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Speaker's Name: Chongchun Zeng

Talk Title: Local dynamics & invariant manifolds of traveling wa manifolds of Hamiltonian PDEs

Date: 10 / 12 / 18 **Time:** 3 : 30 am / [✓ pm] (circle one)

Please summarize the lecture in 5 or fewer sentences: The lecture focuses on local dynamics & invariant manifolds of traveling wave manifolds for the Gross-Pitaevski equation in $\mathbb{R}^3$ & the gKdV equation, with an approach applicable to a general class of problems. Symplectic operators of some of these models are unbounded in energy space allowing a linearized analysis in a newly developed framework. Nonlinearly, the main result is existence of local invariant manifolds of unstable traveling wave manifolds & implications on local dynamics.

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Local dynamics and invariant manifolds of traveling wave manifolds of Hamiltonian PDEs

Chongchun Zeng
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Main Ham PDE example #1: gKdV

$$u_t + (u_{xx} + u^k)_x = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R} \qquad \text{(gKdV)}$$

- Hamiltonian structure:

$$E(u) = \int_{\mathbb{R}} \frac{1}{2} u_x^2 - \frac{1}{k+1} u^{k+1} dx, \qquad J = \partial_x, \qquad u \in H^1$$

- Conserved momentum $\longleftrightarrow$ translation invariance in x:

$$P(u) = \frac{1}{2} \int_{\mathbb{R}} u^2 dx$$

- Scaling invariance: solu $u(t,x) \longrightarrow$ solu

$$u_\lambda(t,x) = \lambda^{\frac{2}{k-1}} u(\lambda^3 t, \lambda x), \qquad P(u_\lambda) = \lambda^{\frac{5-k}{k-1}} P(u)$$

Traveling waves of (gKdV)

- Relative equilibrium: traveling waves (TW) to the right

$$u(t,x) = Q_c(x-ct), \quad Q_c(x) = c^{\frac{1}{k-1}} Q(\sqrt{c}x) \qquad c > 0$$

$$Q(x) = \Big(\frac{k+1}{2} sech^2\big(\frac{k-1}{2}x\big)\Big)^{\frac{1}{k-1}} \in H^1$$

satisfying

$$Q_{xx} - Q + Q^k = 0, \quad Q(\pm\infty) = 0$$

- TWs $\longleftrightarrow$ critical pts of $H(u) = E(u) + cP(u)$
- Due to scaling invariance, fix $c = 1$.

(gKdV) in moving frame

Let

$$u(t,x) = U(t, x-t)$$

(gKdV) $\Longrightarrow$

$$U_t = U_x - (U_{xx} + U^k)_x = JH'(U). \qquad \text{(gKdV-M)}$$

- Hamiltonian $H(U) = E(U) + P(U)$, symplectic structure $J = \partial_x$

- Q an equilibrium of (gKdV-M)

* $H^1 \supset M = \{Q(\cdot + y) \mid y \in \mathbb{R}\} \sim \mathbb{R}$: equilibria of (gKdV-M)

- Linearization at Q for stability of M:

$$U_t = JH''(Q)u, \qquad \text{Morse Index } n^-(H''(Q)) = 1 \qquad \text{(L-gKdV-M)}$$

* GSS not directly applicable for $J^{-1} = \partial_x^{-1}$ unbounded on H^1

Stability of traveling waves

- $k < 5$ (subcritical): stable
  - Orbital stability. (Benjamine, Bona&Souganidis&Strauss, Weinstein)
  - Asymptotic stability in exp. weighted space (Pego&Weinstein, Mizumachi), weakly in H^1 (Martel&Merle)

- $k = 5$ (critical): unstable, starting near M, $\exists$ globally nearby solu, blow-up solu, ... (Martel&Merle, Martel&Merle&Nakanishi&Raphaël)

- $k > 5$ (supercritical): unstable
  - Orbitally unstable. (Bona&Souganidis&Strauss)
  - $\exists$ solu decaying to M (Combet)

- **Goal**: local nonlinear dynamics near M for $k > 5$?

Main Ham PDE example #2: GP in 3-dim

- Gross-Pitaevskii (GP) equation:

$$iu_t + \Delta u + (1 - |u|^2)u = 0, \quad u = u_1 + iu_2 \sim \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3$$

- Hamiltonian structure:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} (1 - |u|^2)^2 dx, \quad J = -i \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Phase 'space'/energy space

$$X_0 = \{u \in H^1_{loc}(\mathbb{R}^3) : E(u) < \infty, \quad \text{i.e. } \nabla u,\ 1 - |u|^2 \in L^2(\mathbb{R}^3)\}.$$

* X_0 *not* a vector space, tangent space $T_u X_0 \approx X_1 \triangleq H^1 \times \dot{H}^1$ for 'nice' u

* Global existence: Gerard 06.

