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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Zensho Yoshida

Talk Title: Singularities in Poisson Manifolds: Bifurcation and Symmetry

Date: 11 / 28 / 2018 **Time:** 9 : 30 **am / pm (circle one)**

Please summarize the lecture in 5 or fewer sentences: Describe physical systems that exhibit chirality (e.g. rattleback) using a symmetric matter in chiral spacetime. What happens to a Lie-Poisson bracket when you deform the measurement? The deformation must be symmetric if it is full rank and the new bracket satisfies the Jacobi identity. If the deformation is not full rank, the new bracket could have chirality and the linearized system near an equilibrium doesn't have to have a Hamiltonian spectrum.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
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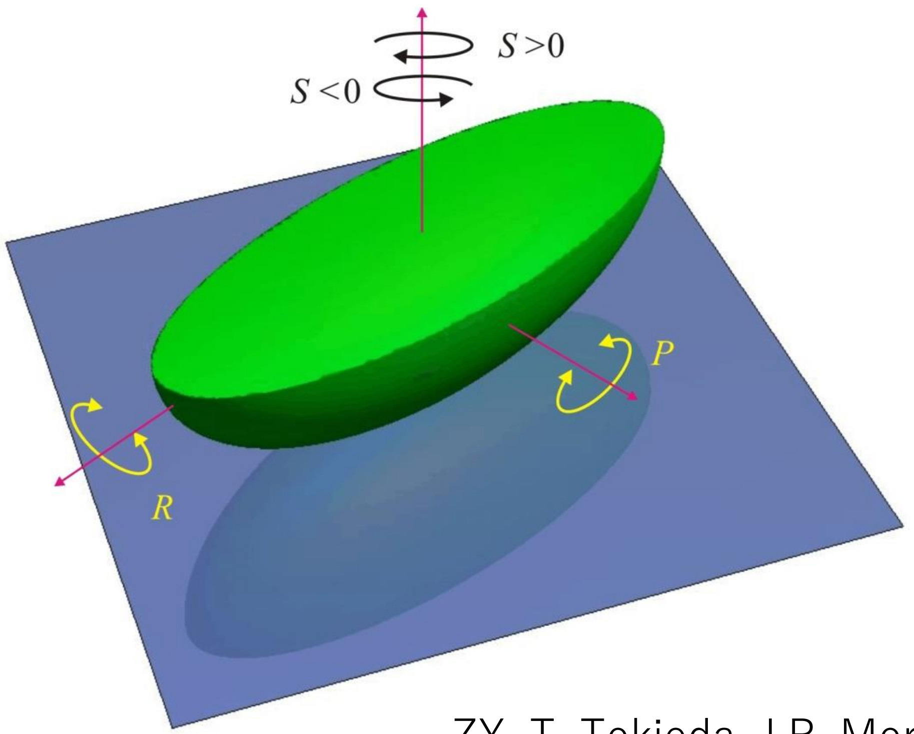
Singularities in Poisson manifolds

bifurcation and symmetry breaking

Z. Yoshida (U. Tokyo)

Collaboration with P. J. Morrison (U. Texas)

