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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

# NOTETAKER CHECKLIST FORM

**(Complete one for each talk.)**

**Name:** ORI KATZ **Email/Phone:** ORIKATZ.OK@gmail.com

**Speaker's Name:** Massimiliano Berti

**Talk Title:** Dynamics of Water Waves

**Date:** 10 / 10 / 18 **Time:** 10 : 30 (am) / pm (circle one)

**Please summarize the lecture in 5 or fewer sentences:** Recent results about the complex dynamics of the water waves equations of 2D fluid with gravity & capillary forces, with space-periodic boundary conditions, are presented. Berti discussed both long-time existence results & bifurcation of small-amplitude, time quasi-periodic solutions.

# CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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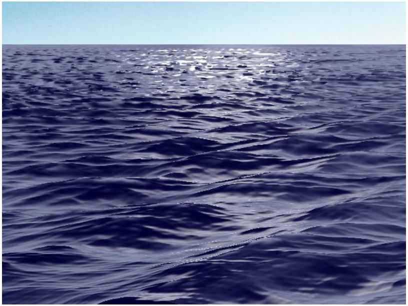
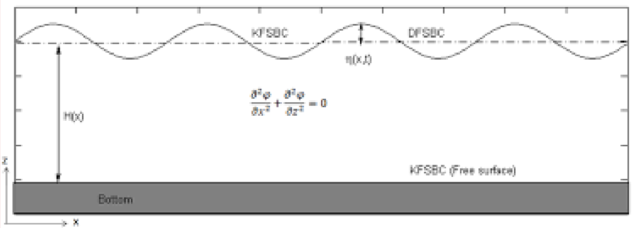
# Long time dynamics of Water Waves

**Massimiliano Berti**, SISSA,
*MSRI, Berkeley*, 10 October 2018



## Time evolution of space periodic water waves in Trieste gulf:




In section it is described by a bidimensional fluid, periodic in $x$

**Water Waves:** Euler equations for an irrotational, incompressible fluid in $S_\eta(t) = \{-h < y < \eta(t,x)\}$ under gravity and capillarity

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + g\eta = \kappa \partial_x \left( \frac{\eta_x}{\sqrt{1+\eta_x^2}} \right) & \text{at } y = \eta(t,x) \\ \Delta \Phi = 0 & \text{in } -h < y < \eta(t,x) \\ \partial_y \Phi = 0 & \text{at } y = -h \\ \partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{at } \ y = \eta(t,x) \end{cases}$$

$$u = \nabla \Phi = \text{velocity field}, \ \ \mathrm{rot}\, u = 0 \text{ (irrotational)},$$
$$\mathrm{div}\, u = \Delta \Phi = 0 \text{ (uncompressible)}$$
$$g = \text{gravity}, \ \kappa = \text{surface tension coefficient}$$
$$\text{Mean curvature} = \partial_x \left( \frac{\eta_x}{\sqrt{1+\eta_x^2}} \right)$$

**Unknowns:**

free surface $y = \eta(t,x)$ and the velocity potential $\Phi(t,x,y)$

# Zakharov formulation '68

Infinite dimensional Hamiltonian system:

$$\partial_t u = J \nabla_u H(u), \quad u := \begin{pmatrix} \eta \\ \psi \end{pmatrix}, \quad J := \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix},$$

**canonical Darboux coordinates:**

$\eta(x)$ and $\psi(x) = \Phi(x, \eta(x))$ trace of velocity potential at $y = \eta(x)$

$(\eta, \psi)$ uniquely determines $\Phi$ in the whole $\{-h < y < \eta(x)\}$ solving the elliptic problem:

**$\Phi$ is harmonic**

$\Delta \Phi = 0 \quad$ in $\{-h < y < \eta(x)\}$, $\quad \Phi|_{y=\eta} = \psi$, $\partial_y \Phi = 0$ at $y = -h$

### Hamiltonian: total energy on $S_\eta = \mathbb{T} \times \{-h < y < \eta(x)\}$

$$H := \frac{1}{2} \int_{S_\eta} |\nabla \Phi|^2 dxdy + \int_{S_\eta} gy \, dxdy + \kappa \int_{\mathbb{T}} \sqrt{1 + \eta_x^2} \, dx$$

*kinetic energy* + *potential energy* + *capillary energy*

### Hamiltonian expressed in terms of $(\eta, \psi)$

$$H(\eta, \psi) = \frac{1}{2} \int_{\mathbb{T}} \psi(x) \, G(\eta)\psi(x) \, dx + \frac{1}{2} \int_{\mathbb{T}} g\eta^2 \, dx + \kappa \int_{\mathbb{T}} \sqrt{1 + \eta_x^2} \, dx$$

### Dirichlet–Neumann operator (Craig-Sulem '93)

$$G(\eta)\psi(x) := \sqrt{1 + \eta_x^2} \, \partial_n \Phi|_{y=\eta(x)} = (\Phi_y - \eta_x \Phi_x)(x, \eta(x))$$