Traveling waves of GP

- (Formal) conservation of the momentum

$$P(u) = \frac{1}{2}\int_{\mathbb{R}^3} \langle i\nabla u, u-1\rangle dx = (P_1, P_2, P_3)(u)$$

* Formal: P is not well-defined on energy space X_0.

- Relative equilibrium: traveling waves (TW)

$$u(t,x) = U_c(x - ct\vec{e_1}), \quad c \in \mathbb{R}$$

*TWs $\longleftrightarrow$ critical pts of $H(u) = E(u) - cP_1(u)$

- Existence of traveling waves $u(t,x) = U_c(x - ct\vec{e_1})$, $U_c = u_c + iv_c$:
  - Formal: Jones, Putterman, Roberts 80s, $c \in (0, \sqrt{2})$
  - Maris 13: rigorous existence, $c \in (0, \sqrt{2})$

Stability

- Traveling wave manifold $M = \{U_c(\cdot_y) \mid y \in \mathbb{R}^3\}$
- For stability of M: linearizing (GP) at U_c in moving frame:

$$u_t = JL_c u, \qquad u = (u_1, u_2)^T \in X_1 = T_{U_c}X_0 = H^1 \times \dot{H}^1 \qquad \text{(LGP)}$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad L_c = \begin{pmatrix} -\Delta - 1 + 3u_c^2 + v_c^2 & -c\partial_{x_1} + 2u_c v_c \\ c\partial_{x_1} + 2u_c v_c & -\Delta - 1 + u_c^2 + 3v_c^2 \end{pmatrix}$$

* $L \in \mathcal{L}(X_1, X_1^*)$, $L^* = L$, $n^-(L) = 1$ (due to the constrained variation)

- $J^* = -J$, but $J^{-1} = -J : X_1 \to X_1^*$ is unbounded since $\dot{H}^1 \subsetneq H^{-1}$.

General linear Hamiltonian PDEs

- Consider:

$$u_t = JLu, \quad u \in X \tag{LH}$$

* X: real Hilbert space

* $L : X \to X^*$, bounded, $R(L) \subset X^*$ closed, $L^* = L$, i.e. $\langle Lu, v\rangle = \langle Lv, u\rangle$

* $J : X^* \to X$, anti-self-dual, i.e. $J^* = -J$

Main assumption: Morse index $n^-(L) < \infty$

- **RK1**: J may not be invertible/Fredholm

- **RK2**: Minor assumptions needed if $\dim \ker L = \infty$ or $R(L)$ not closed.

- **RK3**: non-closed $R(L)$ occurs if L does not have positive lower bound in the 'positive subspace'

Other examples: Euler in 2-d

Euler equa. on a smooth bounded $\Omega \subset R^2$ in vorticity formulation:

$$\omega_t + v \cdot \nabla \omega = 0, \quad \omega = \nabla \times v = -\Delta \psi, \text{ where } v = \nabla^{\perp} \psi \qquad \text{(E)}$$

- A *steady flow* if for some F

$$-\Delta \psi_0 = F(\psi_0) \text{ in } \Omega, \quad \psi_0 = 0 \text{ on } \partial\Omega$$

- **Assume** $F'(\psi_0) > 0$. Linearize (E) at $\omega_0 \Longrightarrow$

$$w_t = JLw, \qquad \text{(E1)}$$

$$L = \frac{1}{F'(\psi_0)} - (-\Delta)^{-1} \in \mathcal{L}(L^2, L^2), \quad L^* = L, \quad n^-(L) < \infty$$

- $J = F'(\psi_0)\, v_0 \cdot \nabla : L^2 \to L^2$, J^{-1} not bounded, $\dim \ker J = \infty$

Other examples: BBM, generalized Bullough–Dodd, ...

