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Name: ORI KATZ Email/Phone: ORI KATZ.OK@gmail.com

Speaker's Name: Gabriela Pinzari

Talk Title: Exponential stability of Euler integral in the 3-body problem

Date: 10/11/18 Time: 1:30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: The first integral characteristic of the Euler 3-center problem is proven to be an approximate integral to the 3-body problem, at least if masses are very different & particles are constrained to a plane. Proof uses a new normal form result [illegible] around [illegible] of problem, & a new study of the phase portrait of the unperturbed problem.

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Exponential stability of Euler integral in the three-body problem

Gabriella Pinzari

Università di Padova

Mathematical Sciences Research Institute

Berkeley, October 8-12, 2018

The three-body problem

3 masses : M_0 , M_1 , M_2 $+$ gravity

$$H_{3B} = \frac{\|p_0\|^2}{2M_0} + \frac{\|p_1\|^2}{2M_1} + \frac{\|p_2\|^2}{2M_2} - \frac{M_0 M_1}{\|q_1 - q_0\|} - \frac{M_0 M_2}{\|q_2 - q_0\|} - \frac{M_1 M_2}{\|q_2 - q_1\|}$$

with

$$G = \text{Gravity constant} = 1$$

$$p_i,\ \ q_i,\ \in \mathbb{R}^d \qquad p_i = M_i \dot{q}_i$$

$$d = 2,\ \ 3$$

The q_0-centric reduction of translations

[Herman, Féjoz, Laskar, Robutel,...]

- linear change:

$$x_0 = q_0 \ , \qquad x' = q_1 - q_0, \qquad x = q_2 - q_0$$

$$y_0 = p_0 + p_1 + p_2 = \sum_i M_i \dot{q}_i$$

$$y' = p_1, \qquad y = p_2$$

- then:

$$x_0 = \text{ cyclic}$$

$$y_0 = \text{first integral}$$

- Fix the center of mass at rest:

$$y_0 = 0$$

- rename masses:

$$M_0 = 1 \ , \qquad M_1 = \mu \ , \qquad M_2 = \mu\varepsilon$$

- rescale:

$$y' \to \mu\varepsilon y', \quad y \to \mu\varepsilon y, \quad H \to (\mu\varepsilon)^{-1} H \ , \quad t \to \varepsilon t$$

- what you get:

$$H = \frac{\varepsilon^2 \|y'\|^2}{2} - \frac{1}{\|x'\|} + \frac{\|y\|^2}{2} - \frac{1}{\|x\|} + \mu\Big(- \frac{1}{\|x' - x\|} + \varepsilon^2 y' \cdot y \Big)$$

a) $\varepsilon \sim 1, \quad \mu \ll 1$ 1 star, 2 planets, star-centric red.

b) $\varepsilon \ll 1, \quad \mu \ll 1$ star, earth, asteroid, star-centric red.

c) $\varepsilon \ll \mu^{-1}, \quad \mu \gg 1$ star, earth, asteroid, earth-centric red.

d) $\varepsilon \gg \mu^{-1}, \quad \mu \gg 1$ star, earth, asteroid, asteroid-centric red.

e) $\varepsilon \sim \mu^{-1}, \quad \mu \gg 1$ 2 stars, 1 planet, star-centric red.

f) $\varepsilon \sim 1, \quad \mu \sim 1$ general 3BP.

a) ''planetary'', star-centric

$$\varepsilon = 1, \qquad \mu \ll 1$$

$$\mathrm{H} = \underbrace{\frac{\|\mathrm{y}'\|^2}{2} - \frac{1}{\|\mathrm{x}'\|}}_{\text{integrable part}} + \underbrace{\frac{\|\mathrm{y}\|^2}{2} - \frac{1}{\|\mathrm{x}\|}}_{\text{integrable part}} + \underbrace{\mu\Big(- \frac{1}{\|\mathrm{x}' - \mathrm{x}\|} + \mathrm{y}' \cdot \mathrm{y}\Big)}_{\text{perturbing function}}$$

[Poincaré, Arnold]

Arnold's Theorem on the stability of planetary motions

general case: 1 sun $+$ N $\geq$ 2 planets

"For the majority of initial conditions under which the instantaneous orbits of the planets are close to circles lying in a single plane perturbations of the planets one another produce in the course of an infinite time little change of these orbits, provided the masses of the planets are sufficiently small"
V. I. Arnold, Russ. Math. Surv. 1963.

- V.I. Arnold, 1963: N=2 planets, d=2
- Laskar & Robutel, 1995: N=2 planets, d=3;
- Herman & Féjoz 2004: $\forall$ N, d=2, 3 (2 resonances; modified Hamiltonian);
- Chierchia & P. 2011: $\forall$ N, d=2, 3: clarify symplectic structure of phase space (rotational degeneracy)+ measure estimates

main tools:

- (properly degenerate) Kolmogorov-Arnold-Moser (KAM) theory: Arnold 1963; Russmann 2000; Herman 2004;
- Chierchia-P.: geometrical analysis: "symplectic" reduction of the angular momentum; necessary to check KAM non-degeneracy conditions when d=3. Deprit 1983; P. 2009-13-15.

Extensions

- Lower dimensional quasi-periodic motions [Biasco, Chierchia, Vadinoci, '2000];
- Quasi-periodic motions with quasi-collisional orbits [Féjoz, Zhao Lei, '2000];
- More elliptic equilibria [Palacyan, Yanguas et. al, 2015];
- Quasi-periodic motions with eccentric orbits [P. 2018];
- Instabilities [Gidea, Guardia, Guzzo, Kaloshin, Laskar, Lega, Seara];
- [...]

Nekhorossev Stability

- planetary model:

stability for exponentially long times for:

- semi-major axes (N. N. Nekhoressev, 1977; L. Niederman, 1995) for any N;

- of eccentricities for N=d=2 (P. 2013).

b) hierarchical, star-centric

$$\varepsilon \ll 1 \ , \qquad \mu \ll 1$$

$$\mathrm{H} = \underbrace{\underbrace{\frac{\|\mathrm{y}\|^2}{2} - \frac{1}{\|\mathrm{x}\|} + \frac{\mu}{\|\mathrm{x}' - \mathrm{x}\|}}_{\texttt{J=two-centre (unperturbed)}} - \frac{1}{\varepsilon\|\mathrm{x}'\|}}_{\texttt{integrable}} + \underbrace{\varepsilon\left(\frac{\|\mathrm{y}'\|^2}{2} + \mu \mathrm{y}' \cdot \mathrm{y}\right)}_{\texttt{f= kinetic terms}}$$

Questions

Is it possible to study the hierarchical, star-centric system as a perturbed 2CP?

What knowledge would we gain?

Unperturbed motions: remind of 2CP

$$J = \frac{\|y\|^2}{2} - \frac{m_-}{\|x - x'\|} - \frac{m_+}{\|x + x'\|}$$

$$x', \quad x, \quad y \in \mathbb{R}^d \qquad x \neq \pm x' \qquad d = 2, \quad 3$$

A ‘‘trivial’’ integral

$$\mathtt{M} = \mathtt{x} \times \mathtt{y} \qquad \Theta = \mathtt{M} \cdot \frac{\mathtt{x}'}{\|\mathtt{x}'\|} \quad (\mathtt{d} = 3) \ .$$

The ‘‘Euler’’ integral

$$E = \|x \times y\|^2 + (x' \cdot y)^2 + 2x \cdot x' \left(\frac{m_+}{\|x + x'\|} - \frac{m_-}{\|x - x'\|} \right) \ .$$

Euler, 1760; Lagrange, Jacobi XVIII century

Integrability of the 2CP - remind

separation of variables

define the ellipsoidal coordinates λ, μ, ω via

$$\|x + x'\| = r(\lambda + \mu) \qquad \|x - x'\| = r(\lambda - \mu) \qquad \omega = \arg(x_2, x_3)$$

with

$$x = (x_1, x_2, x_3) \ , \quad r := \|x'\| \ .$$

Hamilton-Jacobi Equation

Bekov & Omarov, 1978:

$$J = \frac{1}{2r^2(\lambda^2-\mu^2)}\Big[(\lambda^2-1)p_\lambda^2+(1-\mu^2)p_\mu^2 + \Big(\frac{1}{\lambda^2-1}+\frac{1}{1-\mu^2}\Big)p_\omega^2\Big] - \frac{(m_- + m_+)\lambda-(m_- - m_+)\mu}{r(\lambda^2-\mu^2)}$$

ω is cyclic $\qquad p_\omega = \mathrm{cte} = \Theta$

Hamilton-Jacobi Equation

Equation

$$\mathtt{J}(\mathtt{W}_\lambda, \mathtt{W}_\mu, \Theta, \lambda, \mu) = \mathtt{c}$$

with

$$\mathtt{W} = \mathtt{W}_1(\lambda, \Theta, \mathtt{c}) + \mathtt{W}_2(\mu, \Theta, \mathtt{c})$$

splits

$$\mathtt{F}_1(\mathtt{W}_{1\lambda}, \lambda, \Theta, \mathtt{c}) + \mathtt{F}_2(\mathtt{W}_{2\mu}, \mu, \Theta, \mathtt{c}) = 0$$

separates in two equations

$$\mathtt{F}_1(\mathtt{W}_{1\lambda}, \lambda, \Theta, \mathtt{c}) = -\mathtt{F}_2(\mathtt{W}_{2\mu}, \mu, \Theta, \mathtt{c}) = \mathtt{E} = \text{“Euler integral''}$$

Motion Equations

- The solutions

$$W_1 = \int \frac{\sqrt{P_1(\lambda, \Theta, J, E)}}{\lambda^2 - 1} d\lambda \qquad W_2 = \int \frac{\sqrt{P_2(\mu, \Theta, J, E)}}{1 - \mu^2} d\lambda$$

are elliptic integrals.