## Zakharov-Craig-Sulem formulation

$$\begin{cases}\partial_t \eta = G(\eta)\psi = \nabla_\psi^{L^2} H(\eta,\psi) \\ \partial_t \psi = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(G(\eta)\psi + \eta_x\psi_x)^2}{2(1+\eta_x^2)} + \dfrac{\kappa\eta_{xx}}{(1+\eta_x^2)^{3/2}} = -\nabla_\eta^{L^2} H(\eta,\psi)\end{cases}$$

## Dirichlet–Neumann operator

$$G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n \Phi|_{y=\eta(x)}$$

1. $G(\eta)$ is linear in $\psi$, non-local,
2. self-adjoint with respect to $L^2(\mathbb{T}_x)$
3. $G(\eta) \geq 0$, $G(1) = 0$
4. $\eta \mapsto G(\eta)$ nonlinear, smooth,
5. $G(\eta)$ is pseudo-differential, $G(\eta) = D_x \tanh(hD_x) + OPS^{-\infty}$

Calderon, Craig, Lannes, Metivier, Alazard, Burq, Zuily, Delort...

# Symmetries

## Reversibility

$$H(\eta, -\psi) = H(\eta, \psi)$$

## Involution

$$H \circ S = H, \quad S : (\eta, \psi) \to (\eta, -\psi), \quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad S^2 = \mathrm{Id},$$

## Reversible vector field $X_H = J \nabla H$

$$X_H \circ S = -S \circ X_H \quad \Longleftrightarrow \quad \Phi_H^t \circ S = S \circ \Phi_H^{-t}$$

Equivariance under the $\mathbb{Z}/(2\mathbb{Z})$-action of the group $\{\mathrm{Id}, S\}$

# $x$-invariance

**Momentum is a prime integral**

$$M(\eta,\psi) = \int_{\mathbb{T}} \eta_x(x)\,\psi(x)\,dx$$

Noether theorem:

**Associated Hamiltonian vector field generates the translations**

$$J\nabla M = \partial_x \begin{pmatrix} \eta \\ \psi \end{pmatrix}$$

$$\theta \mapsto (\eta(x+\theta), \psi(x+\theta))$$

# Standing Waves

**Invariant subspace: functions even in $x$**

$$\eta(-x) = \eta(x)\,, \quad \psi(-x) = \psi(x)$$

Thus the velocity potential

$$\Phi(-x, y) = \Phi(x, y) \quad \Longrightarrow \quad \Phi_x(0, y) = 0$$

and, using also $2\pi$ periodicity,

$$-\Phi_x(\pi, y) = \Phi_x(-\pi, y) = \Phi_x(\pi, y) \quad \Longrightarrow \quad \Phi_x(\pi, y) = 0$$

$\Longrightarrow$ no flux of fluid outside the walls $\{x = 0\}$ and $\{x = \pi\}$.

**Neumann boundary conditions at $x = 0$ and $x = \pi$**

$$\eta_x(0) = \eta_x(\pi) = 0\,, \quad \psi_x(0) = \psi_x(\pi) = 0$$

## Prime integral: mass

$$\int_{\mathbb{T}} \eta(x)dx$$

## Phase space

$$\eta \in H_0^s(\mathbb{T}) := \{\eta \in H^s(\mathbb{T}) \, : \, \int_{\mathbb{T}} \eta(x)dx = 0\}$$

$$u \in H^s(\mathbb{T}) \; \Leftrightarrow \; u(x) = \sum_{k\in\mathbb{Z}} u_k e^{\mathrm{i}kx}\,, \quad \sum_{k\in\mathbb{Z}} |u_k|^2 \langle k\rangle^{2s} =: \|u\|^2_{H^s} < +\infty$$

The variable $\psi$ is defined modulo constants: only the velocity field $\nabla_{x,y}\Phi$ has physical meaning.

$$\psi \in \dot{H}^s(\mathbb{T}) = H^s(\mathbb{T})/\sim$$

$$u(x) \sim v(x) \quad \Longleftrightarrow \quad u(x) - v(x) = c$$

# Linear water waves theory

**Linearized system at $(\eta, \psi) = (0, 0)$**

$$\begin{cases} \partial_t \eta = G(0)\psi, \\ \partial_t \psi = -g\eta + \kappa \eta_{xx} \end{cases}$$

Dirichlet-Neumann operator at the flat surface $\eta = 0$ is

$$G(0) = D \tanh(hD)\,, \quad D = \frac{\partial_x}{\mathrm{i}} = \mathrm{Op}(\xi)_{\xi \in \mathbb{R}}$$

**Fourier multiplier notation: given $m : \mathbb{Z} \to \mathbb{C}$**

$$m(D)h = \mathrm{Op}(m)h = \sum_{j \in \mathbb{Z}} m(j) h_j e^{\mathrm{i}jx}\,, \quad h(x) = \sum_{j \in \mathbb{Z}} h_j e^{\mathrm{i}jx}$$

