m}{\partial t} = -[\phi, g_m]_x + [A, W_{mn} g_n]_x - W_{mn}\frac{\partial g_n}{\partial z} + \delta_{m1}\frac{\partial \phi}{\partial z} - \sqrt{m!}(\delta_{m0} + \delta_{m2})\frac{\partial A}{\partial z}$$

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 1 & 0 & \sqrt{2} & 0 & ... & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & ... & 0 \\ 0 & 0 & \sqrt{3} & 0 & ... & 0 \\ & ... & & & ... & \\ & ... & & & ... & \\ 0 & 0 & 0 & ... & 0 & \sqrt{N} \\ 0 & 0 & 0 & ... & \sqrt{N} & \alpha\sqrt{N+1} \end{pmatrix}$$

- W symmetric $\Rightarrow \exists$ orthogonal $U : U^T W U = \mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_N)$
- Introduce coordinates $G_i = U^T_{im} g_m, \qquad i = 0, \cdots, N$

$$\frac{\partial G_i}{\partial t} = -[\phi - \lambda_i A, G_i]_x - \lambda_i \frac{\partial G_i}{\partial z} + \sqrt{m!}U^T_{im}\left(\delta_{m1}\frac{\partial \phi}{\partial z} - (\delta_{m0} + \delta_{m2})\frac{\partial A}{\partial z}\right)$$

- Evolution equation for G_i only depends on G_i (apart from ϕ and A)

20. Poisson bracket for reduced fluid models - II

- This suggests a Poisson bracket of the form $\{\,,\,\} = \{\,,\,\}_e + \{\,,\,\}_i$ with

$$\{F,G\}_e = \sum_{i=0}^{N} a_i \int d^3x G_i \left[\frac{\delta F}{\delta G_i}, \frac{\delta G}{\delta G_i}\right] + \sum_{i,j=0}^{N} b_{ij} \int d^3x \frac{\delta F}{\delta G_i}\frac{\partial}{\partial z}\frac{\delta G}{\delta G_j} \tag{9}$$

- Sum of Poisson brackets depending only on G_i and $\delta/\delta G_i \Rightarrow$ Poisson bracket
- Candidate Hamiltonian in new variables $G_0, \cdots, G_N, n_i, D$

$$H(G_0,\cdots,G_N,n_i,D) = \frac{1}{2}\int d^3x \left(n_0 T_e \sum_{n=0}^{N} G_n^2 + \frac{e}{c}A(n_0 v_{te} U_{1l} G_l - \Gamma_0^{1/2} D)\right.$$
$$\left. - e\phi(n_0 U_{0l} G_l - \Gamma_0^{1/2} n_i)\right) + \mathcal{H}_i(n_i, D) \tag{10}$$

- Eqs. from bracket $\{\,,\,\}$ and Hamiltonian (10) correspond to model equations if $a_i = 1/v_i$ and $b_{ij} = \delta_{ij}\lambda_i$, with v_i normalization constant in eigenvectors of U
- Hamiltonian reduced fluid models for an arbitrary number of moments

21. Electron moment dynamics: conservation properties

- Variables $g_0, \cdots, g_N$ direct physical intetrpretation
  Variables $G_0, \cdots, G_N$ more suitable for Poisson bracket properties
- Casimir invariants of $\{\,,\,\}_e$:

$$C_i = \int d^3x\, G_i, \qquad i = 0, \cdots, N$$

- In the 2D limit ($\partial_z = 0$):

$$C_i = \int d^2x\, \mathcal{C}_i(G_i), \qquad i = 0, \cdots, N \qquad \text{with arbitrary } \mathcal{C}_i$$

  $\Rightarrow$ infinite number of constraints

$$\frac{\partial G_i}{\partial t} + \mathbf{v}_i \cdot \nabla G_i = 0, \qquad i = 0, \cdots, N$$

  where $\mathbf{v}_i \equiv \hat{z} \times \nabla(\phi - \lambda_i A)$

  - $G_0, \cdots, G_N$ Lagrangian invariants transported by incompressible velocity fields $\mathbf{v}_0, \cdots, \mathbf{v}_N$

22. Ex. Six-field reduced fluid model

- Hamiltonian closure to derive new models but also to recover old ones (e.g. Schep et al. (1994), Waelbroeck, Hazeltine and Morrison (2009), ..)

