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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Ori Katz **Email/Phone:** ORIKATZ.OK@gmail.com

Speaker's Name: Sergey Bolotin

Talk Title: Jumps of energy near a separatrix in slow-fast Hamiltonian systems.

Date: 10/08/18 **Time:** 3:30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Bolotin considers a Hamiltonian system slowly depending on time, $\mathcal{H} = \mathcal{H}(x, \tau)$, $\dot{\tau} = \varepsilon \ll 1$. For small ε the energy $\mathcal{H}$ changes slowly - $\dot{\mathcal{H}} = \mathcal{O}(\varepsilon)$. For 1 degree of freedom & closed energy level curves, there is an adiabatic invariant except for trajectories near equilibria. Bolotin partially extends Neishtadt's results on quasi-random energy jumps of order ε around hyperbolic equilibria to multidimensional systems.

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(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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- [ ] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [ ] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
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Jumps of energy near a separatrix in slow-fast Hamiltonian systems - Talk by Sergey Bolotin

Lecture notes (Ori S. Katz)

October 11, 2018

Abstract

We consider a Hamiltonian system slowly depending on time: $H = H(x, \tau)$, $\dot{\tau} = \epsilon \ll 1$. For small ϵ the energy H changes slowly: $\dot{H} = O(\epsilon)$. If the frozen autonomous system with Hamiltonian $H(\cdot, \tau)$ has one degree of freedom and energy levels are closed curves, there is an adiabatic invariant I which changes much slower. Then the energy $H = h(\tau, I)$ changes gradually. However, the adiabatic invariant is destroyed for trajectories passing near equilibria. Neishtadt showed that if the frozen system has a hyperbolic equilibrium with a figure eight separatrix, then generically the energy will have quasi-random jumps of order ϵ with frequency of order $1/|\ln \epsilon|$. We partly extend Neishtadt's result to multidimensional systems such that for each τ the frozen system has a hyperbolic equilibrium possessing several transverse homoclinics. The trajectories we construct have quasirandom jumps of energy of order ϵ with frequency $1/|\ln \epsilon|$ while staying distance of order ϵ away from the homoclinic set. Gelfreigh and Turayev showed that if the frozen system has compact uniformly hyperbolic chaotic invariant sets on energy levels, then generically there exist trajectories with energy having quasirandom jumps of order ϵ with frequency of order 1. Thus the energy may grow with rate of order ϵ. However, this result does not work near a homoclinic set, where dynamics of the frozen system is slow, so there is no uniform hyperbolicity.

1 Lecture notes:

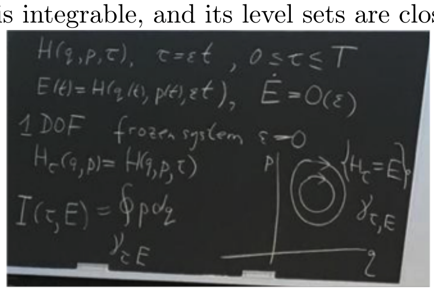
Assume$\mathcal{H}(q, p, \tau)$, $\tau = \epsilon t$, $\epsilon \ll 1$. Then the energy $E(t) = \mathcal{H}(q(t), p(t), \epsilon t)$ will slowly change, $\dot{E} = \mathcal{O}(\epsilon)$.

Can the energy grow? Can it have random jumps, etc.? This is a simplified version of the classical problem of Arnold diffusion.

1 DOF: Then the frozen system, for which $\epsilon = 0$, is an autonomous Hamiltonian and τ is a fixed parameter. Then, the Hamiltonian can be written as

$$\mathcal{H}_\tau(q, p) = \mathcal{H}(q, p, \tau).$$

It is integrable, and its level sets are closed curves.

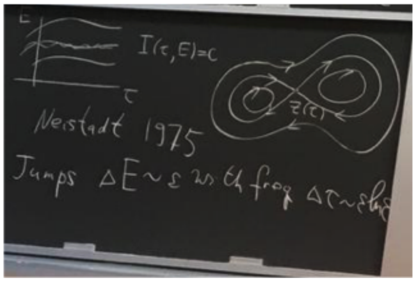


The adiabatic invariant $I(\tau, E) = \int pdq$ is the action of the periodic system.

For an interval $0 \le \tau \le T$ the adiabatic invariant changes at most in order ϵ.