## Linear water waves system

$$\partial_t \begin{bmatrix} \eta \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & G(0) \\ -g+\kappa\partial_{xx} & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \psi \end{bmatrix}$$

## Complex variable

$$u = \Lambda(D)\eta + \mathrm{i}\Lambda^{-1}(D)\psi\,, \quad \Lambda(D) = \left(\frac{g+\kappa D^2}{D\tanh(hD)}\right)^{1/4}$$

## Linear Water Waves

$$u_t + i\omega(D)u = 0, \quad \omega(D) = \sqrt{D\tanh(hD)(g+\kappa D^2)}$$

## Dispersion relation

$$\omega(\xi) = \sqrt{\xi\tanh(h\xi)(g+\kappa\xi^2)}$$

## $\infty$-decoupled harmonic oscillators

$$u(t,x) = \sum_{j\in\mathbb{Z}} e^{-\mathrm{i}\omega(j)t} u_j(0) e^{\mathrm{i}jx}$$

## Linear frequencies of oscillations

$$\omega(j) = \sqrt{j\tanh(hj)(g+\kappa j^2)}\,, \quad j \in \mathbb{Z}\,,$$

All solutions are periodic, quasi-periodic, almost periodic in time according to the irrationality properties of $(\omega_j(h,g,\kappa))_{j\in\mathbb{Z}}$

## The Sobolev norm is constant

$$\|u(t,\cdot)\|_{H^s} = \|u(0,\cdot)\|_{H^s}$$

## Dispersion relation

$$\omega(\xi) = \sqrt{\xi \tanh(h\xi)(g + \kappa\xi^2)}$$

1. Gravity-Capillary water waves

$$\omega(\xi) = \sqrt{\xi \tanh(h\xi)(g + \kappa\xi^2)} \sim \sqrt{\kappa}|\xi|^{\frac{3}{2}} \quad \text{as} \quad |\xi| \to +\infty$$

2. Gravity water waves

$$\omega(\xi) = \sqrt{\xi \tanh(h\xi) g} \sim \sqrt{g}|\xi|^{\frac{1}{2}} \quad \text{as} \quad |\xi| \to +\infty$$

**Remark:** $x \in \mathbb{T}$ and $u(x)$ has zero average $\implies |\xi| \geq 1$

# Nonlinear water waves

## Main questions

1. For which time interval $(-T_{\max}, T_{\max})$ solutions of the nonlinear gravity-capillary water waves equations exist?

2. Are there periodic, quasi-periodic, almost periodic solutions (thus global in time) of the nonlinear gravity-capillary water waves equations?

# Major difficulties:

## Gravity-Capillary WW are quasi-linear PDEs

$$u_t + \mathrm{i}\omega(D)u = N(u, \bar{u}), \quad \omega(D) \sim |D|^{3/2}$$

$N$ = quadratic nonlinearity with derivatives of order $N(|D|^{3/2}u)$

## Gravity WW are fully nonlinear PDEs

$$u_t + \mathrm{i}\omega(D)u = N(u, \bar{u}), \quad \omega(D) \sim |D|^{1/2}$$

$N$ = quadratic nonlinearity with derivatives of order $N(\partial_x u)$

**Singular perturbation of the linear vector field** $\mathrm{i}\omega(D)u$

## Periodic boundary conditions $x \in \mathbb{T}$

*NO dispersive* effects of the linear PDE as for $x \in \mathbb{R}^2$, $x \in \mathbb{R}$ and data decaying at infinity:

**Global well-posedness:** S.Wu, Germain-Masmoudi-Shatah, Ionescu-Pusateri, Alazard-Delort, Ifrim-Tataru, Alazard-Burq-Zuily, . . .

Not available conserved quantities controlling high Sobolev norms

# Nonlinear water waves, main results:

1. **Long time existence Birkhoff normal form result:**
   - Gravity-capillary: M. Berti- J-M. Delort, '17,
     For most $(g, \kappa)$, for *any* small initial condition of size $\varepsilon$ the solutions are defined for long times $T_\varepsilon \geq O(\varepsilon^{-N})$
   - Gravity: M. Berti, R. Feola, F. Pusateri, '18,
     If $\kappa = 0$, $h = +\infty$ then $T_\varepsilon \geq O(\varepsilon^{-3})$

2. **KAM results:** Existence of quasi-periodic solutions for
   - Gravity-capillary: Berti-Montalto, '16,
   - Gravity: Baldi-Berti-Haus-Montalto, '17,

solutions defined for all times, for "*most*" initial conditions

# Almost global existence

**Theorem (M.B., J-M.Delort, 2017)**

*There is a zero measure subset* $\mathcal{N}$ *in* $]0,+\infty[^2$ *such that, for any* $(g,\kappa)$ *in* $]0,+\infty[^2 \backslash \mathcal{N}$*, for any* $N$ *in* $\mathbb{N}$*, there is* $s_0 > 0$ *and, for any* $s \geq s_0$*, there are* $\varepsilon_0 > 0$*,* $c > 0$*,* $C > 0$ *such that, for any* $\varepsilon \in ]0, \varepsilon_0[$*, any even function* $(\eta_0, \psi_0)$ *in* $H_0^{s+\frac{1}{4}}(\mathbb{T},\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T},\mathbb{R})$ *with*

$$\|\eta_0\|_{H_0^{s+\frac{1}{4}}} + \|\psi_0\|_{\dot{H}^{s-\frac{1}{4}}} < \varepsilon$$