- Linearizating the following Ham. PDEs at traveling wave $U(x-ct) \Longrightarrow$

$$u_t = JLu, \quad L^* = L, \quad n^-(L) < \infty, \quad J^* = -J$$

$$u_t + u_x + f(u)_x - u_{txx} = 0, \quad x \in \mathbb{R}, \qquad f(0) = f'(0) = 0 \qquad \text{(BBM)}$$

$$L = -\partial_{xx} + (1 - c^{-1}) - c^{-1} f'(U) \in \mathcal{L}(H^1, H^{-1})$$

$$J = c\partial_x (1 - \partial_{xx})^{-1} : H^{-1} \to H^1, \ J^{-1} \text{ not bounded }, 0 \in \sigma_c(J)$$

$$u_{tx} = au - f(u), \qquad f(0) = f'(0) = 0, \quad a > 0, \quad x \in R \qquad \text{(gBD)}$$

$$L = -c\partial_{xx} + a - f'(U) \in \mathcal{L}(H^1, H^{-1}),$$

$$J = \partial_x^{-1} : H^{-1} \to H^1, \ J^{-1} \text{ not bounded }, 0 \in \sigma_c(J)$$

- Other examples: good Boussinesq type equa., NLS, Klein-Gordon ...

Index notations

- $S \subset X$: a subspace
- $n^{\leq 0}(L|_S)$: $\#$ of nonpositive dim of $\langle Lu, u \rangle$ restricted to S
- Subspace of generalized e-vectors

$$E_\lambda = \{u \in X \mid \exists k > 0,\ s.\ t.\ (JL - \lambda)^k u = 0\}$$

- $k_r = \sum_{\lambda \in \sigma(JL),\ \lambda > 0} \dim(E_\lambda)$
- $k_c = \sum_{\lambda \in \sigma(JL),\ \mathrm{Re}\,\lambda > 0,\ \mathrm{Im}\,\lambda > 0} \dim(E_\lambda)$
- $k_0^{\leq 0} = n^{\leq 0}(L|_{E_0 / \ker L})$
- $k_i^{\leq 0} = \sum_{i\mu \in \sigma(JL),\ \mu > 0} n^{\leq 0}(L|_{E_{i\mu}})$.

Index formula

Theorem (**Lin-Z**) Eigenvalues of JL are symmetric to both real and imaginary axis,

$$k_r + 2k_c + 2k_i^{\leq 0} + k_0^{\leq 0} = n^-(L).$$

- $\exists$ counter examples if $k^{\leq 0}$ replaced by k^-

* There may be eigenvalues embedded in continuous spectrum

* More details on JL, $\langle L\cdot,\cdot\rangle$ restricted on $E_{i\mu}$, generalized Krein signature

* *Conceptually*, $\langle L\cdot,\cdot\rangle \geq \delta > 0$ on the 'subspace' of continuous spectrum

Remarks on $k_0^{\leq 0}$

$$k_r + 2k_c + 2k_i^{\leq 0} + k_0^{\leq 0} = n^- (L) .$$

Corollary:

1. If $k_0^{\leq 0} = n^- (L)$, then (LH) is spectrally stable.
2. If $n^- (L) - k_0^{\leq 0}$ is odd, then JL has a positive e-value $\longrightarrow$ instability.

* Often symmetries contribute to ker L

$\longrightarrow$ corresponding conserved quantities help to compute $k_0^{\leq 0}$.

$\longrightarrow$ Grillakis-Shatah-Strauss (GSS) type stability criterion

- Application to (gKdV) $\longrightarrow$ recover spectral stability/instability

(GP) revisited

- Traveling wave $U_c(x - ct\vec{e_1})$ of (GP) in 3-d:

$$iu_t + \Delta u + (1 - |u|^2)u = 0, \quad u = u_1 + iu_2 \sim \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3$$

* Linearized equation $u_t = JLu$, $J = -i \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- Jones, Putterman, Roberts 80s: conjectured a (GSS) type criterion:
  - linearly stable if $\frac{dP_1(U_c)}{dc} > 0$ (lower branch)
  - linearly unstable if $\frac{dP_1(U_c)}{dc} < 0$ (upper branch)

$$P_1(u) = \frac{1}{2} \int_{\mathbb{R}^3} \langle i\partial_{x_1} u, u - 1 \rangle dx$$

* $P_1(u)$ can be extended from H^1 to the energy space X_0

- Grillakis-Shatah-Strauss 87, 91, *etc.* do *not* apply since

$$J^{-1} : X_1 = H^1 \times \dot{H}^1 \to X_1^* \text{ is unbounded}$$

Linear stability criterion of (GP)

Theorem (**Lin-Wang-Z**) Suppose $\exists$ a family of TW U_c smooth in c where $E - cP_1$ has Morse index 1 ($\Leftrightarrow n^-(L) = 1$), then