$N = 3, \alpha = 0, 2D \rightarrow$ six-field model (Grasso and Tassi (2015))

$$\frac{\partial n_e}{\partial t} + [\phi, n_e] - [A, u_e] = 0,$$
$$\frac{\partial F}{\partial t} + [\phi, F] + \rho_s^2 [A, T_\parallel + n_e] = 0,$$
$$\frac{\partial T_\parallel}{\partial t} + [\phi, T_\parallel] - 2[A, q_\parallel + u_e] = 0,$$
$$\frac{\partial q_\parallel}{\partial t} + [\phi, q_\parallel] - \frac{3}{2}\frac{\rho_s^2}{d_e^2}[A, T_\parallel] = 0,$$
$$\frac{\partial n_i}{\partial t} + [\Phi, n_i] + [u_i, \mathscr{A}] = 0,$$
$$\frac{\partial D}{\partial t} + [\Phi, D] + \rho_{thi}^2 [n_i, \mathscr{A}] = 0,$$
$$n_e = \Gamma_0^{1/2} n_i + \left(\frac{\Gamma_0 - 1}{\rho_{thi}^2}\right)\phi, \qquad \nabla_\perp^2 A = u_e - \Gamma_0^{1/2} u_i,$$

- Normalized variables: $n_e \propto g_0, \quad F \propto g_1, \quad T_\parallel \propto g_2, \quad q_\parallel \propto g_3$
- Parameters: $\tau \equiv T_i/T_e, \qquad \rho_s^2 \equiv \rho_{thi}^2/\tau, \qquad d_e$ electron skin depth
- Some redefined functions : $\Phi \equiv \Gamma_0^{1/2}\phi, \qquad \mathscr{A} \equiv \Gamma_0^{1/2} A, \qquad F \equiv A - d_e^2 u_e$

23. Hamiltonian of the six-field model

$$H(n_e, F, T_\parallel, q_\parallel, n_i, D) = \mathcal{H}_e(n_e, F, T_\parallel, q_\parallel) + \mathcal{H}_i(n_i, D) + \mathcal{H}_c(n_e, F, n_i, D) \quad (11)$$

$$\mathcal{H}_e = \frac{1}{2}\int d^2x \left(\rho_s^2 n_e^2 + \frac{F^2}{d_e^2} + \frac{\rho_s^2}{2}T_\parallel^2 + \frac{2}{3}d_e^2 q_\parallel^2\right),$$

$$\mathcal{H}_i = \frac{1}{2}\int d^2x \left(\rho_i^2 n_i^2 + \frac{D^2}{d_i^2}\right),$$

$$\mathcal{H}_c = -\frac{1}{2}\int d^2x \left(\phi(n_e - \Gamma_0^{1/2} n_i) + A\left(\frac{F}{d_e^2} - \frac{\Gamma_0^{1/2} D}{d_i^2}\right)\right)$$

Hamiltonian (11) can also be rewritten as

$$\frac{1}{2}\int d^2x \left(\rho_i^2 n_i^2 + \rho_s^2 n_e^2 + d_i^2 u_i^2 + d_e^2 u_e^2 + |\nabla_\perp A|^2 - \phi\frac{\Gamma_0 - 1}{\rho_i^2}\phi + \frac{\rho_s^2}{2}T_\parallel^2 + \frac{2}{3}d_e^2 q_\parallel^2\right)$$

- Thermal free energy + kinetic energy + magnetic energy + electrostatic energy (due to polarization)

24. Poisson bracket of the six-field model

$$\{\mathcal{F},\mathcal{G}\} = \{\mathcal{F},\mathcal{G}\}_e + \{\mathcal{F},\mathcal{G}\}_i,$$