*the gravity-capillary water waves equations have a unique classical solution, even in space,*

$$(\eta,\psi) \in C^0(]-T_\varepsilon, T_\varepsilon[, H_0^{s+\frac{1}{4}}(\mathbb{T},\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T},\mathbb{R}))$$

*with*

$$T_\varepsilon \geq c\varepsilon^{-N}$$

*satisfying the initial condition* $\eta|_{t=0} = \eta_0$*,* $\psi|_{t=0} = \psi_0$

# Remark 1) Time of existence

1. $N = 1$, time of existence $T_\varepsilon = O(\varepsilon^{-1})$, local existence theory, Beyer-Gunther, Coutand-Shkroller, Alazard-Burq-Zuily
2. $N = 2$, time of existence $T_\varepsilon = O(\varepsilon^{-2})$, S. Wu, Ifrim-Tataru, if $h = +\infty$ there are no "triple wave interactions" + quasi-linear modified energy

**No solutions $k_1, k_2, k_3 \in \mathbb{Z} \setminus 0$ of**

$$\begin{cases} |k_1|^{\frac{1}{2}} \pm |k_2|^{\frac{1}{2}} \pm |k_3|^{\frac{1}{2}} = 0 \\ k_1 \pm k_2 \pm k_3 = 0 \end{cases}$$

3. For $N \geq 2$, to get time of existence $T_\varepsilon = O(\varepsilon^{-N})$, we erase parameters $(g, \kappa)$ to avoid multiple wave interactions

*Ionescu-Pusateri:* $x \in \mathbb{T}^2$, $T_\varepsilon = O(\varepsilon^{-\frac{5}{3}})$ *for most values of* $(g, \kappa)$

$N = 3$, Berti, Feola, Pusateri, '18, $x \in \mathbb{T}$,
Gravity waves ($\kappa = 0$) with infinite depth $h = +\infty$: $T_\varepsilon = O(\varepsilon^{-3})$,

- There are nontrivial 4-order wave interactions (Benjamin-Fair)
- Nevertheless Zakharov-Dyanchenko, Craig-Workfolk proved that the FORMAL 4-th order Birkhoff normal form Hamiltonian $\mathrm{i}\partial_{\bar z} H^{(4)}_{BNF}$ is *integrable* and well posed on $H^s$ ("null condition")
- Using bounded changes of variables:

**Poincaré-Birkhoff Normal Form for pure gravity WW**

$$\partial_t z = \mathrm{i}\partial_{\bar z} H^{(4)}_{BNF} + \mathcal{X}_{\geq 4}(z)$$

where $\mathcal{X}_{\geq 4}(z)$ admits energy estimates in $H^s$

$\Longrightarrow$ WE JUSTIFY USE OF THESE FORMAL NORMAL FORM EXPANSIONS USED SUCCESSFULLY BY PHYSICISTS!
*In same spirit that Lindsted formal series in celestial mechanics were rigorously justified a-posteriori by KAM theorem (Moser)*

# Remark 2) Parameters

1. **Internal parameters: fixed equation!** We can also think to fix $(\kappa, g, h)$ and the result holds for most space "wave-length": periodic boundary conditions $x \in \lambda\mathbb{T}$, $\lambda \in \mathbb{R}$.
   The linear frequencies depend non-trivially w.r.t $\lambda$
   $$\omega_j = \sqrt{\lambda j \tanh(h\lambda j)(g + \kappa\lambda^2 j^2)}$$
2. We fixed $h = 1$. We can not use $h$ as a parameter:
   $$h \to \omega_j(h) = \sqrt{j \tanh(hj)(g + \kappa j^2)}$$
   is **not** sub-analytic. The parameter $h$ moves just exponentially the frequencies:
   $$\omega(j) = \sqrt{j \tanh(hj)} = \sqrt{|j|}(1 + O(e^{-hj}))$$

# Remark 3) Reversible and Hamiltonian structure

Algebraic property to exclude "growth of Sobolev norms"

1. Hamiltonian
2. Reversibility (Poincaré, Moser)

Dynamical systems heuristic explanation:

### Water waves

$$u_t = \mathrm{i}\omega(D)u + N_2(u,\bar{u})\,, \quad N_2(u,\bar{u}) = O(u^2)$$

### Fourier and Action-Angle variables $(\theta, I)$

$$u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}\,, \quad u_j = \sqrt{I_j}e^{\mathrm{i}\theta_j}$$

$$\text{Sobolev norm } \|u\|_{H^s}^2 = \sum_{j\in\mathbb{Z}}(1+j^2)^s I_j$$

### Small amplitude solutions

Rescaling $u \mapsto \varepsilon u$

$$u_t = \mathrm{i}\omega(D)u + \varepsilon O(u^2)$$

in action-angle variables reads

$$\frac{d}{dt} I_j = \varepsilon f_j(\varepsilon,\theta,I)\,, \quad \frac{d}{dt}\theta_j = \omega(j) + \varepsilon g_j(\varepsilon,\theta,I)$$

angles $\theta_j = \omega(j)t$ “rotate fast", actions $I_j(t)$ “slow" variables

## "Averaging principle":

The effective dynamics of the actions is expected to be governed by

$$\frac{d}{dt} I_j = \varepsilon \langle f_j \rangle(\varepsilon, I) \,, \quad \langle f_j \rangle(\varepsilon, I) := \int_{\mathbb{T}^\infty} f_j(\varepsilon, \theta, I) d\theta$$

## Necessary condition for QP solutions and long time existence

$$\langle f_j \rangle(I) = 0$$

## Hamiltonian case: $f(\theta, I) = (\partial_\theta H)(\theta, I)$

$$\Longrightarrow \quad \int_{\mathbb{T}^\infty} (\partial_\theta H)(\theta, I) d\theta = 0$$

## Reversible vector field (Moser)

$$\frac{d}{dt}\theta = g(I, \theta) \,, \ \frac{d}{dt} I = f(I, \theta) \,, \quad f(I, \theta) \, \mathrm{odd\ in} \, \theta, \ g(I, \theta) \, \mathrm{even\ \ in} \, \theta$$

$$\Longrightarrow \quad \int_{\mathbb{T}^\infty} f(\theta, I) d\theta = 0$$

The water waves equations (written in complex variables) are reversible with respect to the involution

$$S : u(x) \mapsto \bar{u}(x)$$

that on the subspace

$$u(-x) = u(x)\,, \quad u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx} = \sum_{j\in\mathbb{Z}} \sqrt{I_j} e^{\mathrm{i}\theta_j} e^{\mathrm{i}jx}\,,$$

is

**Moser reversibility**

$$(\theta, I) \mapsto (-\theta, I)$$

**Alinhac “good unknown" which has to be introduced to get energy estimates (local existence theory) preserves the reversible structure, not the Hamiltonian one**

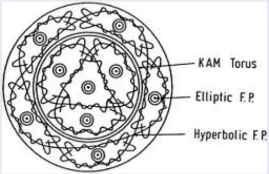
# Remark 4) Global existence?

Question: **Do these solutions exist for all times?**

Probably not

Craig-Workfolk: for $\kappa = 0$, $h = +\infty$ the water-waves PDEs are not integrable at the fifth order Birkhoff normal form

## Expected scenario for nearly-integrable Hamiltonian systems




1. **KAM results**: There are many solutions defined for all times: **selection of "initial conditions" giving rise to global solutions**
2. **Almost global existence:** $|t| \le c_N \varepsilon^{-N}$.
   Exponential estimates?
3. **Arnold diffusion:** *What happens to a solution which does not start on a KAM torus for times* $|t| > c_N \varepsilon^{-N}$*?*
   Chaos? Growth of Sobolev norms?

# Quasi-periodic solution with $n$ frequencies of $u_t = X(u)$

**Definition**

$u(t,x) = U(\omega t, x)$ where $U(\varphi, x) : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$,
$\omega \in \mathbb{R}^n$($=$ *frequency vector*) is irrational $\omega \cdot k \neq 0$, $\forall k \in \mathbb{Z}^n \setminus \{0\}$
$\Longrightarrow$ the linear flow $\{\omega t\}_{t \in \mathbb{R}}$ is DENSE on $\mathbb{T}^n$

- Global in time
- If $n = 1$ then $U(\omega t, x)$ is time-periodic with period $T = 2\pi/\omega$

**Periodic solutions:** $n = 1$

- **Plotnikov-Toland**: '01
  Gravity Water Waves with Finite depth
- **Iooss-Plotnikov-Toland** '04, **Iooss-Plotnikov** '05-'09
  Gravity Water Waves with Infinite depth
  Completely resonant, infinite dimensional bifurcation equation
- **Alazard-Baldi** '15,
  Capillary-gravity water waves with infinite depth

**Quasi-Periodic solutions:** $n \geq 2$

- **Berti-Montalto** '16,
  Gravity-Capillary Water Waves
- **Baldi-Berti-Haus-Montalto**
  Gravity Water Waves '17