1. spectrally stable if $\frac{dP_1(U_c)}{dc} \geq 0$ ($\Leftrightarrow k_0^{\leq 0} = 1$, lower branch)
2. linearly unstable if $\frac{dP_1(U_c)}{dc} < 0$ (upper branch), under a non-degeneracy condition

$$\ker(E'' - cP_1'')(U_c) = span\{\partial_{x_1} U_c, \partial_{x_2} U_c, \partial_{x_3} U_c, \} \qquad \text{(N-deg)}$$

- TWs found by Maris 13, *etc.* has Morse index 1
- extension to other dim and general nonlinearity e.g. cubic-quintic NLS
- existence of slow traveling waves and their instabilities
- transversal instabilities of TW of 2-dim (GP)
- nonlinear orbit stability/instability

RK: Compare with (GSS) criterion

Other references on index formula

- dim-$X < \infty$: Mackay (1986) ...
- dim-$X \leq \infty$, mostly assuming J invertible and $\langle L\cdot, \cdot\rangle$ non-degenerate restricted to $(JL)^{-1}(\ker(L)) / \ker L$:
  Grillakis, Kapitula, Kevrekidis, Sandstede, Pelinovsky, Chugunova, Stefanov, Bronski, Johnson, Haragus, Pego, Kollar, Gurski, ...
- KDV type equa.: some work by Kapitula-Stefnov, Pelinovsky, (2013-2014)
- ...

RK. 1. Any anti-self-dual J allowed, even with $\dim\ker J = \infty$ or $0 \in \sigma_c(J)$.

RK. 2. some more detailed results seem (?) to be new even in the finite dimensional case.

Exponential trichotomy (ET) of e^{tJL}

Theorem (**Lin-Z**) X is decomposed into closed subspaces

$$X = E^u \oplus E^c \oplus E^s.$$

- $e^{tJL}(E^{u,s,c}) = E^{u,s,c}$, $\forall t$
- $\exists\ M > 0,\ \Lambda > 0$, such that

$$\begin{aligned} |e^{tJL}|_{E^s}| &\leq Me^{-\Lambda t}, \quad \forall\ t \geq 0, \\ |e^{tJL}|_{E^u}| &\leq Me^{\Lambda t}, \quad \forall\ t \leq 0. \end{aligned}$$

and

$$|e^{tJL}|_{E^c}| \leq M(1 + |t|^K),\ \forall\ t \in \mathbf{R}.$$

where

$$K \leq 1 + 2n^-(L)$$

RK. $E^c = \{u \mid \langle Lu, v\rangle = 0, \forall\ v \in E^s \oplus E^u\}$.

Remarks

- (ET) does not follow directly from the spectral gap of $\sigma(JL)$ even though $\sigma_{\text{ess}}(JL) \subset i\mathbf{R}$ (spectral mapping?) or resolvent estimates
- Ingredients of the proof:
    - Invariance of $\langle L\cdot, \cdot\rangle$ under e^{tJL}
    - $n^{-}(L) < \infty \longrightarrow$ Pontryagin invariant subspace theorem
    - Carefully decompose JL blockwisely.
- Exponential dichotomy (ET) on X can be extended to $D\big((JL)^k\big) \subset X$
- (ET) allows one to construct local invariant manifolds for the nonlinear problem.

- **Other results**: Upper triangular form of JL, structural stability of JL

Local invariant manifolds of semi linear equa.

- Consider

$$u_t = Au + F(u), \qquad u \in Y, \quad Y: \text{ Banach space} \qquad \text{(NL)}$$

Theorem (See e.g. Chow-Lu) Suppose

- e^{tA} has exponential dichtomy on $Y = E^+ \oplus E^-$ with $\alpha_- < \alpha_+$ s. t.

$$e^{tA}(E^\pm) \subset E^\pm, \quad \left| e^{tA}|_{E^\pm} \right| \leq Ce^{\alpha_\pm t}, \quad \mp t \geq 0$$

- $F \in C^k(Y, Y)$, $F(0) = 0$, and $F'(0) = 0$.

Then $\forall \beta_- < \beta_+$, $\beta_\pm \in (\alpha_-, \alpha_+)$, $\exists$ (possibly not uniquely) smooth local invariant manifolds $M^\pm$ s. t.

- $0 \in M^\pm$, $T_0 M^\pm = E^\pm$
- If $u(0) \in M^\pm$, then $u(t) \in M^\pm$, for t in some $(t_-, t_+) \ni 0$, and $u(t)$ can exit $M^\pm$ only through $\partial M^\pm$
- Before exiting $M^\pm$, $|u(t)| \leq Ce^{\beta_\pm t}$ for $\mp t \geq 0$.