$$\{\mathcal{F},\mathcal{G}\}_e = \int d^2x \left(n_e \left([\mathcal{F}_{n_e},\mathcal{G}_{n_e}] + \rho_s^2 d_e^2 [\mathcal{F}_F,\mathcal{G}_F] + 2[\mathcal{F}_{T_\parallel},\mathcal{G}_{T_\parallel}] + \frac{3}{2}\frac{\rho_s^2}{d_e^2}[\mathcal{F}_{q_\parallel},\mathcal{G}_{q_\parallel}] \right) \right.$$
$$+F \left([\mathcal{F}_F,\mathcal{G}_{n_e}] + [\mathcal{F}_{n_e},\mathcal{G}_F] + 2([\mathcal{F}_F,\mathcal{G}_{T_\parallel}] + [\mathcal{F}_{T_\parallel},\mathcal{G}_F]) - \frac{3}{d_e^2}([\mathcal{F}_{q_\parallel},\mathcal{G}_{T_\parallel}] + [\mathcal{F}_{T_\parallel},\mathcal{G}_{q_\parallel}]) \right)$$
$$+T_\parallel \left([\mathcal{F}_{T_\parallel},\mathcal{G}_{n_e}] + [\mathcal{F}_{n_e},\mathcal{G}_{T_\parallel}] + \rho_s^2 d_e^2 [\mathcal{F}_F,\mathcal{G}_F] \right.$$
$$\left. -\frac{3}{2}\rho_s^2([\mathcal{F}_{q_\parallel},\mathcal{G}_F] + [\mathcal{F}_F,\mathcal{G}_{q_\parallel}]) + 4[\mathcal{F}_{T_\parallel},\mathcal{G}_{T_\parallel}] + \frac{3}{2}\frac{\rho_s^2}{d_e^2}[\mathcal{F}_{q_\parallel},\mathcal{G}_{q_\parallel}] \right)$$
$$\left. +q_\parallel \left([\mathcal{F}_{q_\parallel},\mathcal{G}_{n_e}] + [\mathcal{F}_{n_e},\mathcal{G}_{q_\parallel}] - 2d_e^2([\mathcal{F}_{T_\parallel},\mathcal{G}_F] + [\mathcal{F}_F,\mathcal{G}_{T_\parallel}]) + 2([\mathcal{F}_{q_\parallel},\mathcal{G}_{T_\parallel}] + [\mathcal{F}_{T_\parallel},\mathcal{G}_{q_\parallel}]) \right) \right)$$

$$\{\mathcal{F},\mathcal{G}\}_i = -\int d^2x \left(n_i \left([\mathcal{F}_{n_i},\mathcal{G}_{n_i}] + \rho_{thi}^2 d_i^2 [\mathcal{F}_D,\mathcal{G}_D] \right) + D \left([\mathcal{F}_D,\mathcal{G}_{n_i}] + [\mathcal{F}_{n_i},\mathcal{G}_D] \right) \right)$$

in particular for electrons, expressed in terms of the Hermite moment variables $((g_0,\cdots,g_3) \rightarrow (n_e, F, T_\parallel, q_\parallel))$

25. Formulation in terms of Lagrangian invariants

$$
\begin{aligned}
\frac{\partial G_i}{\partial t} + \mathbf{v}_i \cdot \nabla G_i = 0, \qquad & i = 0, \cdots, 3, \\
\frac{\partial I_\pm}{\partial t} + \mathbf{v}_\pm \cdot \nabla I_\pm = 0, &
\end{aligned}
$$

where

$$
\begin{aligned}
G_0 &= F - \frac{d_e \rho_s}{\sqrt{3+\sqrt{6}}} n_e - d_e \rho_s \sqrt{\frac{1}{2} + \frac{1}{\sqrt{6}}} T_\parallel - d_e^2 \sqrt{\frac{2}{3}} q_\parallel, \\
G_1 &= F + \frac{d_e \rho_s}{\sqrt{3+\sqrt{6}}} n_e + d_e \rho_s \sqrt{\frac{1}{2} + \frac{1}{\sqrt{6}}} T_\parallel - d_e^2 \sqrt{\frac{2}{3}} q_\parallel, \\
G_2 &= F + \frac{d_e \rho_s}{\sqrt{3-\sqrt{6}}} n_e - d_e \rho_s \sqrt{\frac{1}{2} - \frac{1}{\sqrt{6}}} T_\parallel + d_e^2 \sqrt{\frac{2}{3}} q_\parallel, \\
G_3 &= F - \frac{d_e \rho_s}{\sqrt{3-\sqrt{6}}} n_e + d_e \rho_s \sqrt{\frac{1}{2} - \frac{1}{\sqrt{6}}} T_\parallel + d_e^2 \sqrt{\frac{2}{3}} q_\parallel, \\
I_\pm &= D \pm d_i \rho_{thi} n_i
\end{aligned}
$$