What happens to the energy? To first approximation, the energy changes along the level curves of $I(\tau, E) = Const.$.

But this isn't true if there are fixed points. Suppose a fixed point $z(\tau)$. Away from the separatrix, there will be adiabatic invariants, but close to it this will break. There are many works that have dealt with this situation, the lecturer's inspiration has been Neistadt 1975.



Near the separatrix, the adiabatic invariant can have quasi-random jumps $\Delta E \sim \epsilon$ with frequency $\Delta\tau \sim \epsilon \ln \epsilon$.

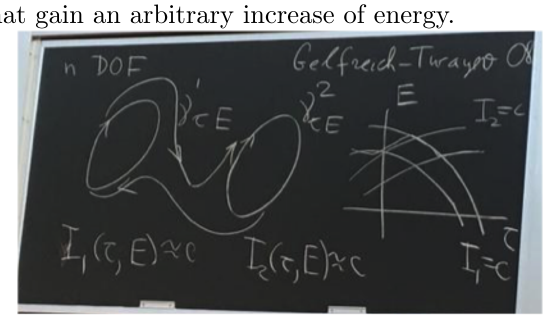
Note: τ will be used for the time variable instead of t.

The goal of this work is to obtain similar results for large dimensions. Assume an n degree of freedom (DOF) system. If it's integrable, there are similar results. But if the frozen system is not integrable, there are generally no adiabatic invariants. However, there is something that works. Assume a periodic orbit of the non-integrable frozen system $\gamma_{\tau E}$. Around it, we can define an adiabatic invariant by the same formula. However, since we are at higher dimensions, this procedure only makes sense close to the periodic orbit. Thus we obtain an approximate adiabatic invariant. However, periodic trajectories in non-integrable n-dimensional systems are generally not stable.

What can be done? Assume a chaotic system, for example a Smale horseshoe, with 2 periodic trajectories, and trajectories that follow these orbits closely transitioning from one to the other. This is an example of symbolic dynamics - denote how many times a trajectory circles one periodic trajectory before transitioning to the other periodic trajectory.

Then there will be an adiabatic invariant related to each periodic orbit, $I_{1,2}(\tau, E) \approx Const.$. As usual in hyperbolic dynamics, a close trajectory will jump from one periodic trajectory to the other.

This work was done by Gelfreich - Turayev - 2008 (GT08) - using this, they proved the existence of solutions that gain an arbitrary increase of energy.



In GT08, they considered no critical point. The lecturer will talk of work on systems with hyperbolic fixed points. This is related to many works on Arnold diffusion.

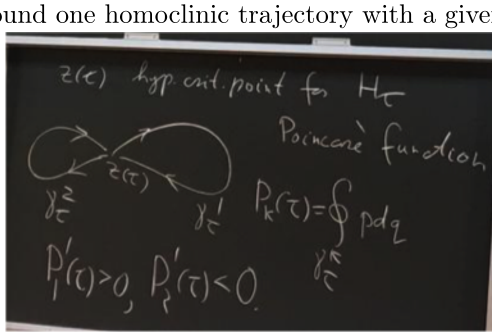
The results require many assumptions to be made:

Assume an N DOF Hamiltonian system with a hyperbolic critical point $z(\tau)$ for the frozen system $\mathcal{H}_\tau$, and at least 2 transverse homoclinic orbits (figure 8 orbits). (This is very similar to GT08 but with homoclinic trajectories.)

For each orbit, we derive the Poincare function

$$P_k(\tau) = \oint_{\gamma_\tau^k} pdq.$$

Assume these functions are non-trivial, for example assume $P_1'(\tau) > 0$, $P_2'(\tau) < 0$. Construct trajectories that go around one homoclinic trajectory with a given energy, and then the other (with a different energy generally).