- The motion equations

$$\begin{cases} \partial_J \Big[W_1(\lambda, \Theta, J, E) + W_2(\mu, \Theta, J, E) \Big] = t - t' \\ \\ \partial_E \Big[W_1(\lambda, \Theta, J, E) + W_2(\mu, \Theta, J, E) \Big] = 0 \end{cases}$$

requires to invert a 2×2 system involving elliptic integrals.

Is 2CP Liouville-Arnold integrable?

What are periodic motions?

What are the action-angle coordinates?

- Dullin, Waalkens & Richter, 2006: bifurcation diagrams, actions for the symmetric and a-symmetric problem (resp. $m_+ = m_-$, $m_+ \neq m_-$) + regularization

- Dullin & Montgomery, 2016: Syzygies + regularization

- Biscani & Izzo, 2016: explicit solution for d=3

- Terracini & al, 2017: parabolic orbits (any number of centers >=2)

Regularization (Dullin & al)

- Planar problem: $\Theta = 0$, $x = (x_1, x_2)$, $y = (y_1, y_2)$
- Fix a energy level $J = E$
- change coordinates $x_1 + i x_2 = \sin(\eta + i\xi)$
- change time $t \to \tau$: $\frac{dt}{d\tau} = \frac{1}{\cosh^2\xi - \cos^2\eta}$: regularization

- new Hamiltonian

$$\breve{J}(p_\eta, p_\xi, \eta, \xi, E) = \breve{J}_\eta(p_\eta, \eta, E) + J_\xi(p_\xi, \xi, E)$$

with

$$\begin{cases} \breve{J}_\eta(p, q, E) = \dfrac{p^2}{2} - (m_+ + m_-)\cosh q - E\cosh^2 q \\ \\ \breve{J}_\xi(p, q, E) = \dfrac{p^2}{2} + (m_+ - m_-)\sin q + E\sin^2 q \\ \\ \breve{J}_\eta(\eta, p_\eta, E) + \breve{J}_\xi(\xi, p_\xi, E) \equiv 0 \end{cases}$$

The "asymmetric" 2CP

Write the 2CP Hamiltonian in the form:

$$\mathrm{J} = \frac{\|\mathrm{y}^2\|}{2} - \frac{1}{\|\mathrm{x}\|} - \frac{\mu}{\|\mathrm{x} - \mathrm{x}'\|} = \mathrm{J}_0 + \mu \mathrm{J}_1$$

with

$$\mu \ll 1$$

The Euler Integral

$$\mathtt{E}(\mathtt{x}', \mathtt{x}, \mathtt{y}) = \|\mathtt{M}\|^2 - \mathtt{L} \cdot \mathtt{x}' + \mu \frac{\mathtt{x}' - \mathtt{x}}{\|\mathtt{x}' - \mathtt{x}\|} \cdot \mathtt{x}'$$

$$\mathtt{M} = \mathtt{x} \times \mathtt{y} \qquad \mathtt{L} = \mathtt{y} \times \mathtt{M} - \frac{\mathtt{x}}{\|\mathtt{x}\|}$$

The canonical setting

We use the canonical coordinates

$$\mathcal{K} = \Big((\mathtt{Z}, \mathcal{C}, \mathtt{G}, \Theta, \mathtt{R}, \mathtt{L}) \ , \ (\mathtt{z}, \gamma, \mathtt{g}, \vartheta, \mathtt{r}, \lambda) \Big)$$

$$\Omega = \mathrm{d}\mathtt{Z} \wedge \mathrm{d}\mathtt{z} + \mathrm{d}\mathcal{C} \wedge \mathrm{d}\gamma + \mathrm{d}\mathtt{G} \wedge \mathrm{d}\mathtt{g} + \mathrm{d}\Theta \wedge \mathrm{d}\vartheta + \mathrm{d}\mathtt{R} \wedge \mathrm{d}\mathtt{r} + \mathrm{d}\mathtt{L} \wedge \mathrm{d}\lambda$$

9 coordinates (P. 2013-2015)

The definition of the 9 coordinates

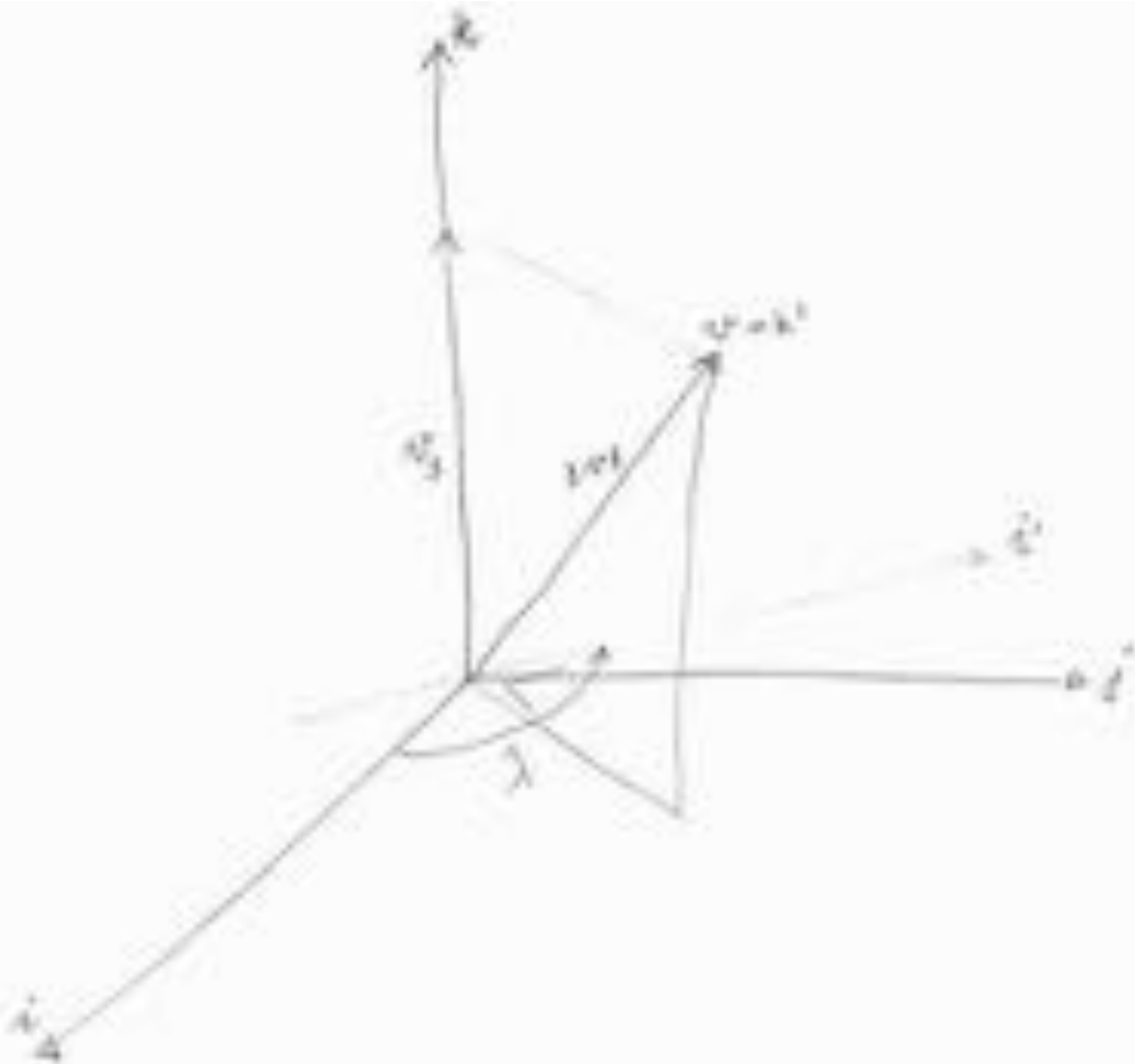
$$\mathtt{Z},\quad \mathcal{C},\quad \Theta,\quad \mathtt{G},\quad \mathtt{z},\quad \vartheta,\quad \gamma,\quad \mathtt{r},\quad \mathtt{g}$$

is based on the iteration of always the same same construction

$$(\mathtt{F}, \mathtt{v}) \in \{\mathtt{orthogonal\ frames}\} \times \mathbb{R}^3 \Longrightarrow (\mathtt{n},\ \ \mathtt{v}_3,\ \ \|\mathtt{v}\|,\ \ \lambda) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{T}$$

where

$$\mathtt{v} \not\parallel \mathtt{k} \quad \mathtt{if} \quad \mathtt{F} = (\mathtt{i}, \mathtt{j}, \mathtt{k})$$