26. Casimir invariants

- Lagrangian invariants G_i and $I_\pm$ advected by $\mathbf{v}_i = \hat{z} \times \nabla \phi_i$ for $i = 0, \cdots, 3$ and $\mathbf{v}_\pm = \hat{z} \times \nabla \Phi_\pm$, respectively, with

$$\phi_0 = \phi - \sqrt{3+\sqrt{6}}\frac{\rho_s}{d_e}A, \qquad \phi_1 = \phi + \sqrt{3+\sqrt{6}}\frac{\rho_s}{d_e}A,$$
$$\phi_2 = \phi + \sqrt{3-\sqrt{6}}\frac{\rho_s}{d_e}A, \qquad \phi_3 = \phi - \sqrt{3-\sqrt{6}}\frac{\rho_s}{d_e}A,$$
$$\Phi_\pm = \Phi \mp \frac{\rho_{thi}}{d_i}\mathscr{A}$$

- Infinite number of topological conservation laws associated with Casimir invariants

- Magnetic field can change topology, but topology of contour lines of Lagrangian invariants is preserved
  $\Rightarrow$ Magnetic reconnection (solar flares, magnetospheric substorms, "sawtooth" oscillations in tokamaks)
  in the presence of "hidden" topological conservation laws

27. Further physical effects for space plasmas applications

- For space plasmas (solar wind, planetary magnetospheres) important physical effects we neglected so far (e.g. Schekochihin et al. (2009), Kunz et al. (2015)):

  - Finite Larmor radius

  - Perturbations also along guide field:
    $\mathbf{B}(\mathbf{x}, t) = \nabla \times (A(\mathbf{x}, t)\hat{z}) + (B + B_{\|}(\mathbf{x}, t))\hat{z}$

  - More general equilibrium distribution functions (e.g. bi-Maxwellian)

28. More general gyrokinetic parent model

$$\frac{\partial g_s}{\partial t} + \frac{c}{B}\left[J_0\phi - \frac{v_\parallel}{c}J_0 A + 2\frac{\mu_s B}{q_s}\frac{J_1}{a_s}\frac{B_\parallel}{B}, g_s\right] + v_\parallel \frac{\partial}{\partial z}\left(g_s + \frac{q_s}{T_{\parallel_s}}F_{eq_s}\left(J_0\phi - \frac{v_\parallel}{c}J_0 A + 2\frac{\mu_s B}{q_s}\frac{J_1}{a_s}\frac{B_\parallel}{B}\right)\right) = 0, \qquad s = e, i,$$

- More general gyrokinetic equations (Kunz et al. (2015))
- $g_s(\mathbf{x}, v_\parallel, \mu_s, t) \equiv \mathsf{f}_s(\mathbf{x}, v_\parallel, \mu_s, t) + \frac{q_s}{T_{\parallel_s}}\frac{v_\parallel}{c}F_{eq}(v_\parallel, \mu_s)J_0 A$

  with $F_{eq}(v_\parallel, \mu_s) \equiv \left(\frac{m_s}{2\pi}\right)^{3/2} \frac{n_0}{T_{\parallel_s}{}^{1/2} T_{\perp_s}} \mathrm{e}^{-\frac{m_s v_\parallel^2}{2T_{\parallel_s}} - \frac{\mu_s B}{T_{\perp_s}}}$
- Gyroaverage operators (finite Larmor radius effects)

  $J_0 \equiv J_0(a_s), \qquad J_1 \equiv J_1(a_s)$ with $a_s \equiv k_\perp v_\perp/\omega_{cs}$
- $B_\parallel$ parallel magnetic fluctuations

29. More general gyrokinetic parent model - II

$$\sum_{s=e,i} q_s \int d\mathcal{W}_s\, J_0 g_s = \sum_s \frac{q_s^2}{T_{\perp_s}} \int d\mathcal{W}_s\, F_{eq} \left(1 - J_0^2\right) \phi - \sum_s q_s \int d\mathcal{W}_s\, 2\frac{\mu_s B}{T_{\perp_s}} F_{eq} J_0 \frac{J_1}{a_s} \frac{B_\parallel}{B}, \tag{12}$$

$$\sum_{s=e,i} q_s \int d\mathcal{W}_s\, v_\parallel J_0 \left(g_s - \frac{q_s}{T_{\parallel_s}} \frac{v_\parallel}{c} F_{eq_s} J_0 A \right) = -\frac{c}{4\pi} \nabla_\perp^2 A + \sum_s \frac{q_s^2}{m_s} \int d\mathcal{W}_s\, F_{eq_s} \left(1 - \frac{1}{\Theta_s} \frac{v_\parallel^2}{v_{ts}^2} \right) \left(1 - J_0^2\right) \frac{A}{c}, \tag{13}$$