Linearized analysis of (gKdV) at Q

- $H^1 \supset M = \{Q(\cdot + y) \mid y \in \mathbb{R}\} \sim \mathbb{R}$: traveling wave manifold
- Exp. trichotomy splitting for linearized (gKdV) at Q: $u_t = JLu$

$$H^1 = X^+ \oplus X^- \oplus X^c, \quad X^c = X^e \oplus X^T, \quad X^T = span\{\partial_x Q\} = T_Q M$$

Moreover

$$\dim X^\pm = span\{V^\pm\}; \quad JLV^\pm = \pm\lambda V^\pm,\ \lambda > 0, \quad \mathcal{L}_{X^e} \geq \delta > 0$$

* Translation invariance $\longrightarrow$ exp. trichotomy splitting at $U_c(\cdot + y)$:

$$H^1 = X_y^+ \oplus X_y^- \oplus X_y^c, \quad X_y^c = X_y^e \oplus X_y^T,$$

$$X_y^{\pm,c,T,e} = \{u(\cdot + y) \mid u \in X^{\pm,c,T,e}\}, \quad y \in \mathbb{R}$$

associated with projections $\Pi_y^{\pm,e,T}$.

Local dynamics near traveling wave Q of (gKdV)

Theorem

*(**Jin-Lin-Z**) $\exists!$ smooth locally invariant (under (gKdV)) manifolds $W^u, W^s, W^c, W^{cs}, W^{cu} \supset M$, s. t. at $\forall Q(\cdot + y) \in M$,*

$$T_{Q(\cdot+y)} W^u = X_y^+ \oplus T_{Q(\cdot+y)} M, \quad T_{Q(\cdot+y)} W^s = X_y^- \oplus T_{Q(\cdot+y)} M$$

$$T_{Q(\cdot+y)} W^{cu} = X_y^+ \oplus X_y^c, \quad T_{Q(\cdot+y)} W^{cs} = X_y^- \oplus X_y^c, \quad T_{Q(\cdot+y)} W^c = X_y^c$$

$$M = W^u \cap W^s, \quad W^u \subset W^{cu}, \quad W^s \subset W^{cs}, \quad W^c = W^{cu} \cap W^{cs}$$

- *$W^{u,s,c,cu,cs}$ are invariant under x-translation and rescaling, $W^{u,s} \subset C^\infty$*
- *M is orbitally stable on W^c*

- As usual, invariant manifolds ⟶ organized local dynamics near M

* Local invariance: orbits starting on $W^{u,s,cs,cu,c}$ can leave them only through their boundaries

Construction: stable/unstable manifolds

- First, stable/unstable manifolds of Q, i.e. for $y = 0$.

$$H^1 = X^+ \oplus X^- \oplus X^c \longrightarrow V = a^+ V^+ + a^- V^- + v^c, \; v^c \in X^c = X^T \oplus X^e$$

Rewrite (gKdV) in terms of $(a^\pm, V^c)$:

$$\begin{cases} \partial_t a^\pm = \pm\lambda a^\pm + F^\pm(a^\pm, v^c) \\ \partial_t v^c = JL_0 v^e + \partial_x F^c(a^\pm, v^c) + F_1^c(a^\pm, v^c) \end{cases}$$

* $F^\pm(a^\pm, v^c) \in \mathbb{R}$: quadratic terms
* $F^c(a^\pm, v^c), F_1^c(a^\pm, v^c) \in H^1$: quadratic terms with nice spatial decay

- Lyapunov-Perron approach for W^s (W^u similar)

$$\begin{cases} a^-(t) = e^{-\lambda t} a^-(0) + \int_0^t e^{-\lambda(t-s)} F^-(a^\pm, v^c)(s) ds \\ a^+(t) = \int_t^{+\infty} e^{\lambda(t-s)} F^+(a^\pm, v^c)(s) ds \\ v^c(t) = \int_t^{+\infty} e^{(t-s)JL_0} \big(\partial_x F^c(a^\pm, v^c)(s) + F_1^c(a^\pm, v^c)(s)\big) ds \end{cases}$$

- $\partial_x F^c \in L^2$ loses regularity

Smoothing estimates (Kenig&Ponce&Vega)

Lemma

Let $W(t)$ be the solu group of $u_t + u_{xxx} = 0$.