Consider:

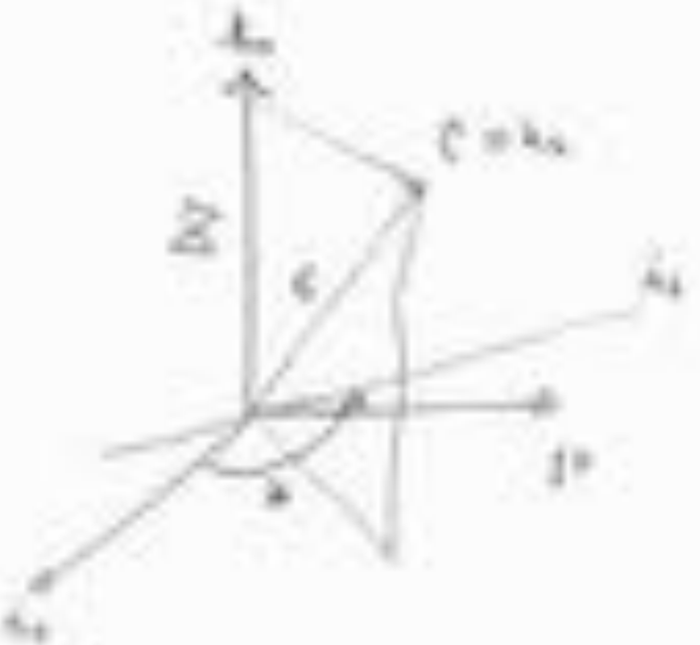
- $\mathrm{M}' = \mathrm{x}' \times \mathrm{y}'$, $\mathrm{M} = \mathrm{x} \times \mathrm{y}$: angular momenta of the smaller bodies;

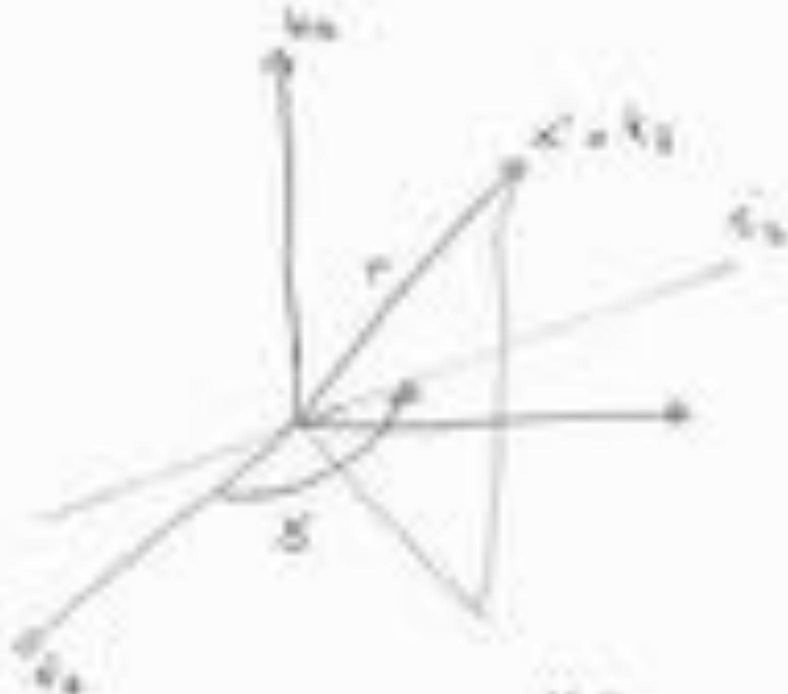
- $\mathrm{C} = \mathrm{M} + \mathrm{M}'$: total angular momentum (first integral of motion);

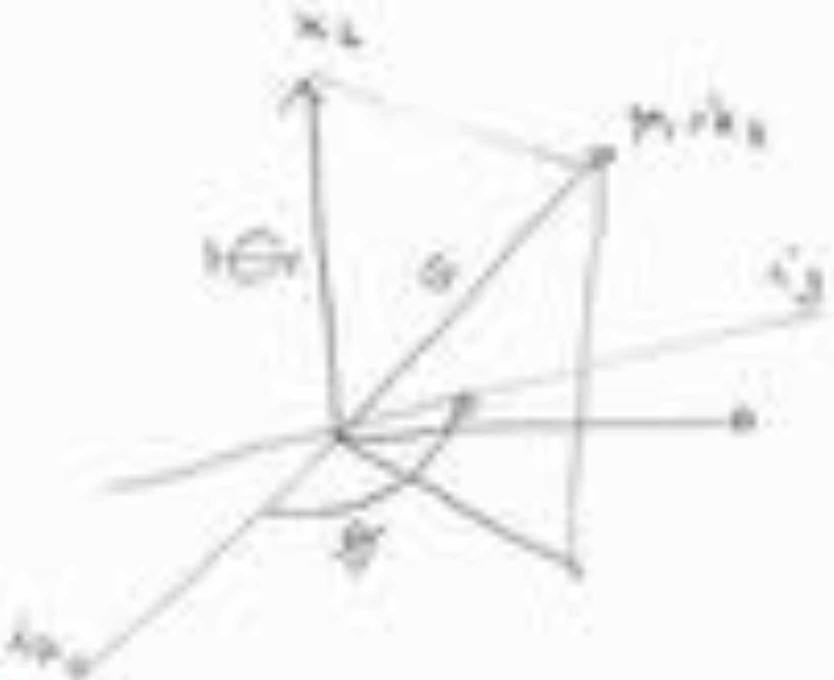
- $\mathcal{E}$: the Keplerian motion generated by the Keplearian term $\mathrm{K} = \frac{\|\mathrm{y}\|^2}{2} - \frac{1}{\|\mathrm{x}\|}$

- P, with , $\|\mathrm{P}\| = 1$: the perihelion direction of $\mathcal{E}$ (supposing it is definite)

then repeat the construction above 4 times as follows.







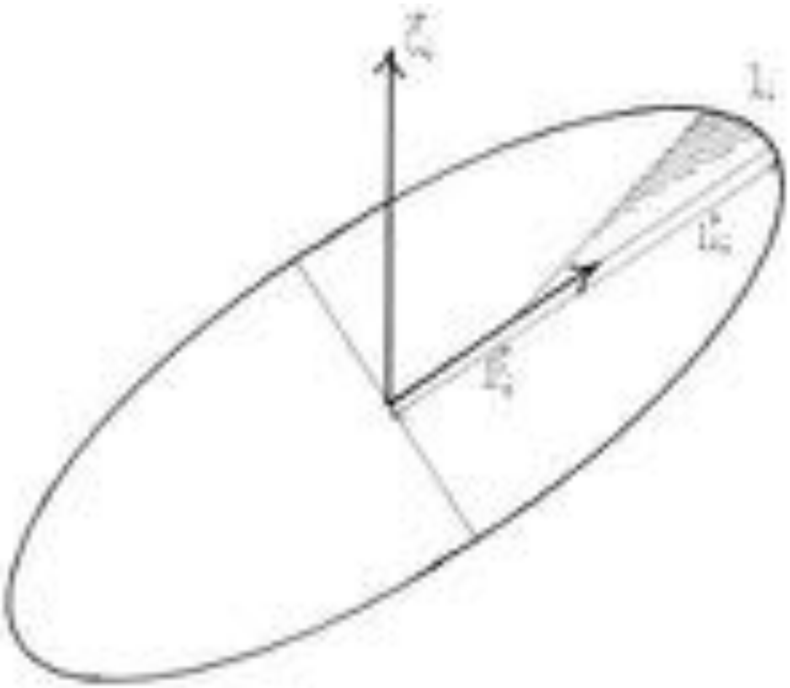
$$\left.\begin{array}{l} \mathtt{F}_0 = (\mathtt{i}_0, \mathtt{j}_0, \mathtt{k}_0) \\ \\ \mathtt{v}_0 = \mathtt{C} = \mathtt{k}_1 \end{array}\right\} \Longrightarrow \quad (\mathtt{i}_1,\ \mathtt{Z},\ \mathcal{C},\ \mathtt{z}) \quad \mathtt{Z} = \mathtt{C}\cdot\mathtt{k}_0 \ ,\ \mathcal{C} = \|\mathtt{C}\|$$