$$\sum_{s=e,i} \frac{\beta_{\perp_s}}{n_0} \int d\mathcal{W}_s\, 2\frac{\mu_s B}{T_{\perp_s}} \frac{J_1}{a_s} g_s = -\sum_s \frac{\beta_{\perp_s}}{n_0} \frac{q_s}{T_{\perp_s}} \int d\mathcal{W}_s\, 2\frac{\mu_s B}{T_{\perp_s}} F_{eq_s} J_0 \frac{J_1}{a_s} \phi - \left(2 + \sum_s \frac{\beta_{\perp_s}}{n_0} \int d\mathcal{W}_s\, F_{eq_s} \left(2\frac{\mu_s B}{T_{\perp_s}} \frac{J_1}{a_s} \right)^2 \right) \frac{B_\parallel}{B}. \tag{14}$$

with $d\mathcal{W}_s \equiv 2\pi (d\mu_s B/m_s) dv_\parallel$, $\quad \Theta_s \equiv T_{\perp_s}/T_{\parallel_s}$, $\quad \beta_{\perp_s} \equiv 8\pi \frac{n_0 T_{\perp_s}}{B^2}$

- Quasi-neutrality (12) and perpendicular Ampère's law (14):
  $\phi = \phi(g_e, g_i), \qquad B_\parallel = B_\parallel(g_e, g_i)$
- Parallel Ampère's law (13) : $A = A(g_e, g_i)$

30. Example : two-field model

- Hamiltonian closures can be applied also in this more general context (Tassi, Passot, Sulem (2018))
- Two-field model accounting for $B_\parallel$ in electron dynamics (Passot, Sulem, Tassi (2018))

$$\frac{\partial n_e}{\partial t} + [\phi, n_e] - [B_\parallel, n_e] - \frac{2}{\beta_e}[A, \nabla_\perp^2 A] + \frac{2}{\beta_e}\frac{\partial \nabla_\perp^2 A}{\partial z} = 0, \tag{15}$$

$$\frac{\partial}{\partial t}\left(1 - \frac{2\delta^2}{\beta_e}\nabla_\perp^2\right) A + \left[\phi - B_\parallel, \left(1 - \frac{2\delta^2}{\beta_e}\nabla_\perp^2\right) A\right] + [A, n_e] + \frac{\partial}{\partial z}(\phi - n_e - B_\parallel) = 0, \tag{16}$$

$$n_e = \left(\frac{\Gamma_0 - 1}{\tau} + \delta^2 \nabla_\perp^2\right)\phi - (1 - \Gamma_0 + \Gamma_1) B_\parallel, \tag{17}$$

$$\nabla_\perp^2 A = \frac{\beta_e}{2} u_e \tag{18}$$

$$\left(\frac{2}{\beta_e} + (1 + 2\tau)(\Gamma_0 - \Gamma_1)\right) B_\parallel = \left(1 - \frac{\Gamma_0 - 1}{\tau} - \Gamma_0 + \Gamma_1\right)\phi, \tag{19}$$

with $\delta^2 = m_e/m_i$

- Retained $B_\parallel$ and finite Larmor radius effects in quasi-neutrality and Ampère's law

31. Two-field model - II

$$H(n_e, A_e) = -\frac{1}{2}\int d^3x \left(\frac{2}{\beta_e}A_e\nabla_\perp^2 A + n_e(\phi - n_e - B_\parallel)\right),$$

$$\{F, G\} = \int d^3x \, \left(n_e([F_{n_e}, G_{n_e}] + \delta^2[F_{A_e}, G_{A_e}]) + A_e([F_{n_e}, G_{A_e}] + [F_{A_e}, G_{n_e}])\right.$$
$$\left. + F_{n_e}\frac{\partial}{\partial z}G_{A_e} + F_{A_e}\frac{\partial}{\partial z}G_{n_e}\right)$$

where $A_e \equiv A - (2\delta^2/\beta_e)\nabla_\perp^2 A$

- In 2D Lagrangian invariant formulation:

$$\frac{\partial G_\pm}{\partial t} + \mathbf{v}_\pm \cdot \nabla G_\pm = 0, \qquad (20)$$

  with $G_\pm \equiv A_e \pm \delta n_e$ and $\mathbf{v}_\pm = \hat{z} \times \nabla\phi_\pm$ with $\phi_\pm \equiv \phi - B_\parallel \pm (1/\delta)A$