$$|\partial_x W(t)u_0|_{L_x^\infty L_t^2} + |\partial_x^{1/4} W(t)u_0|_{L_t^4 L_x^\infty} \leq C|u_0|_{L^2}.$$

$$|W(t)u_0|_{L_x^2 L_{[0,T]}^\infty} \leq C_{s,\rho}(1+T)^\rho |u_0|_{H^s}, \quad s > 3/4, \ \rho > 3/4, \ T \leq \infty$$

$$|\partial_x \int_0^t W(t-s)g(s)ds|_{L_{[0,T]}^\infty L_x^2} \leq C|g|_{L_x^1 L_{[0,T]}^2}, \quad T \leq \infty$$

$$|\partial_{xx} \int_0^t W(t-s)g(s)ds|_{L_x^\infty L_{[0,T]}^2} \leq C|g|_{L_x^1 L_{[0,T]}^2}, \quad T \leq \infty$$

- Smoothing estimates + decay of $\partial_x F^c$ in x + Lyapunov-Perron framework $\Longrightarrow$ stable/unstable mani. $W_0^{u,s}$ of Q;
- Stable/unstable mani. $W_y^{u,s}$ of $Q(\cdot + y)$ via translation;
- $W^{u,s} = \cup_{y \in \mathbb{R}} W_y^{u,s}$.
- Construction can be done in $H^k \Longrightarrow W^{u,s} \subset H^k$ due to uniqueness

Center manifold: global construction

- One can construct local invariant mani $W_y^{cs,cu,c}$ of $Q(\cdot + y)$ similarly
- Lack of uniqueness of $W_y^{cs,cu,c} \longrightarrow$ the local invariant mani. $W^{cu,cs,c}$ of the whole M can not be obtained by patching $W_y^{cs,cu,c} \longrightarrow$
- $W^{cs,cu,c}$ should be constructed near but globally along M

• Recall the natural local coord. near M:

$$V = \Phi(y, a^\pm, v^e) = (Q + a^\pm V^\pm + v^e)(\cdot + y), \quad y, a^\pm \in \mathbb{R}, \ v^e \in X^e$$

(gKdV-M) $\Longrightarrow$

$$\begin{cases} \partial_t y = A_{Te} V^e + \widetilde{F}^T(y, a^\pm, v^e) \\ \partial_t a^\pm = \pm\lambda a^\pm + \widetilde{F}^\pm(y, a^\pm, v^e) \\ \partial_t v^e + \partial_t y \Pi_y^e \partial_x v^e = A_e v^e + \partial_x \widetilde{F}^e(y, a^\pm, v^e) + \widetilde{F}_1^e(y, a^\pm, v^e) \end{cases}$$

• $\partial_x v^e$ loses regularity and does not have enough decay in x to be handled by smoothing estimates

Center manifold: a bundle coordinates near M

- Revisit $\partial_t y \Pi_y^e \partial_x v^e$:

$$\Phi(y, a^{\pm}, v^e) = (Q + a^{\pm} V^{\pm} + v^e)(\cdot + y) : \mathbb{R}^3 \times X^e \to H^1$$

is homeomorphic, but not smooth in y:

$$\partial_y \Phi(y, a^{\pm}, v^e) = (\partial_x Q + a^{\pm} \partial_x v^{\pm} + \partial_x v^e)(\cdot + y)$$

$$L^2 \ni \partial_x v^e \notin H^1$$

- Resolution for (NKG) by Nakanishi&Schlag 12: nonlinear 'quasi-distance'

- Our approach: a bundle coordinate system (Bates&Lu&Z, Jin&Lin&Z)

$$\Psi(y, a^{\pm}, Z^e) = (Q + a^{\pm} V^{\pm})(\cdot + y) + Z^e,$$

  - $Z^e \in X_y^e = \{v \in H^1 \mid v(\cdot - y) \in X^e\}$, but do not parametrize Z^e by $v^e = Z^e(\cdot - t) \in X^e$

Construction of the center manifold

- Recall $\Pi_y^{T,\pm,e}$ are smooth in $y \Longrightarrow$

$$\tilde{X}^e = \{(y, v) \mid v \in X_y^e\} \text{ is a smooth bundle over } M \sim \mathbb{R}$$

and

$$\Psi : \tilde{X}^e \times \mathbb{R}^2 \to H^1 \text{ is smooth!}$$

- Smoothing estimates (with weak exp. growth) + Lyapunov-Perron framework $\longrightarrow$ $W^{cu,cs,c}$ of $\mathcal{M}$
- $\langle L_y Z^e, Z^e \rangle \geq a|Z^e|_{H^1}^2$ + conservation of $H = (E + P) \Longrightarrow$ orbitally stability of M inside W^c