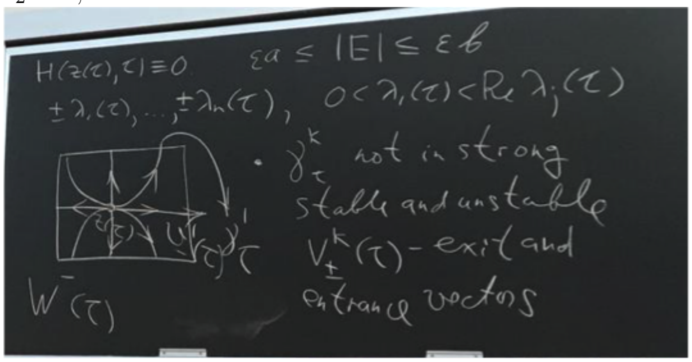


Assume $\mathcal{H}(z(\tau), \tau) = 0$, and the energy in the considered trajectory is bounded - there exist $0 < a < b$ such that $\epsilon a \leq |E| \leq \epsilon b$, so we're not too close or far from the homoclinic trajectory, along which the energy is 0.

Assume there are some eigenvalues $\{\pm\lambda_i(\tau)\}_{i=1}^{n}$, and assume the smallest is positive and real, $0<\lambda_1(\tau)<Re(\lambda_j(\tau))\,\forall j\neq 1$. Then the unstable manifold will have a strong unstable direction. Assume the trajectory we consider γ_τ^k is not in the strong stable/unstable directions.

Denote $v_\pm^k(\tau)$ as the exit (+) and entrance (-) vectors.

Another condition - the symplectic form $\omega = dp\wedge dq$ satisfies $\omega\left(v_+^{k_1}(\tau), v_-^{k_2}(\tau)\right)>0$ for any combination of $k_1,k_2=1,2$.



Then, there is a theorem:

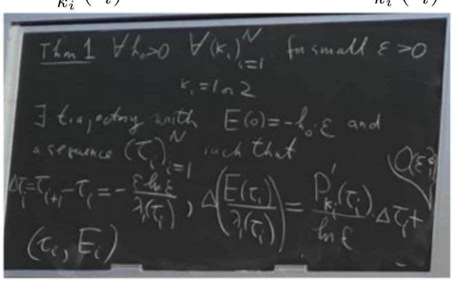
- Theorem 1:

For all $h_0>0$, for all sequences $\{k_i\}_{i=1}^{N}$, $k_i\in\{1,2\}$, for a small enough $\epsilon>0$, there exists a trajectory with energy satisfying $E(0)=-h_0\cdot\epsilon$, and a sequence of times $\{\tau_i\}_{i=1}^{N}$s.t.

$$\Delta\tau_i \equiv \tau_{i+1}-\tau_i = -\frac{\epsilon\ln\epsilon}{\lambda_1(\tau_i)}$$

$$\Delta\left(\frac{E_i(\tau_i)}{\lambda_1(\tau_i)}\right) \equiv \frac{E(\tau_{i+1})}{\lambda_1(\tau_{i+1})}-\frac{E(\tau_i)}{\lambda_1(\tau_i)} = \frac{P'_{k_i}(\tau_i)}{\ln\epsilon}\Delta\tau_i+\mathcal{O}\left(\epsilon^2\right),$$

where $P'_{k_i}(\tau_i)$ is the derivative of $P_{k_i}(\tau_i)$.

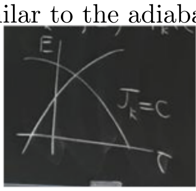


So there is a trajectory with these energy leaps. Note that in this approach, we take a fixed sequence and then choose the small ϵ.

Thus, for $k_1=k_2=...=k_N=k$, then

$$J_k(\tau,E)=P_k(\tau)-\frac{E\ln\epsilon}{\lambda_1(\tau)}\approx Const.$$

This means that J_k is almost constant, as can be seen by the theorem, along the sequence (τ_i,ϵ_i). This looks very similar to the adiabatic invariance.



Note: This works only for trajectories that remain close to the homoclinic trajectories, $\epsilon a\leq|E|\leq\epsilon b$. Therefore, N must be finite, else we may exit this energy regime. So in order for the energy to stay in this interval, the sequence $\{k_i\}$ must be chosen carefully at each step.