**Theorem (Baldi, Berti, Haus, Montalto, Inventiones Math. 2018 )**

*For every choice of the tangential sites* $\mathbb{S} \subset \mathbb{N} \setminus \{0\}$, *there exists* $\bar{s} > \frac{|\mathbb{S}|+1}{2}$, $\varepsilon_0 \in (0,1)$ *such that: for all* $\xi_j \in (0, \varepsilon_0^2)$, $j \in \mathbb{S}$,
$\exists$ *a Cantor like set* $\mathcal{G}_\xi \subset [h_1, h_2]$ *with asymptotically full measure as* $\xi \to 0$, *i.e.* $\lim_{\xi \to 0} |\mathcal{G}_\xi| = h_2 - h_1$, *such that, for any depth* $h \in \mathcal{G}_\xi$, *the* GRAVITY WATER WAVES EQUATION *has a reversible, quasi-periodic standing wave solution* $(\eta, \psi) \in H^{\bar{s}}$ *of the form*

$$\eta(\tilde{\omega}_j t, x) = \sum_{j \in \mathbb{S}} \sqrt{\xi_j} \cos(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|})$$

$$\psi(\tilde{\omega}_j t, x) = -\sum_{j \in \mathbb{S}} \sqrt{\xi_j} \omega_j^{-1} \sin(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|})$$

*with frequencies* $\tilde{\omega}_j$ *satisfying* $\tilde{\omega}_j - \omega_j(h) \to 0$ *as* $\xi \to 0$.
*The solutions are* **linearly stable**.

# Linear stability: perturbative Floquet theory

There exist coordinates

$$(\phi, y, v) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times (H^s_x \cap L^2_{\mathbb{S}^c})$$

in which the linearized equation $\partial_t h = J\partial_u \nabla H(u(\omega t))h$ reads

$$\begin{cases} \dot{\phi} = by \\ \dot{y} = 0 \\ v_t = \mathrm{i}\mu^\infty(D)v\,, \quad v = \sum_{j \notin \mathbb{S}} v_j e^{\mathrm{i}jx}\,, \quad \mu^\infty(j) \in \mathbb{R}\,, \ \dot{v}_j = \mathrm{i}\mu_j^\infty v_j\,, \end{cases}$$

$y(t) = y_0, v_j(t) = v_j(0)e^{\mathrm{i}\mu_j^\infty(j)t} \Longrightarrow \|v(t)\|_{H^s_x} = \|v(0)\|_{H^s_x}$: stability

$0, \{\mathrm{i}\mu^\infty(j)\}_{j\in\mathbb{S}^c}$ = **Floquet exponents**

1. Sharp **asymptotic expansion** of the **Floquet exponents**

$$\mu^{\infty}(j) = m_{\frac{1}{2}}(j\tanh(\mathtt{h}j))^{\frac{1}{2}} + r_j(\mathtt{h})$$

where $m_{\frac{1}{2}} \in \mathbb{R}$ is a constant satisfying

$$m_{\frac{1}{2}} \sim 1\,, \quad r_j \sim cj^{-\frac{1}{2}}\,, \quad c \sim O(|\xi|^a)$$

2. Bounded change of variables $\Phi(\varphi) : H^s \to H^s\,, \quad \forall s \geq s_0$

# Ideas of Proof for long time existence: normal form

1. Quadratic nonlinearity

$$u_t = \mathrm{i}\omega(D)u + P_2(u)\,, \quad P_2(u) = O(u^2)$$

Time of existence of solution with $u(0) = \varepsilon u_0$ is $T_\varepsilon = O(\varepsilon^{-1})$

2. Cubic nonlinearity

$$u_t = \mathrm{i}\omega(D)u + P_3(u)\,, \quad P_3 = O(u^3)$$

Time of existence of solution with $u(0) = \varepsilon u_0$ is $T_\varepsilon = O(\varepsilon^{-2})$

**Poincaré-Dulac Normal form idea**

Look for a change of variable s.t. the nonlinearity becomes smaller

# Poincaré-Dulac-Birkhoff

1. Perform change of variable to decrease the size of nonlinearity. Required non-resonance conditions among linear frequencies
$$\omega(j_1) \pm \omega(j_2) \pm \omega(j_3) \pm \omega(j_4) \neq 0$$
2. At higher degrees of homogeneity –yet at degree 4– there remains "*resonant terms*" $P_4$ which can not be eliminated
$$\omega(j_1) - \omega(j_2) + \omega(j_3) - \omega(j_4) = 0 \text{ for } j_1 = j_2,\ j_3 = j_4$$
3. Check that these *resonant terms* do not contribute to Sobolev energy. Algebraic structure of PDE, i.e. Hamiltonian
4. For Hamiltonian *semilinear* PDEs –*Birkhoff normal form*– Bambusi, Grebert, Delort, Szeftel '02-'07,

For *quasi-linear* PDEs this procedure gives unbounded *formal* transformations, like $u \mapsto u + \varepsilon(\partial_x u)^2$