Local dynamics of (GP)

$$iu_t + \Delta u + (1 - |u|^2)u = 0, \quad x \in \mathbb{R}^3, \quad u(t, \infty) = 1 \qquad \text{(GP)}$$

- (**Jin-Lin-Z**) Invariant manifolds of traveling wave manifold M

* Energy X_0 space non-flat: use a coordinate system due to P. Gerard

* Local bundle coordinates near M used to avoid loss of regularity

* Stritchartz space-time $L^p_{t,loc} L^q_x$ estimates (with weak exp. growth)

RK: Regularity issue occurs due to spatial translation (also Lorenz, Galileo, *etc.*), but not phase symmetry

- The above approach applicable to a large class of Hamiltonian PDEs
- As usual, invariant manifolds $\longrightarrow$ organized local dynamics near M

Local dynamics and invariant manifolds of traveling wave manifolds of Hamiltonian PDEs - Talk by Chongchun Zeng

Blackboard + Lecture Notes (Ori S. Katz)

October 15, 2018

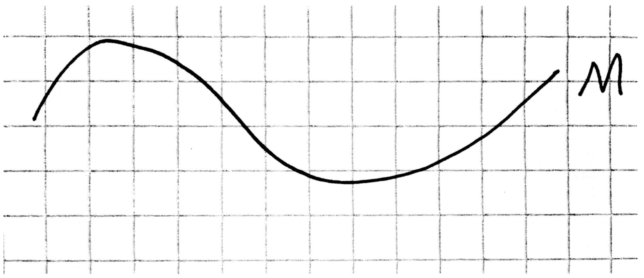
Abstract

Some Hamiltonian PDEs which are invariant under spatial translations possess traveling wave solutions which form finite dimensional invariant manifolds parametrized by their spatial locations. Extensive studies have been carried out for their stability analysis. In this talks we shall focus on local dynamics and invariant manifolds of the traveling wave manifolds for the Gross-Pitaevskii equation in R^3 and the gKdV equation as our main PDE models, while our approach works for a general class of problems. Noting that the symplectic operators of some of these models happen to be unbounded in the energy space, violating a commonly assumed assumption for the study of the linearized systems at these traveling waves, we could carry out linearized analysis in a general framework we developed recently. Nonlinearly our main results are the existence of local invariant manifolds of unstable traveling waving manifolds and the implications on the local dynamics. In addition to applying certain space-time estimates, we use a bundle coordinate system to handle an issue of a seemingly regularity loss caused by the spatial translation parametrization.

1 Blackboard + lecture notes

general KdV equation - Hamiltonian structure

Question - we want to study stability of traveling waves - can be described as non-isolated steady states in the co-moving space. Obtain a curve of traveling wave states. What about the stability?



The problem with stability analysis is that the inverse of J is not bounded on the energy space. J^{-1} corresponds to the symplectic form, so this is not an unreasonable assumption.

Another example: Gross-Pitaevskii (GP) equation on $\mathbb{R}^3$. The phase space is not a linear space, the constraint is not linear. Gerard showed the problem is nevertheless well-posed.

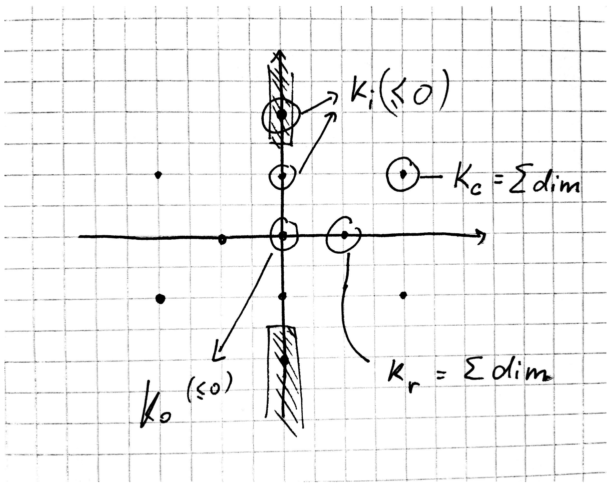
Traveling waves of GP: traveling waves must have velocity $c \in \left(0, \sqrt{2}\right)$. Maris '13 - proved rigorous existence.