$$\left.\begin{array}{l} \mathtt{F}_1 = (\mathtt{i}_1, \mathtt{j}_1, \mathtt{k}_1) \\ \\ \mathtt{v}_1 = \mathtt{x}' = \mathtt{k}_2 \end{array}\right\} \Longrightarrow \quad \left(\mathtt{i}_2,\ \mathtt{x}'\cdot\frac{\mathtt{C}}{\|\mathtt{C}\|},\ \mathtt{r},\ \gamma\right) \quad \mathtt{r} = \|\mathtt{x}'\|$$

$$\left.\begin{array}{l} \mathtt{F}_2 = (\mathtt{i}_2, \mathtt{j}_2, \mathtt{k}_2) \\ \\ \mathtt{v}_2 = \mathtt{M} = \mathtt{k}_3 \end{array}\right\} \Longrightarrow \quad (\mathtt{i}_3,\ \Theta,\ \mathtt{G},\ \vartheta) \quad \Theta = \mathtt{M}\cdot\frac{\mathtt{x}'}{\|\mathtt{x}'\|} \ ,\ \mathtt{G} = \|\mathtt{M}\|$$

$$\left.\begin{array}{l} \mathtt{F}_3 = (\mathtt{i}_3, \mathtt{j}_3, \mathtt{k}_3) \\ \\ \mathtt{v}_3 = \mathtt{P} \end{array}\right\} \Longrightarrow \quad (\mathtt{i}_4,\ 0,\ 1,\ \mathtt{g})$$

3 more coordinates



$$R = \frac{y' \cdot x'}{\|x'\|}$$

$$L = \sqrt{a}$$

$$l = \text{mean anomaly}$$

Features

- $(\mathcal{C}, \mathtt{Z}, \mathtt{z})$: first integrals to 2CP & 3BP

$\Longrightarrow$ get rid of $\mathtt{SO}(3)$ invariance

- $\mathtt{r} = \|\mathtt{x}'\|$, $\Theta = \mathtt{M} \cdot \frac{\mathtt{x}'}{\|\mathtt{x}'\|}$: first integral to 2CP

$\Longrightarrow$ get rid of invariance by $\mathtt{x}'-$rotations for 2CP

- The coordinates $(\mathtt{L}, \mathtt{G}, \mathtt{l}, \mathtt{g})$ describe the variations of the semi-axis and the eccentricity of $\mathcal{E}$.

- The coordinates (Θ, ϑ) describe the inclination of $\mathcal{E}$.

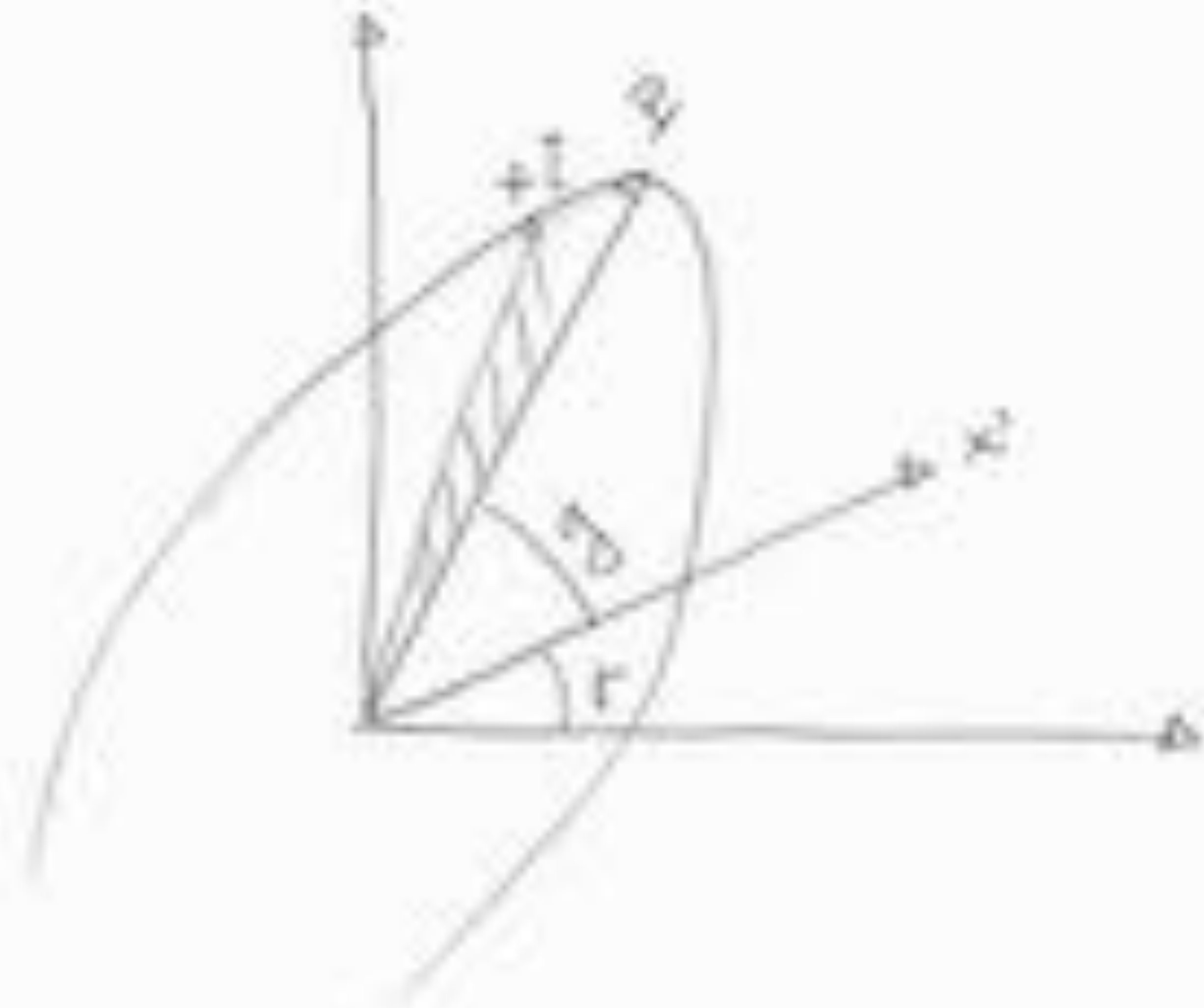
- The coordinates $(\mathtt{R}, \mathtt{r})$ describe the motions of the Earth.

The planar case

- The planar limit is well defined:

$$\begin{cases} \Theta = 0 \ , \ \vartheta = \pi \ (\text{prograde}) \\ \Theta = 0 \ , \ \vartheta = 0 \ (\text{retrograde}) \end{cases}$$

Note: a similar regularity does not hold for the classical Jacobi reduction/Deprit coordinates



$$\begin{aligned}
\mathtt{J}(\mathtt{L},\mathtt{G},\mathtt{l},\mathtt{g},\mathtt{r},\Theta) &= -\frac{1}{2\mathtt{L}^2} - \frac{\mu}{\sqrt{\mathtt{r}^2 + 2\mathtt{r}\mathtt{a}\varrho\sqrt{1-\frac{\Theta^2}{\mathtt{G}^2}}\cos(\mathtt{g}+\nu) + \mathtt{a}^2\varrho^2}} \\
&= \mathtt{J}_0(\mathtt{L}) + \mu \mathtt{J}_1(\mathtt{L},\mathtt{l},\mathtt{G},\mathtt{g},\mathtt{r},\Theta)
\end{aligned}$$

$$\begin{aligned}
\mathtt{E}(\mathtt{L},\mathtt{G},\mathtt{l},\mathtt{g},\mathtt{r},\Theta) &= \mathtt{G}^2 + \mathtt{r}\sqrt{1-\frac{\Theta^2}{\mathtt{G}^2}}\sqrt{1-\frac{\mathtt{G}^2}{\mathtt{L}^2}}\cos\mathtt{g} \\
&+ \mu\mathtt{r}\frac{\mathtt{r} + \mathtt{a}\varrho\sqrt{1-\frac{\Theta^2}{\mathtt{G}^2}}\cos(\mathtt{g}+\nu)}{\sqrt{\mathtt{r}^2 + 2\mathtt{r}\mathtt{a}\varrho\sqrt{1-\frac{\Theta^2}{\mathtt{G}^2}}\cos(\mathtt{g}+\nu) + \mathtt{a}^2\varrho^2}} \\
&= \mathtt{E}_0(\mathtt{L},\mathtt{G},\mathtt{g},\Theta) + \mu \mathtt{E}_1(\mathtt{L},\mathtt{G},\mathtt{l},\mathtt{g},\mathtt{r},\Theta)
\end{aligned}$$