# New procedure for quasi-linear PDEs:

1. First transform the water waves system to a **diagonal, constant coefficients in $x$** system up to **smoothing remainders**
2. Then implement a "semilinear" normal form procedure which reduces the *size* of the nonlinear terms
3. Check that the “resonant terms" left do not contribute to energy estimates. Here **Reversibility** for example

$$\dot{u}_j = \mathrm{i}\omega_j u_j + \mathrm{i}(\textstyle\sum_n a_n |u_n|^2) u_j$$

$$f(Su) = -Sf(u)\,,\ S : u_j \mapsto \bar{u}_j$$

$$\Longrightarrow \ a_n \in \mathbb{R} \ \Longrightarrow \ |u_j(t)|^2 \ \textit{constant}$$

Introducing Alinhac good unknown, paracomposition, and paradifferential non-linear changes of variables, **bounded** in $H^s$, we transform (WW) into

**Paradifferential Normal form for gravity-capillary WW**

$$u_t = \mathrm{i}\big((1+\zeta_3(u))\omega(D) + \zeta_1(u)|D|^{1/2} + r_0(u;D)\big)u + R(u)u$$

where

1. $\omega(\xi) = (\xi \tanh(\xi)(1+\kappa\xi^2))^{1/2}$, linear dispersion relation
2. $\zeta_3(u), \zeta_1(u)$ are real valued, of size $O(u)$, **constant in** $x$,
$$\zeta_3(u) = \int_{\mathbb{T}} \eta_x^2\, dx = \int_{\mathbb{T}} \Big(\omega(D)\partial_x\big(\tfrac{u-\bar{u}}{2\mathrm{i}}\big)\Big)^2 dx$$
3. $r_0(u;\xi)$ is a symbol of order 0 **constant in** $x$
4. $R(u)$ is a regularizing operator: for any $\rho$ it maps $H^s \to H^{s+\rho}$, for $\rho \le s - \frac{1}{2}$ large, $\|R(u)[u]\|_{H^{s+\rho}} \le C\|u\|_{H^\rho}\|u\|_{H^s}$

$\Longrightarrow$ we are back to a *semilinear* PDE situation

# Long time existence: energy estimates

The PDE

$$u_t = \mathrm{i}\big((1+\zeta_3(u))\omega(D) + \zeta_1(u)|D|^{1/2} + \mathrm{Re}(r_0(u;D))\big)u$$

preserves for all times $t \in \mathbb{R}$ the $L^2_x$ and $H^s_x$ norms since the symbol

$$(1+\zeta_3(u))\omega(\xi) + \zeta_1(u)|\xi|^{1/2} + \mathrm{Re}(r_0(u;\xi))$$

is **real** (self-adjointness) and has **constant coefficients in** $x$

## Normal form

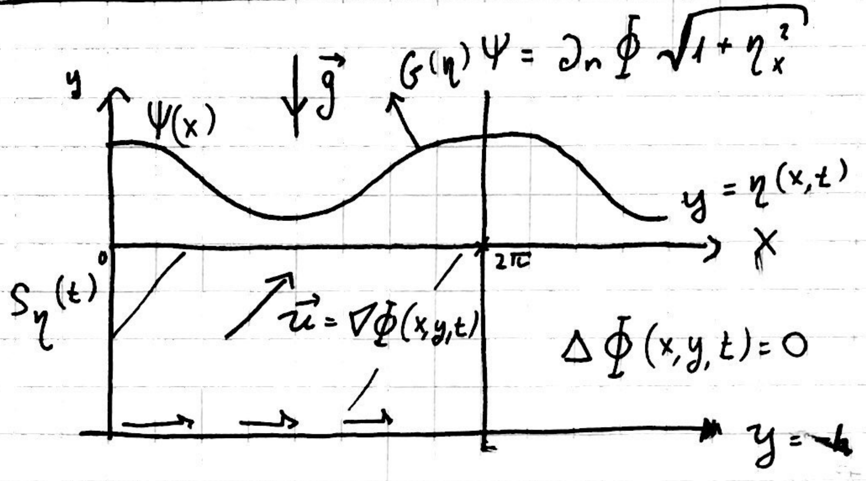
Reduce the size of $\mathrm{Im}(r_0(u;\xi))$ and $R(u)$ up to $O(\|u\|^N)$

by **Reversibility** the normal form has norm $\| \ \|_{H^s}$ as prime integrals $\Longrightarrow$ the Sobolev norm $\|u(t)\|_{H^s}$ of the solution with $u(0) = O(\varepsilon)$ remains bounded up to times $|t| \leq O(\varepsilon^{-N})$

*Thanks for your attention!*

Blackboard + lecture notes - talk by Massimiliano Berti

(notes by Ori (Soporta) Katz) 3.2




$\eta(x)$ - Canonical variables

$\Psi(x,t) = \Phi(x, \eta(x,t), t)$

Given $(\eta, \Psi)$, $\Phi$ is uniquely determined.