Stability: We are talking about the 3D manifold of traveling waves M, not a single traveling wave. Because of the translational invariance, we know L have 3 kernel directions, there may be more.

General linear Hamiltonian PDEs: We consider the case of the linear Hamiltonian being a symmetric quadratic form. The main assumption, $n^-(L) < \infty$, is often true but not always.

What is the general framework that covers these problems?

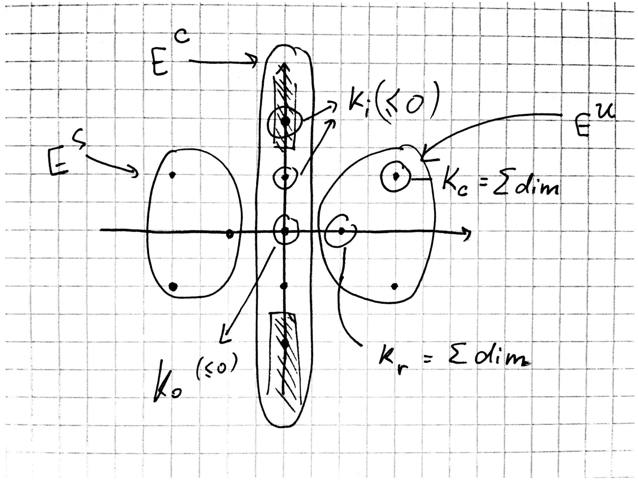
To present the main results, we need to introduce some notations.



Can we get information about this distribution from the energy functional L? We need to do some counting. Count the total dimension of all the eigenvalues in the first point, and make sure it is finite, and call it k_c. Count all the dimensions of the point on the real axis k_r. Count the non-positive eigenvalues on the imaginary axis $k \leq 0$. How to define the eigenspace on the embedded (shaded) part of the imaginary axis? Finally, there is the generalized kernel (at the origin) - k_0, and count the total non-positive dimensions of the energy functional. Because of symmetry, only need to calculate for 4 points and multiply by their multiplicity. The sum is the Morse index $n^-(L)$.

Because of the multiplicity, $n^- - k_0^{\leq 0}$ odd or even signifies existence of k_r, signifying stability/non-stability.

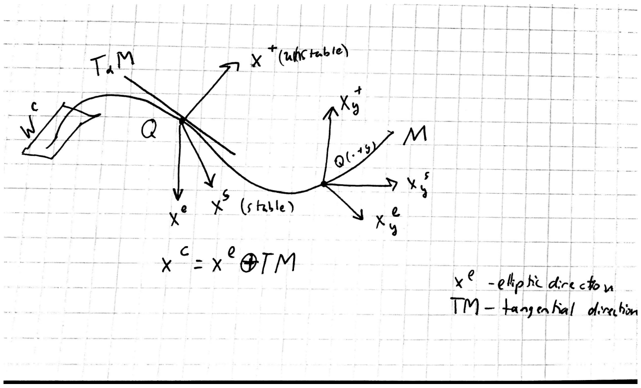
Exponential trichotomy (ET) of e^{tJL}: Is the system really stable on the eigenspace on the imaginary axis? Can obtain the stable subspace from the spectral theory E^S, the unstable subspace E^U and the center subspace E^C.



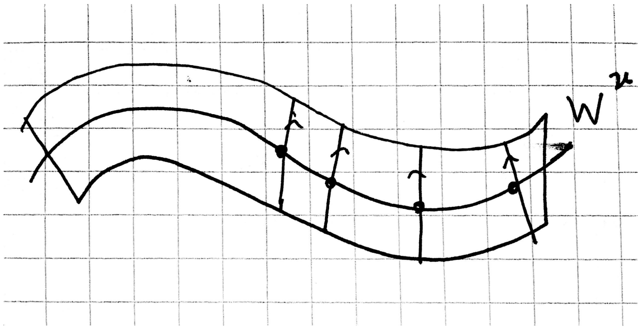
The stable and unstable subspaces are finite-dimensional invariant subspaces under the linear solution flow, so the space can be reduced by them. They are isotropic subspaces.

However, the central direction is more complicated. On the center subspace, the linear subspace has no growth. This is the best upper bound one can get.

Linearized analysis of (gKdV) at Q:



Expect the dynamics around the W^c manifold to be stable.
(Near a saddle - there is a "stable manifold on ice".)
Near the stable and unstable manifold there is a foliation:



Center manifold: a bundle coordinates near M: The problem is that in these very natural coordinates, the transformation is a local homeomorphism, not a local diffeomorphism.