$\nu(\mathtt{L},\mathtt{G},\mathtt{l}) = $ true anomaly

$\varrho(\mathtt{L},\mathtt{G},\mathtt{l}) = \|\mathtt{x}\|/\mathtt{a}$

The unperturbed motion

Proposition Given an integrable, 2 d.o.f. Hamiltonian

$$J(I, \varphi, p, q) = J_0(I) + \mu J_1(I, \varphi, p, q; \mu)$$

equipped with a first integral

$$E(I, \varphi, p, q) = E_0(I, p, q) + \mu E_1(I, \varphi, p, q; \mu) \ .$$

Assume J, E are real-analytic. Let

$$\bar{J}_1(I, p, q) = \frac{1}{2\pi} \int_0^{2\pi} J_1(I, \varphi, p, q) d\varphi$$

and let $\phi^{(n)}$ be a r.a. canonical, transformation such that

$$J \circ \phi^{(n)} = J^{(n)}(I, p, q) + O(\mu^{n+1})$$

where

$$J^{(n)}(I, p, q) = J_0(I) + \mu \bar{J}_1(I, p, q) + \cdots$$

Then:

- $\mathtt{E}_0$ is a first integral to $\overline{\mathtt{J}}_1$;

- $\mathtt{E}\circ\phi^{(\mathtt{n})} = \mathtt{E}^{(\mathtt{n})}(\mathtt{I},\mathtt{p},\mathtt{q}) + \mathtt{O}(\mu^{\mathtt{n}+1})$

- $\mathtt{J}^{(\mathtt{n})}(\mathtt{I},\mathtt{p},\mathtt{q}))$ and $\mathtt{E}^{(\mathtt{n})}(\mathtt{I},\mathtt{p},\mathtt{q}))$ commute up to order $\mathtt{O}(\mu^{\mathtt{n}+1})$

Corollary

$J^{(n)}(I, p, q)$ is a function of $E^{(n)}(I, p, q)$ and I, up to $O(\mu^{n+1})$. Therefore:

- The dynamics of (p, q) under $J^{(n)}(I, p, q)$ or $E^{(n)}(I, p, q)$ is the same, up to a rescaling of time (depending on I).

In particular:

- the dynamics of (p, q) under

$$J^{(1)}(I, p, q) = -\frac{1}{2L^2} + \mu \bar{J}_1 \quad \text{and} \quad E^{(1)}(I, p, q) = E_0 + \mu \bar{E}_1$$

are the same up to $O(\mu^2)$

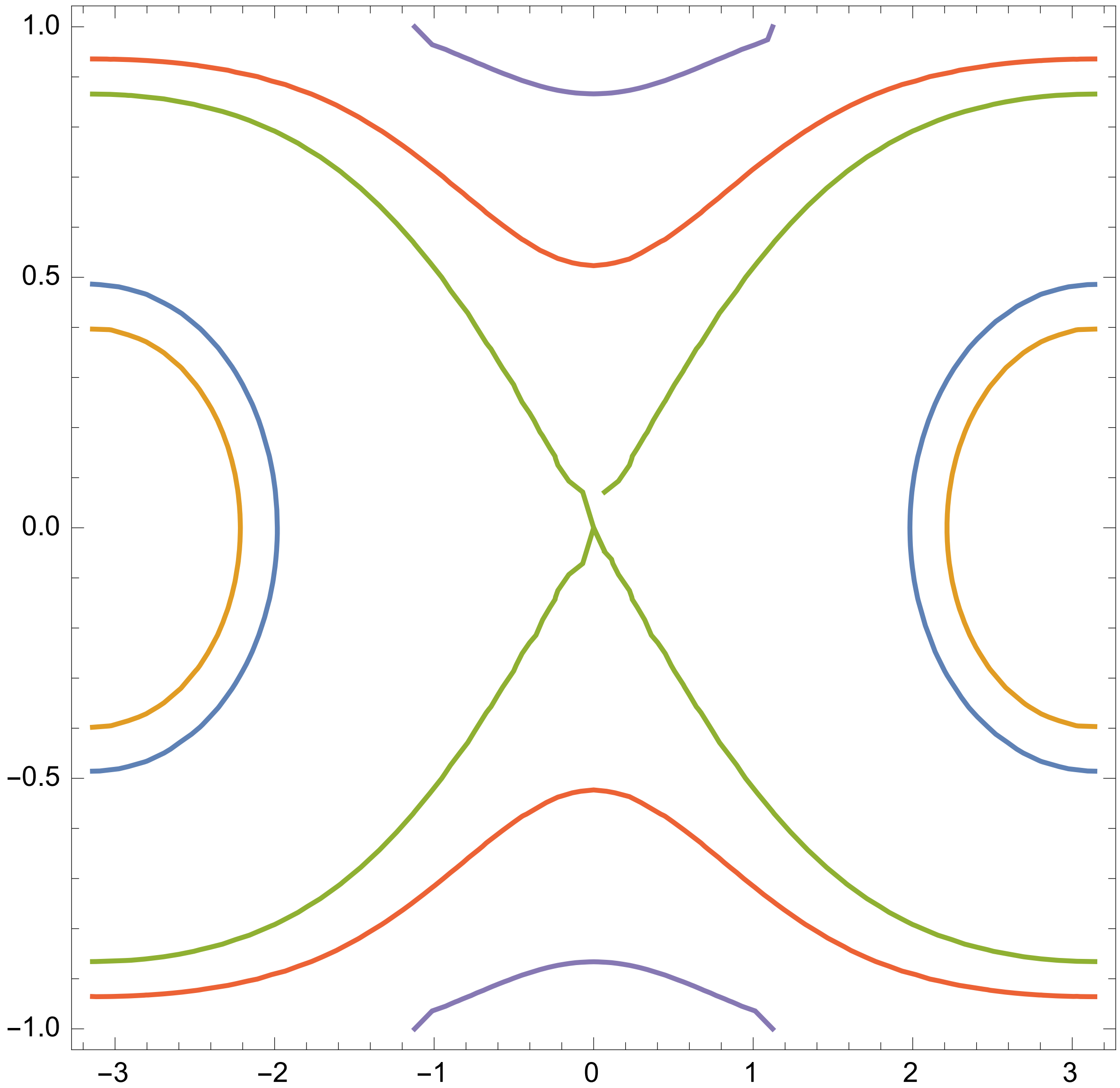
Phase portraits of $E_0(L,\cdot,\cdot)$ (planar case)