- Casimir invariants adopted to study absolute equilibrium states (Passot et al. (2018))

- Foreseen application to modelling of turbulence in space plasmas

32. Conclusions

- Hamiltonian structure of reduced fluid models for plasmas not always easy to identify
- Devised procedure to derive Hamiltonian reduced fluid models from parent Hamiltonian gyrokinetic models
- Applicable to moments with respect to $v_\parallel$ coordinate at any order in the hierarchy
- Set of variables $(G_0, \cdots, G_N)$ simplifying the Poisson bracket (and thus identification of Casimir invariants)
- In 2D models can be cast in terms of transport equations for Lagrangian invariants $G_0, \cdots, G_N$
- Shown explicitly for drift-kinetic model but amenable to further extensions including parallel magnetic fluctuations, finite Larmor radius effects and equilibrium temperature anisotropy

Hamiltonian reduced fluid models for non-dissipative plasmas - Talk by Emanuele Tassi

Lecture notes (Ori S. Katz)

October 10, 2018

Abstract

Progress in the understanding of several phenomena occurring in plasmas greatly benefited from the use of continuum models based on a fluid description of plasmas. In the absence of dissipative effects, all such models are supposed to possess a Hamiltonian structure. The existence of such structure for a given model is, however, not guaranteed, in general, unless it is implied in its derivation or shown a posteriori.

In this talk I will consider a class of so-called reduced fluid models for plasmas, which are applicable in the situation where the magnetic field can be written as the sum of a uniform and constant component (guide field) with a fluctuating contribution depending on space and time. The amplitude of the fluctuating contribution is also assumed to be much smaller than that of the guide field. Such very commonly adopted assumption has led, together with further assumptions, to the derivation, over the last decades, of a number of reduced fluid models, a considerable part of which were shown to possess a Hamiltonian structure.

In this context, I will first recall earlier results on the derivation of a class of reduced fluid models, which guarantees the existence of a Hamiltonian structure. Such derivation is based on a closure of the hierarchy of fluid equations evolving moments of the perturbation of the distribution function satisfying a Hamiltonian drift-kinetic equation. In the remaining part of the talk I will consider recent extensions of this procedure, addressed to applications to collisionless space plasmas.

1 Lecture notes

The plasma state is natural in many astrophysical environments, but can also be created in the laboratory. In the image, examples of plasma for different temperatures and number density.

Theoretical modeling of plasma dynamics: Because of the large number of particles, continuum models are often adopted, specifically (i) kinetic description - distribution function $f(x, v, t)$ describing the probability density of finding the species at x with velocity v (Vlasov-Maxwell, Vlasov-Poisson); (ii) multifluid description, in which the particle species are treated as fluids interacting with electromagnetic fields; and (iii) magnetohydrodynamics (MHD) - single conducting fluid.

Reduced fluid models: For example, starting with a parent model of the ideal MHD system (no dissipation). The pressure is determined by the non-compressibility constraint $\nabla \cdot v = 0$. This is a useful model for describing phenomena such as turbulence, reconnection, instabilities... It is possible to obtain a simplified fluid model, good for numerics and analytics, by reduction.

One example of a possible reduction is in the case of a strong, constant and uniform component of the magnetic field along a certain direction. The magnetic field along the strong component is termed the guide field. If in addition to this assumption assume some other conditions, specifically low-frequency phenomena, strong anisotropy, small pressure and negligible velocity, obtain the reduced MHD equations (Strauss 1976, 1977).

In this model the magnetic field is a sum of the strong constant component and a small perturbation along the perpendicular direction that can depend on time. This reduced model is less general but is easier to analyze and simulate - reduced the dynamical variables from 6 to 2 (A, ϕ).

There are many other examples of low-frequency reduced fluid models.

In all previous models, we have neglected the dissipative term. It is important that the model be Hamiltonian in this case. These reduced fluid models are typically formulated in Hamiltonian terms of infinite dimensions and noncanonical variables.

Example - Reduced MHD (RMHD) - Morrison Hazeltine 1984.

The field variables are $A(x, t)$ (magnetic potential) and $\omega(x, t)$ (vorticity). Indeed the Poisson bracket is not the canonical Poisson bracket for fields. The Casimir invariants in the two dimensional limit become two infinite families of Casimir invariants.