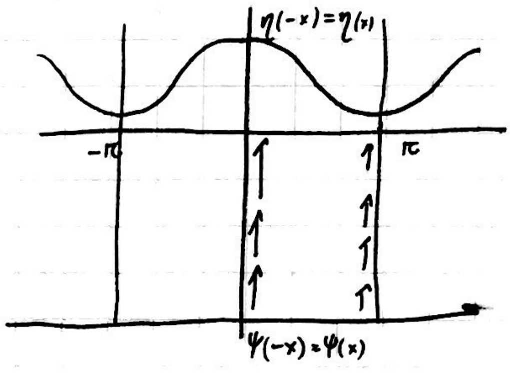
Hamiltonian - potential energy is the gravity.

$G(\eta)\Psi$

In 1 space dimension, $G(\eta) = D_x \tanh(hD_x) + OPS^{-\infty}$

Since the bottom is flat, the problem is x-invariant.

If $\eta(-x) = \eta(x)$, $\Psi(-x) = \Psi(x)$:




flow is vertical at $x=0, \pm\pi$

Linear system at $(\eta, \psi) = (0,0)$ can be written with one complex variable:

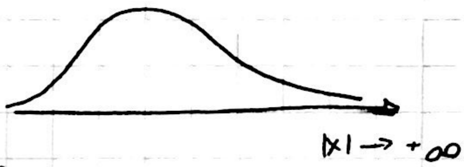
$$u = \Lambda(D)\eta + i\Lambda^{-1}(D)\psi$$

$$u_t + i\omega(D)u = 0, \quad \omega(\xi) = \sqrt{\xi \tanh(\xi)(g + K\xi^2)}$$

$K$ - capillary parameter. Asymptotic difference between $\begin{cases} K = 0 \\ K \neq 0 \end{cases}$.

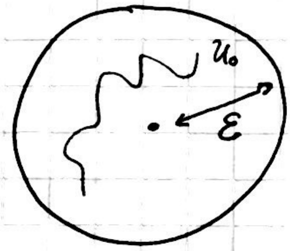
Non-linearity: $u_t + i\omega(D)u = N_2(u)$

* We work with $\int_{\pi} \eta(x)dx = 0$, so the zero element of Fourier is not relevant.

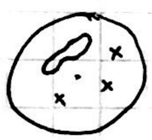
* In the case 

i.e. data decaying at $\infty$, no dispersive effects.

Long time existence Birkhoff normal form result:

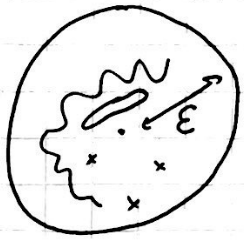



solutions exist, for long times.

Quasiperiodic results - exist for most initial conditions (except for singular points x  )

Theorem : $T_\varepsilon > \varepsilon^{-N}$

Almost global existence.



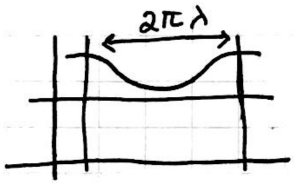

* Remark 1 : Plane waves have form

$$e^{i\omega(k)x} \quad e^{ikx}$$

$$\omega(k) = |k|^{1/2}$$

In the case of infinite depth $h = +\infty$,
the formal $4^{th}$ order Birkhoff normal form
is integrable ( $5^{th}$ order is not ).

* Remark 2 ; Periodicity in $2\pi\lambda$




* Fourier formulation of equations:

$$u_t + i\omega(D)u = N_2(u)$$

$$u(x) = \sum_j u_k e^{ikx} = \sum_J \sqrt{I_J}\, e^{i\theta_J} e^{ikx}$$

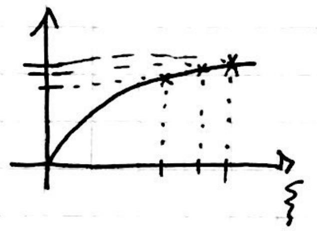
Hamiltonian case: $f(\theta, I)$ is a total derivative.

What happens for times $T_\varepsilon > \varepsilon^{-N}$ ?

We proved that until $\varepsilon^{-N}$ the solutions remain $\varepsilon$- bounded

I present results for gravity water waves.
This is the more difficult case, because when $\kappa = 0$ the dispersion is faster:

$$\omega(\xi) = \sqrt{\xi \tanh(\xi)(g + \cancel{\kappa}\xi^2)} \sim |\xi|^{1/2}$$



Existence proof: $u(\omega t, x)$

Linearize Hamiltonian system

$$h_t = J D^2 H(\omega t, x) h$$

=> Floquet exponents are purely imaginary.
This implies stability.

Ideas of proof for long-time existence:

$$u_0 = \varepsilon, \quad T_\varepsilon = \varepsilon^{-1}$$

* Poincaré - Dulac - Birkhoff:

  Non-resonance condition among linear frequencies

  <=> no 4-wave interactions.

* Sobolev norm ~ energy.

Paradifferential normal form for gravity-capillary WW:

$\zeta_3(u)$ contains all nonlinear effects,

explicitly calculated, highest order term.