Let

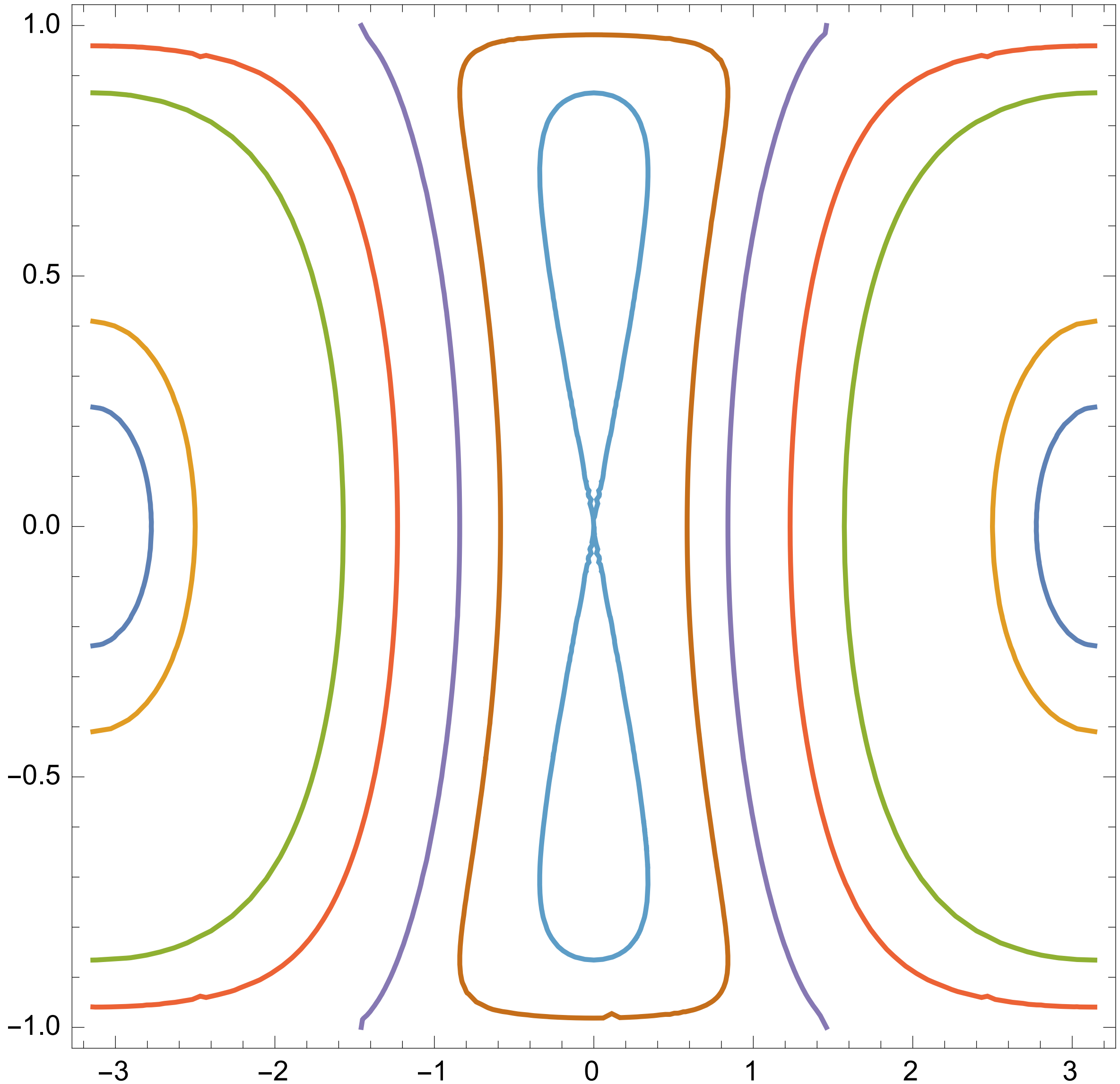
$$\delta = \frac{r}{a} \ .$$

Two cases:

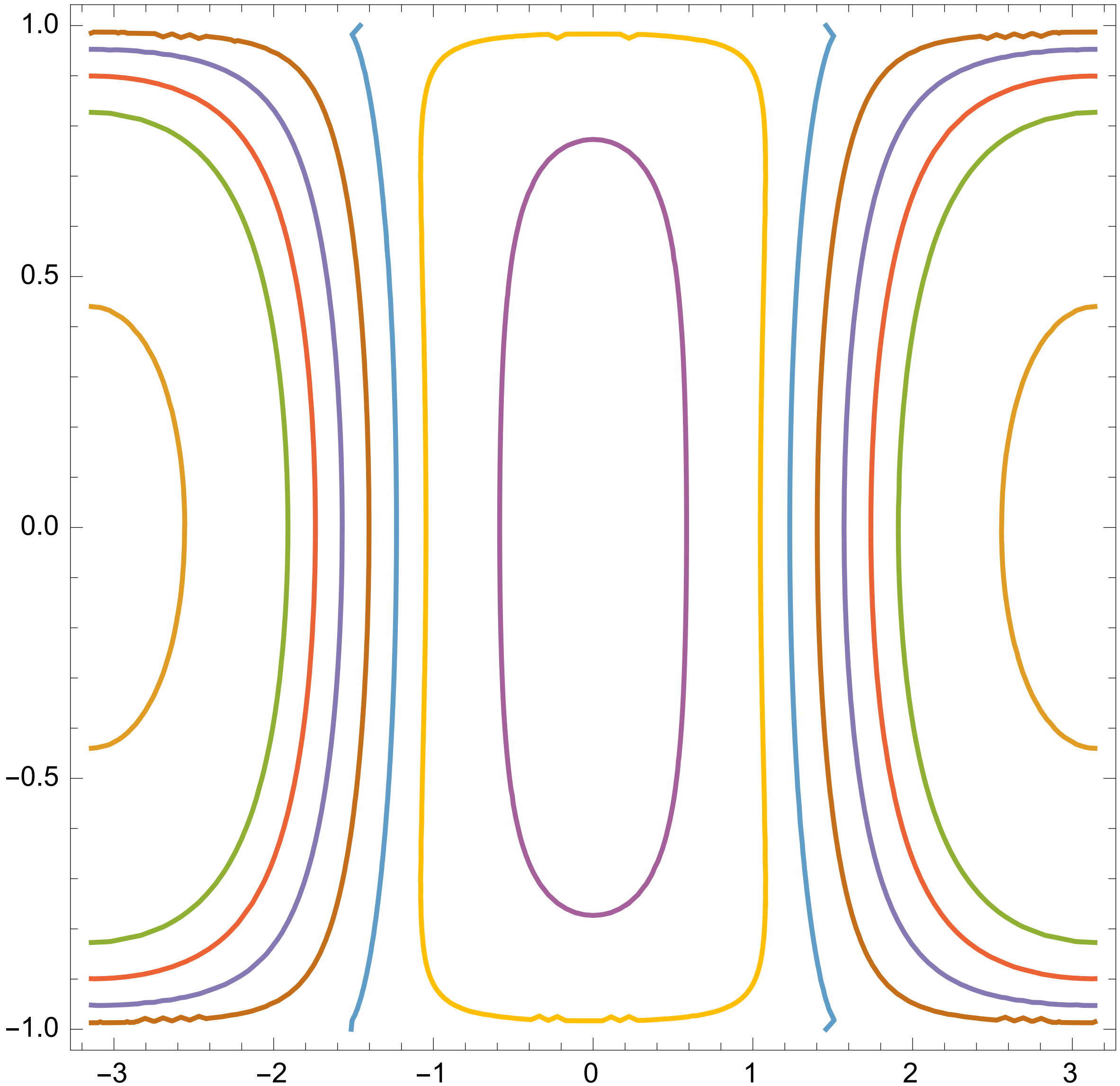
- $0 < \delta < 2$ collisions $x = x'$ are possible;
- $\delta > 2$ collisions $x = x'$ are not possible.



$0 < \delta < 1$



$1 < \delta < 2$



$\delta > 2$

2 separatrices

- Separatrix 1: level set through G=L ($\forall\ \delta$)

- Separatrix δ: level set through the saddle (G,g)=(0,0) ($0 < \delta < 2$)

- RK: the "eye" of the pendulum has strength $\sqrt{\delta}$.

Equation of the δ-separatrix
$(0 < \delta < 2)$

$$\mathrm{G}^2 + \mathrm{r}\sqrt{1 - \frac{\mathrm{G}^2}{\mathrm{L}^2}}\cos \mathrm{g} = \mathrm{r} \quad \Leftrightarrow \quad \mathrm{r} = \frac{\mathrm{G}^2}{1 - \sqrt{1 - \frac{\mathrm{G}^2}{\mathrm{L}^2}}\cos \mathrm{g}}$$

=collisional manifold

E_0-motion on the separatrix δ

$$\begin{cases} G(t) = \pm \dfrac{\sigma L}{\cosh \sigma L(t-t')} \\ \\ g(t) = \pm \cos^{-1} \dfrac{1 - \frac{\beta^2}{\cosh^2 \sigma L(t-t')}}{\sqrt{1 - \frac{\sigma^2}{\cosh^2 \sigma L(t-t')}}} \end{cases}$$

with

$$\sigma^2 := \delta(2-\delta) \qquad \beta^2 := 2-\delta \qquad \delta := \frac{r}{a}$$

For $\delta = 1$ reduces to the classical pendulum

Equation of the separatrix 1

$$\begin{cases} \mathtt{G} = \mathtt{L} \\ \\ \mathtt{G} = \mathtt{L}\sqrt{1 - \delta^2 \cos^2 \mathtt{g}} \end{cases}$$

The full problem

$$\mathrm{H} = \underbrace{\underbrace{\frac{\|\mathrm{y}\|^2}{2} - \frac{1}{\|\mathrm{x}\|} + \frac{\mu}{\|\mathrm{x}' - \mathrm{x}\|}}_{\mathrm{J=two-centre\ \ (unperturbed)}} - \frac{1}{\varepsilon\|\mathrm{x}'\|}}_{\mathrm{integrable}} + \varepsilon\underbrace{\Big(\frac{\|\mathrm{y}'\|^2}{2} + \mu \mathrm{y}' \cdot \mathrm{y}\Big)}_{\mathrm{f=\ kinetic\ terms}}$$

The full problem

$$\mathtt{H} = \mathtt{J}(\mathtt{L}, \mathtt{G}, \mathtt{l}, \mathtt{g}, \mathtt{r}, \Theta; \mu) - \frac{1}{\varepsilon \mathtt{r}} + \frac{\varepsilon \mathtt{R}^2}{2} + \frac{\varepsilon \Phi^2}{2\mathtt{r}^2} + \mathtt{O}(\mu\varepsilon)$$

with

$$\Phi^2 = \mathtt{G}^2 + \mathtt{C}^2 + 2\sqrt{\mathtt{G}^2 - \Theta^2}\sqrt{\mathtt{C}^2 - \Theta^2}\cos\vartheta \quad \text{(spatial case)}$$

$$\Phi^2 = (\mathtt{C} \mp \mathtt{G})^2 \quad \text{(planar case)}$$

Result

Theorem [P. 2018] Assume μ, ε, δ are small and that the eccentricity of middle body lies in a annular neighborood of zero. Then the Euler integral varies a little in the course of an exponentially long time, which can be quantified if terms of μ, ε, δ and the maximum radius of such neighborood.