Importance of the Hamiltonian structure: energy conservation, helps identify conservation laws due to the Casimir invariants, energy-Casimir method for stability, structure-preserving numerical algorithms. However, it is generally not easy to identify the Hamiltonian structure.

This is the main framework for the following results. I will consider reduced fluid models from taking moments of δf gyrokinetic equations. A reminder of the idea of gyrokinetic models: Consider low-frequency phenomena $\omega/\omega_{ci} \sim \epsilon \ll 1$. Can remove the gyration angle from the dynamics and push it to a higher order and obtain a reduced description of the gyrocenters. This preserves magnetic moment. Thus the gyrokinetic equations can be derived.

So a reduced description can be obtained by considering the moments, such as the definition in slide (12). By taking moments of the gyrokinetic equations we obtain an infinite hierarchy of the evolution equations for p_{mn_s}.To obtain a closed reduced fluid model, the hierarchy must be truncated. In order to obtain a Hamiltonian parent gyrokinetic model, the truncation must be chosen carefully.

I will consider the parent model with electron drift-kinetic equations, with a magnetic and electric field. For simplicity, I will show a simplified case of the system for electrons, although we have proved an extension to more general models. Note that the evolution function g_e is not the distribution function, but a linear combination of $\tilde{f}$, an average of the distribution function, and a term that takes into account the magnetic potential.

The parent model with drift-kinetic electrons is taken from Scott 2010. The first three are the evolution equations, the fourth expresses the quasi-neutrality, and the last is Amperes law in the direction parallel to the guiding field. Recall we are looking at periodic boundary conditions, so the Fourier space gyroaverage operator Γ_0 has a simple structure.

Thus, the parent model has a Hamiltonian structure. It can be written as a sum of three contributions - $\mathcal{H}_{ek}$, $\mathcal{H}_i$ and $\mathcal{H}_{ck}$ (coupling between electrons and ions). The Poisson bracket can also be decomposed into two terms - the electrokinetic and the ion terms.

To obtain a reduced model from this parent model, it is convenient to take the moments with respect to Hermite polynomials. Note we take moments w.r.t. parallel direction to the guiding field. Moments defined this way have an immediate physical meaning. Obtain a hierarchy of models along the moments. The truncated model, assuming a truncation condition, retains a Hamiltonian structure.

The resulting model has a Hamiltonian consisting of three terms - the first one depending on the first $N + 1$ moments, the second considering the contribution of the ions on the fluids, and the last a coupling term. This is a conserved quantity and is a good candidate for a Hamiltonian. To show it truly is, one must find the Poisson brackets.

Using the new coordinates from diagonalizing the symmetric W matrix, obtain a new form for the evolution equation for the m'th moment. These new variables suggest an Anzatz for the Poisson bracket, decomposed into a sum of an electron and an ionic contribution. This can be checked with the candidate Hamiltonian, and indeed the equations of motion of the reduced Hamiltonian emerge. Thus, writing the Hamiltonian with the new variables shows that the reduced model obtained by this truncation possesses a Hamiltonian structure.

So the previous moments g_j have a direct physical interpretation, but the new variables G_j are more suitable for the Poisson bracket properties. In the 2D limit, the entire equations can be written in a simplified scalar form.

Example: 6-field reduced fluid model. This is an example of a previously derived model, re-derived with the new method.

The Hamiltonian can also be rewritten as a sum of four physically-intuitive terms - thermal free energy, kinetic energy, magnetic energy and electrostatic energy.

The Poisson bracket is again written as a sum of two contributions. The variables here are proportional to the Hermite polynomials and has a complicated form. Using the alternative variables, the formulation is much simpler.

The Casimir invariants are important in the study of reconnections, related to (hidden) topological symmetries.

This procedure can be extended to a more general gyrokinetic parent model, taking into account some of the effects we neglected previously. The evolution equations can be then closed by quasi-neutrality and perpendicular Ampere's law.

In this context, we recently derived, for example, a 2-field model with finite Larmor radius effects. This model was supposed to describe turbulence, absolute equilibrium states (Passot 2018).

2 Questions:

- Plans to incorporate non-uniformity in the guide field, like magnetic drifts?

Might not be relevant for the applications we considered, but maybe for other applications.

- Is the truncation assumption - that the $N + 1$ moment is proportional to the N moment - justified?

It provides the Hamiltonian structure, however the physical constraint related to this assumption depends on the model and the truncation number, for example it could be akin to imposing 0 temperature. In some cases, the physical meaning of the closure is not clear.