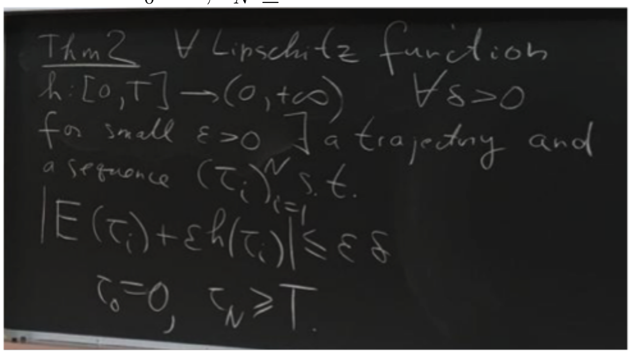
This argument produces the following theorem:

- Theorem 2:

For any Lipschitz function $h : [0, T] \to (0, t\infty)$, for any $\delta > 0$, for a small enough $\epsilon > 0$, there exists a trajectory and a sequence $\{\tau_i\}_{i=1}^{N}$ s.t.

$$|E(\tau_i) + \epsilon h(\tau_i)| \leq \epsilon\delta.$$

Note: Here, the number of steps N depends on ϵ.
We choose $\tau_0 = 0$, $\tau_N \geq T$.

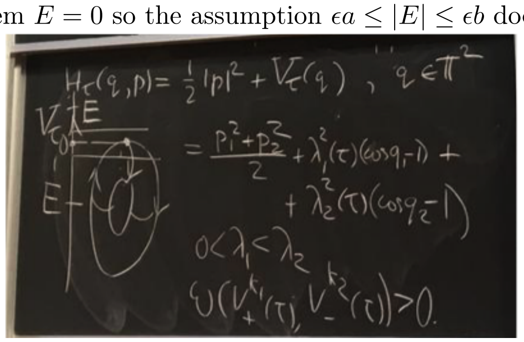


Thus, we may move from one curve to the other. Along one curve the energy grows and along the other the energy decreases, and the jumps are quasi-random.

An example in which this doesn't work:
Motion along the 2-torus

$$\mathcal{H}_\tau(q, p) = \frac{1}{2}|p|^2 + V_\tau(q), \; q \in \mathbb{T}^2(two - torus)$$

There are at least 2 homoclinic orbits. If the energy is far from them, the Turayev theorem works. But next to them $E = 0$ so the assumption $\epsilon a \leq |E| \leq \epsilon b$ does not hold and the theorem doesn't apply.



The initial goal of this work was to track the transition around $E = 0$. But here this cannot work.

Another example where nothing works:
Consider a pendulum

$$\mathcal{H}_\tau = \frac{p_1^2 + p_2^2}{2} + \lambda_1^2(\tau)(\cos q_1 - 1) + \lambda_2^2(\tau)(\cos q_2 - 1), \; 0 < \lambda_1 < \lambda_2$$

Then the condition $\omega\left(v_+^{k_1}(\tau), v_-^{k_2}(\tau)\right) > 0$ is not satisfied except inside the homoclinic trajectories, so there is no trajectory moving between loops.
In fact, only when the system has chaotic behavior this condition can be satisfied.

A few words about the proofs:
Introduce the extended symplectic form

$$\hat{\omega} = dp \wedge dq + dh \wedge d\tau,$$

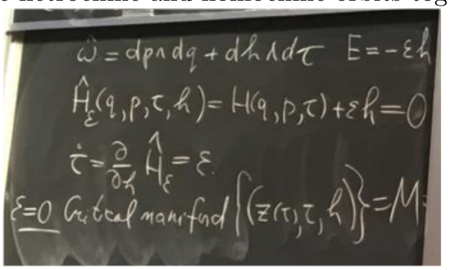
and the extended Hamiltonian

$$\hat{\mathcal{H}}_\epsilon(q, p, \tau, h) = \mathcal{H}(q, p, \tau) + \epsilon h \equiv 0,$$

where we consider the zero energy level set. Then

$$\dot{\tau} = \frac{\partial}{\partial h}\hat{\mathcal{H}}_\epsilon = \epsilon.$$

For $\epsilon = 0$, critical manifold $M = \{(z(\tau), \tau, h)\}$. It is a symplectic, normal-hyperbolic manufold. For $\epsilon = 0$, there are a lot of homoclinic orbits on the manifold, each giving rise to a family of hetroclinic orbits. Then, by shadowing, glue hetroclinic and homoclinic orbits together, $E = \epsilon h$.



2 Questions

- What are $v_\pm^k(\tau)$?

$$v_+^k(\tau) = \lim_{t\to\infty} \dot{\gamma}_\tau^k(t)\, e^{-t|\lambda_1(\tau)|}.$$

It's a vector tangent to the homoclinic orbit in the future (past for (-)).