Matter vs. Space



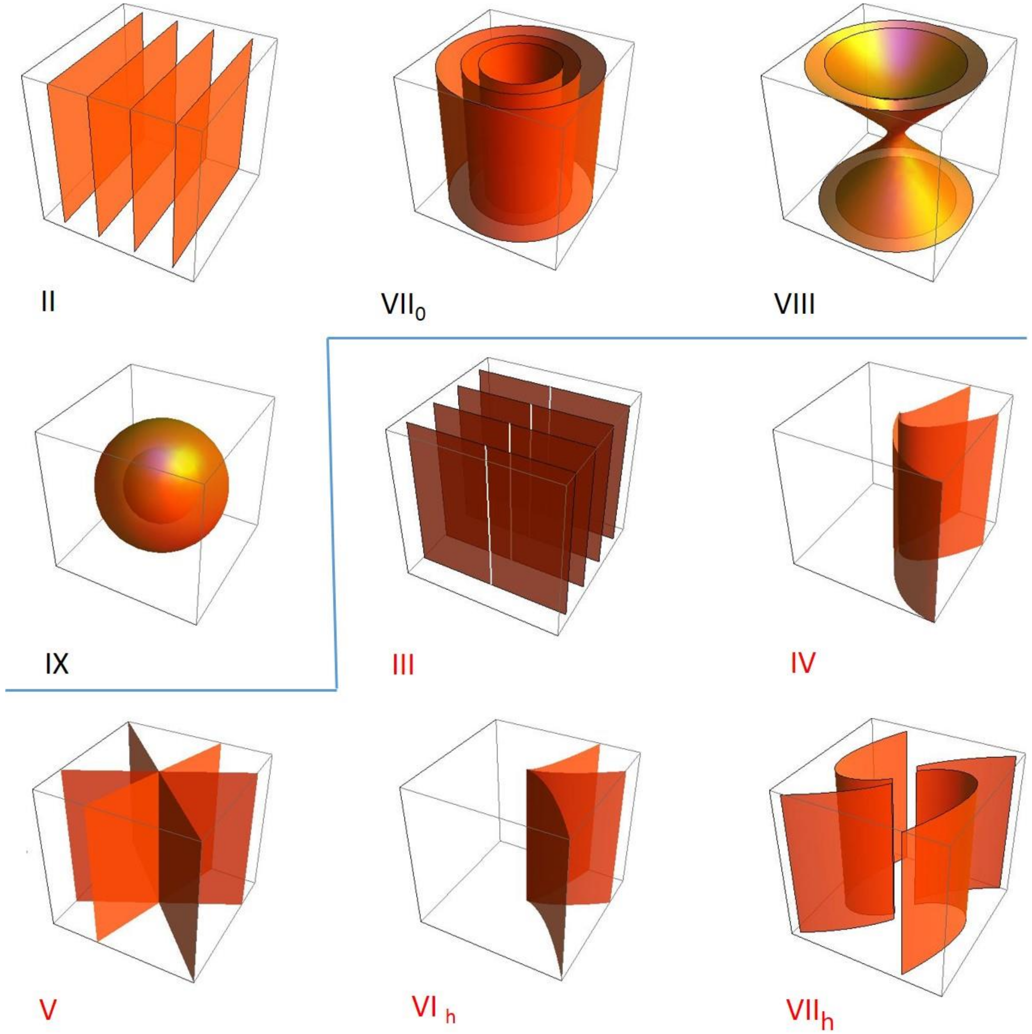
Symmetry breaking in matter

Symmetry breaking in space

ZY, T. Tokieda, J.P. Morrison, Phys. Lett. A 381 (2017), 2772—2777.

3D Casimir leaves (Bianchi classification)

Class A
symplectic leaves



Class B
Singularities in leaves

Deformation of 3D Lie algebras

Class	Type	M	Rank M
A	I	0	0
A	II	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1
A	VI_{-1}	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
A	VII_0	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
A	VIII	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	3
A	IX	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3

Non-symmetric deformations

B	III	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1
B	IV	$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
B	V	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
B	$\mathrm{VI}_{h\neq -1}$	$\begin{pmatrix} 0 & h & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
B	$\mathrm{VII}_{h\neq 0}$	$\begin{pmatrix} -1 & h & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2

Summary

- Deformation of Lie algebra though Lie-Poisson bracket
- Take so(3) (and its bundle) as the "origin"
- Ker(M) = Coker(M^t) = singularity
- At singularity, Casimir ←→ Hamiltonian
- Non-symmetric M → class-B, chiral systems

SINGULARITIES IN POISSON MANIFOLDS
BIFURCATION AND SYMMETRY BREAKING

ZENSHO YOSHIDA   Notes by Jeffrey Heninger

Rattleback

chirality - preferential direction of rotation

rotate clockwise → unstable → rattles → reverses direction

rotate counterclockwise → stable

unsymmetric matter in symmetric (non-chiral) spacetime vs.

symmetric matter in chiral spacetime ← we will describe it this way

possible nonsymmetric spaces - 3D Casimir leaves (Bianchi classification)

Class B - has singularities on the leaves

manifold is not globally symplectic - strange things can happen near singularities

1 Lie - Poisson Manifold

formulated on the dual space $C^\infty(V^*)$

(1) General relation between a space and its dual $V \longleftrightarrow V^* = \mathrm{Hom}(V, \mathbb{R})$

$$\underset{\text{phase space}}{V} = \underset{\text{state vectors}}{\{\mathbb{Z} = \text{state}\}} \longleftrightarrow \underset{\text{dual}}{V^*} = \underset{\text{observations}}{\{u = \text{measurement}\}}$$

If a "point" in the phase is given, $V \ni \mathbb{a}$: $u^j \mapsto u^j(\mathbb{a})$

it tells us the result of the measurement

Deform measurement. What happens to the states?