Remark: no need to check steepness or SDM or similar.

The estimates on the variation of E do not allow to infer that, a motion that begins with (G, g) "inside the eye" of the pendulum remains there.

Application

At a collision,

$$\mathtt{E} = \mathtt{r} + \mathtt{O}(\mu \mathtt{r}) \ .$$

If, at a certain time, E is "sufficiently" far from r, then collisions are excluded for an exponentially long time.

Ideas of proof for planar, prograde

Expand the leading part of the perturbing term around its equilibrium

$$\mathtt{R}=0\ ,\quad \mathtt{r}=\varepsilon^2(\mathtt{C}-\mathtt{G})^2=:\mathtt{r}_0$$

get

$$-\frac{1}{\varepsilon\mathtt{r}}+\frac{\varepsilon\mathtt{R}^2}{2}+\frac{\varepsilon\Phi^2}{2\mathtt{r}^2}=-\frac{1}{2\varepsilon^3(\mathtt{C}-\mathtt{G})^2}+\mathcal{Q}(\mathtt{R},\mathtt{r}-\mathtt{r}_0)$$

where $\mathcal{Q}(\mathtt{x},\mathtt{y})$ begins with a quadratic form in (x,y).

Case δ large

Observe that the frequency ratio of smallest body to the middle is:

$$\frac{\varepsilon^3(\mathrm{C}-\mathrm{G})^2}{\Lambda^3} = \delta^{3/2}$$

So, the kinetic terms are really "small" only for $\delta \gg 1$.

In the case $0 < \delta < 2$ an extra argument is needed.

Case δ small

To get rid of the robustness of the kinetic terms:

to discard terms depending on "action variables" only, and evaluate the size of the remaining part.

Two settings where this is/arguably might be done:

1) $\delta \to 0$ [P. 2018; arXiv: 1808.07633];

2) $G/\mathcal{C} \to 0$, any δ [conjecture].

Normal form lemma without small denominators

Let

$$H(y,I,x,\varphi)=h(y,I)+f(y,I,x,\varphi)\qquad (I,\varphi)\in B^n_r\times\mathbb{T}^n\ ,\ \ \|x\|\le\chi$$

be such that

$$CN\chi\left\|\frac{\partial_I h}{\partial_y h}\right\|\le 1\ ,\quad CN\chi\left\|\frac{f}{\partial_y h}\right\|\le 1$$

Then it is possible to conjugate, via a canonical transformation ϕ, H to

$$H'=H\circ\phi=h(y,I)+\overline{f}(y,I,x)+2^{-N}f'(y,I,x,\varphi)$$

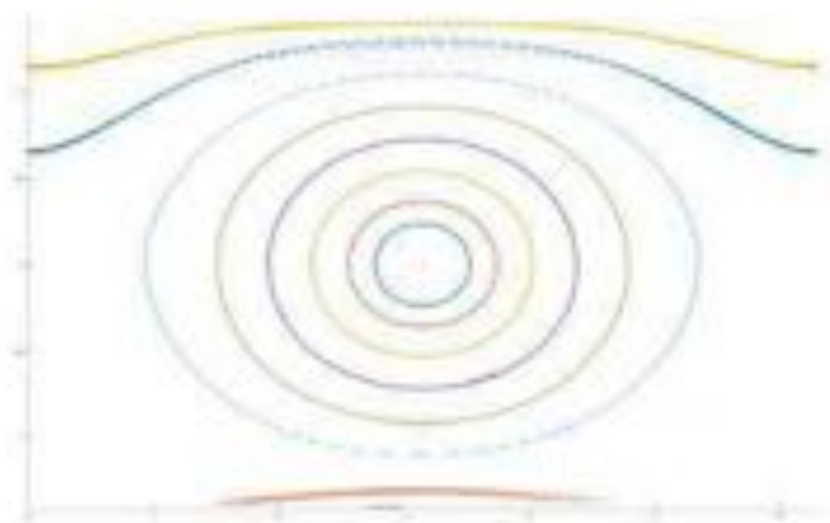
where

$$\overline{f}=\langle f\rangle_\varphi+\text{h.o.t}$$

[Fortunati-Wiggins 2016]: exponential decay of f w. r. t. x

Some numerical experiments

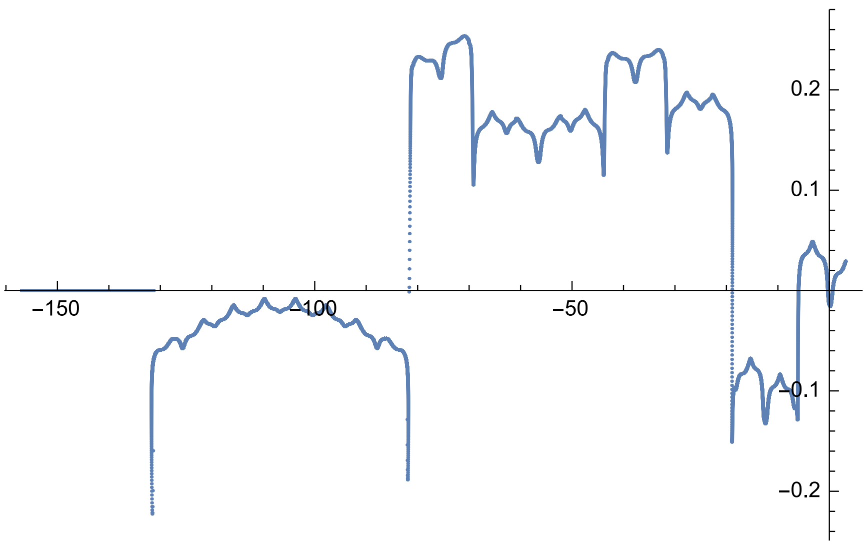
$\delta \ll 1 +$ artificial parameter σ

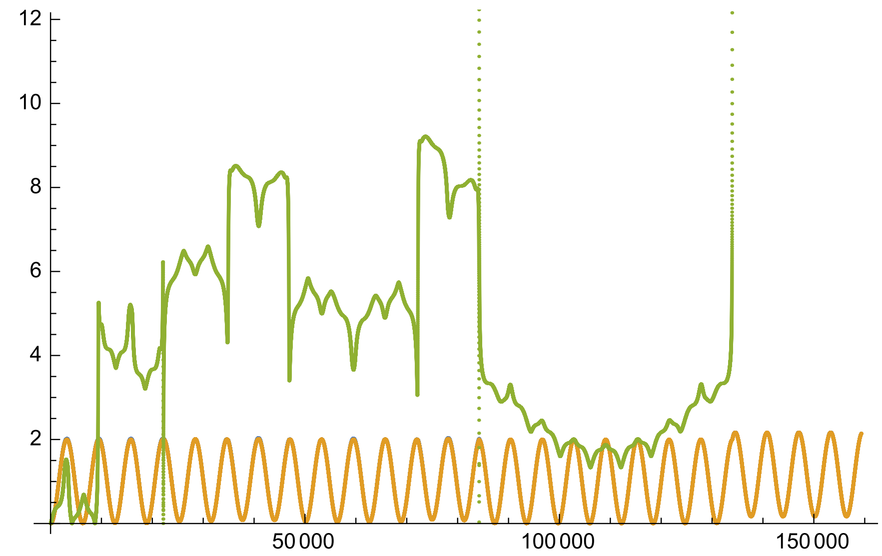


$$\mathrm{H}_\sigma = \mathrm{J} + \sigma \mathrm{f} \qquad \sigma \le \varepsilon^3 \ (\sigma = \varepsilon: \ \text{3BP})$$