Ex. $V = \mathrm{span}\{e_1 \cdots e_n\}$ $V^* = \mathrm{span}\{e^1 \cdots e^n\}$

$$u^j = \langle \mathbb{Z}, e^j \rangle$$

remove one of your measurements $\cancel{e^k}$

this means you can't measure one of your states anymore

Ex. Things become more complicated in infinite dimensions

$$V = \mathcal{D}(\Omega) \longleftrightarrow V^* = \mathcal{D}'(\Omega)$$
$$V = L^2(\Omega) \longleftrightarrow V^* = L^2(\Omega)$$
$$V = L^q(\Omega) \longleftrightarrow V^* = L^p(\Omega)$$
$$V = \mathcal{D}'(\Omega) \longleftrightarrow V^* = \mathcal{D}(\Omega)$$

If you change the resolution of the measurement, the resolution of the state changes

(2) Lie-Poisson

V : Lie algebra $[\ ,\ ]$ $\quad ad_H = [\ , H]$

$\updownarrow$

V^* : observation. On $C^\infty(V^*)$, define a Poisson bracket

$$\{F, G\} = \langle [\partial_u F, \partial_u G], u \rangle = \langle \partial_u F, [\partial_u G, u]^* \rangle$$
$$J(u) = [\circ, u]^*$$

For a Hamiltonian, this gives dynamics

$$\dot{F} = \{F, H\} \iff \dot{u} = J(u)\, \partial_u H$$

Modify $u \to Mu$. What happens?

(3) Examples

① $so(3)$ complete lie algebra in 3-dim.

$V = \mathbb{R}^3$, $V^* = \mathbb{R}^3 \simeq V$ $\qquad \langle\ ,\ \rangle$ Euclidean

$[\mathbf{a}, \mathbf{b}] = \mathbf{a} \times \mathbf{b}$ $\qquad \omega \in V$

$$\{F, G\} = \langle [\partial_\omega F, \partial_\omega G], \omega \rangle$$
$$= \langle \partial_\omega F, [\partial_\omega G, \omega]^* \rangle$$

$$[\mathbf{g}, \omega]^* = \mathbf{g} \times \omega$$
$$[\ ,\ ] \simeq [\ ,\ ]^*$$

$$\dot{\omega} = J(\omega)\, \partial_\omega H = (\partial_\omega H) \times \omega$$

rigid body motion

② Poisson

$V = \mathcal{D}(\Omega)$ or $C(\Omega)$ or ...
$= \{f(\mathbf{z})\}$

$\mathbf{z} = \binom{x}{y} \in \Omega$ base space
"phase space" of a single particle

$$[f, g]_p = \partial_x f\, \partial_y g - \partial_y f\, \partial_x g$$

$$\langle f, g \rangle = \int_\Omega f g\, d^2x$$

Integrate by parts $\Rightarrow [\ ,\ ]_p \simeq [\ ,\ ]_p^*$

$$\{F, G\} = \langle [\partial_u f, \partial_u g], u \rangle$$
$$= \langle \partial_u f, [\partial_u g, u]^* \rangle$$

For a "pure state" $u = \delta(\mathbf{z} - \boldsymbol{\xi})$

$$F(u) = \langle z^j, u \rangle$$
$$\dot{F} = \dot{\xi}^j = [\xi^j, \partial_u H]_p^*$$

This is the single particle equation of motion

(4) Deformation

$J(u) = [0, u]^* \longrightarrow J(Mu) = [0, Mu]^* \qquad M \in End(V^*)$

this means that the Poisson bracket is modified:

$$\{F, G\} = \langle [\partial_u F, \partial_u G], u \rangle \longrightarrow \langle [\partial_u F, \partial_u G], Mu \rangle = \langle \underbrace{M^+ [\partial_u F, \partial_u G]}_{\text{Call this } [\partial_u F, \partial_u G]_M}, u \rangle$$

Does the new bracket also satisfy Jacobi?