courtesy of Edoardo Legnaro,
University of Padua

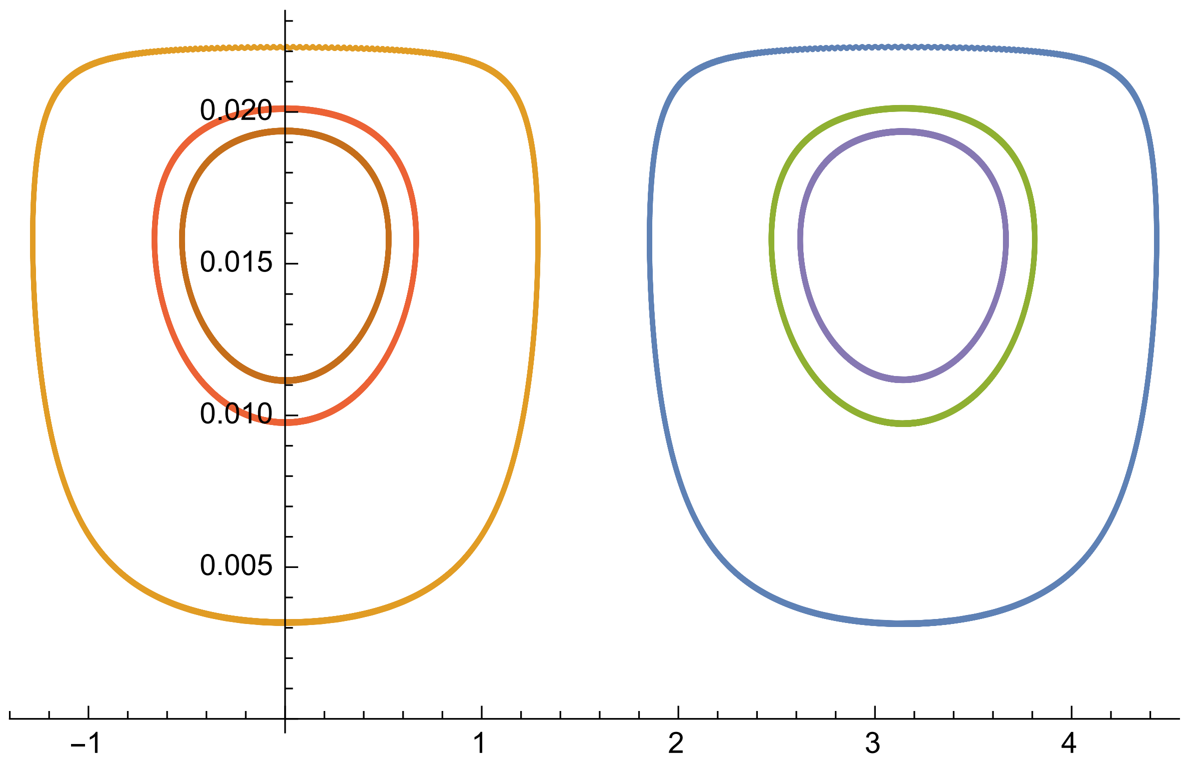
One experiment with $0 < \delta < 2$





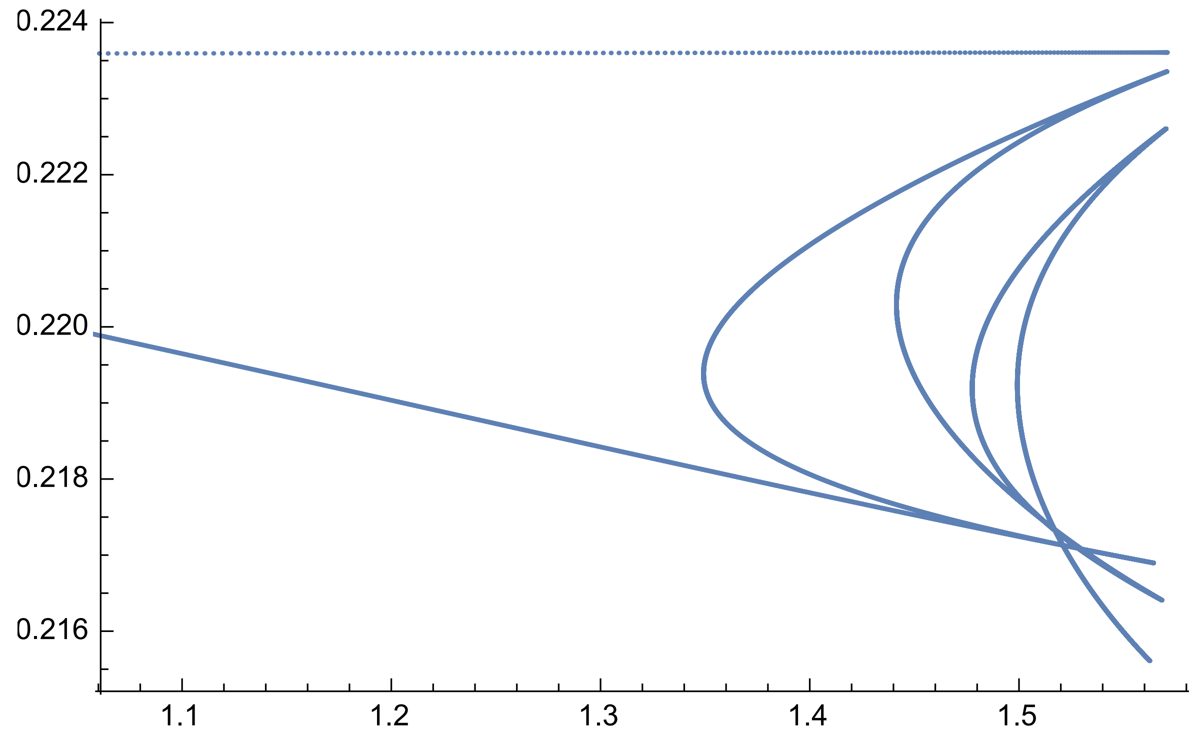
$$\delta = 1.5\ ,\quad \mu = 0.01\ ,\quad \varepsilon = 10^{-15}$$

One experiment with $\delta > 2$



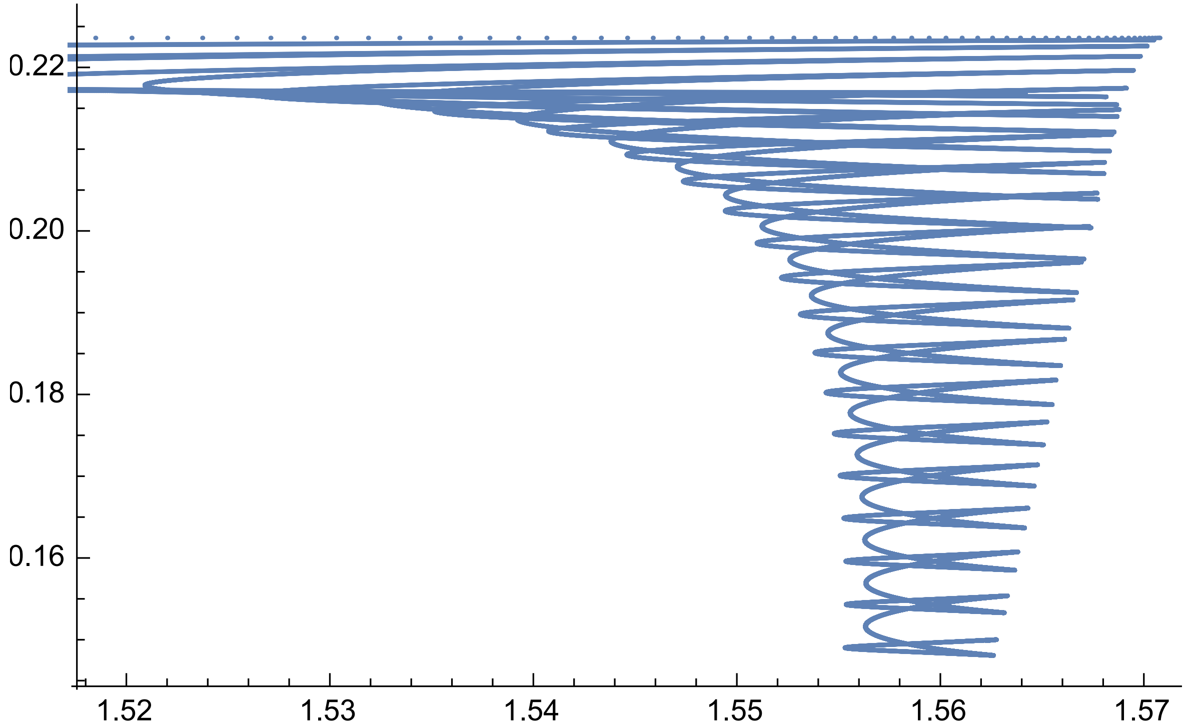
$\delta = 10^5\ , \quad \mu = 1\ , \quad \varepsilon = 1$

One experiment with $\delta > 2$



$\delta = 10^3 \,, \quad \mu = 1 \,, \quad \varepsilon = 1$

One experiment with $\delta > 2$



$\delta = 10^3 \,, \quad \mu = 1 \,, \quad \varepsilon = 1$

Open problems

- provide a rigorous, quantitative comparison on the variations of E and G;

- prove existence of librations for (G,g) when $\delta > 2$;

- prove existence of librations and rotations for (G,g) when $0 < \delta < 2$;

- prove Arnold diffusion bifurcating from periodic orbits, in the case $\delta > 2$ (S. Bolotin lecture).