2. 3D Systems

(1) $SO(3)$ as the "original" and its deformation:

$$\begin{array}{ccc} J(\omega)\, b & \longrightarrow & J(M\omega)\, b \\ \| & & \| \\ b \times \omega & & b \times M\omega \end{array} \qquad [a, b] \longrightarrow M^+ [a, b]$$

Bianchi classification of Lie algebras
gives us possible M
class A – symmetric matrices
class B – nonsymmetric matrices – at most rank 2.

Jacobi

$$[[e_1, e_2]_M, e_3] + \circlearrowleft$$
↑ not modified

$$= (M^+_{23} - M^+_{32})\, e_1 + (M^+_{31} - M^+_{13})\, e_2 + \cdots$$

If this is full rank, ~~then~~ then M must be symmetric
rank $M = 3$, fully symmetric
rank $M < 3$, possibility of nonsymmetric M's

(2) Singularity $\quad \Sigma = Ker(M) = CoKer(M^+)$

if singularity, $J(Mu)$ could become zero

rank 2: Suppose $M = N \oplus 0 \qquad$ i.e. $M = \left(\begin{array}{c|c} N & \\ \hline & 0 \end{array} \right)$

$\Rightarrow g_M'$: abelian $\qquad (\dim g_M' = \text{rank } M^+)$

$[e_1, e_2] = M^+ [e_1, e_2] = M^+ e_3 = 0$. so abelian.

(3) Linearization (near singularity)

$\dot{\omega} = J_M(\omega)\, \partial_u H \qquad$ Perturb: $\omega = \omega_e + \tilde{\omega}$

$$\dot{\tilde{\omega}} = J_M(\tilde{\omega})\, \partial_u H\big|_e + \underline{J_M(\omega_e)\, \tilde{H}(\tilde{\omega})}$$

usually, ↑ 0 $\qquad$ ↑ Hamiltonian system
Hamiltonian ~~s~~ spectrum
no chirality

$$\dot{\tilde{\omega}} = \underbrace{J_m(\tilde{\omega})\, \partial_u H|_e}_{\text{these are the eqtns.}} + \underbrace{J_m(\omega_e)\, \hat{H}(\tilde{\omega})}_{0}$$

at singularity,

Class A:

$$\dot{\tilde{\omega}} = [h_0, M\omega]^*_{so(3)}$$

$$= [M\tilde{\omega}, -h_0]^*$$

$$= J(-h)\, M\tilde{\omega} \qquad C(\tilde{\omega}) = \tfrac{1}{2}\langle M\omega, \omega\rangle$$

Casimir

$H(\tilde{\omega}) = \langle h_e, \tilde{\omega}\rangle \rightarrow$ Casimir

Hamiltonian & Casimir exchange. Symmetric $M \Rightarrow$ no chirality

Class B: $C(\tilde{\omega})$ such that $J(M\tilde{\omega})\, \partial_{\tilde{\omega}} C(\tilde{\omega}) = 0$

this is a constant of motion

since M is not symmetric, there can be chirality

3. Arbitrary (∞) Dimension

(1) Symmetric Deformation

<u>Thm</u> V is a Lie algebra $[\ ,\ ]$

M is a self adjoint operator with point spectra only

then $M[\ ,\ ] = [\ ,\ ]_M$ is a Lie bracket

<u>Remark</u>

① For self adjoint M, define the set of operators that commute with M.

$B_M = \{A ;\ AM = MA\}$ ← commutative ring

B_M / I_λ = projection onto eigenspace

↑ max ideal of B_M

$V = \{I_\lambda\}$

② If M is degenerate, we have a singularity $\Sigma = \mathrm{Ker}(M)$

$\Rightarrow$ Projector [Chandre et al. 2013]

(2) Reduction M self adjoint

$V^* = V_M^* \oplus \mathrm{Ker}(M)$

Denote $\omega = Mu \in V_M^*$

For $F(\omega)$, $\partial_\omega F = M\partial_u F$

Reduced Lie-Poisson

$$\{F, G\} = \langle M\partial_\omega F, [M\partial_\omega G, \omega]^*\rangle$$

$$J(\omega) = M[M\circ, \omega]^*$$

Can also probably non-symmetric ∞-dim Lie-Poisson.