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# NOTETAKER CHECKLIST FORM

**(Complete one for each talk.)**

**Name:** Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

**Speaker's Name:** James Meiss

**Talk Title:** Drift and Diffusion in Symplectic and Volume-Preserving Maps

**Date:** 11 / 29 / 2018 **Time:** 9 : 30 **am** / pm (circle one)

**Please summarize the lecture in 5 or fewer sentences:** Compares volume preserving maps to symplectic maps. There is a KAM theory, but we don't know what the most robust torus is and we don't have an Aubrey-Mather theory to describe what happens after breakup. When the map is symplectic, the drift along a resonance channel is diffusive because the action along the resonance is approximately invariant, as guaranteed by Nekhoroshev. When the map is non-symplectic, the drift along a resonance channel is ballistic and can switch between different resonance channels, leading to much higher transport.

# CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

# Drift and Diffusion in Symplectic and Volume-Preserving Maps

James Meiss
University of Colorado at Boulder

Collaborators:
Holger Dullin, Adam Fox,
Nathan Guillery, Hector Lomelí

MSRI Nov 29,2018

# Hamiltonian Flow is Symplectic

- Hamiltonian Dynamics:

$$\frac{dq}{dt} = \frac{\partial}{\partial p} H$$
$$\frac{dp}{dt} = -\frac{\partial}{\partial q} H$$

$$\dot{z} = J \nabla H$$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

- Flow φ on $z = (q,p)$

$$z(t) = \varphi_t(z(0)) \Rightarrow D\varphi_t^T J D\varphi_t = J$$

- A map is symplectic if:

$$Df^T J Df = J \qquad Df = \frac{\partial(x', y')}{\partial(x, y)}$$

- $\Rightarrow$ preservation of the Poincaré loop action $\mathcal{A} = \oint_{\mathcal{L}} p \cdot dq$

# Symplectic Maps

- A map $f(x,y) \to (x',y')$ is symplectic if

coordinates
momenta

$$Df^T J Df = J \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

- Standard (Chirikov-Froeschlé) form on $\mathbb{T}^n \times \mathbb{R}^n$

$$\begin{aligned} x' &= x + \Omega(y') \mod 1 \\ y' &= y + F(x) \end{aligned}$$

Angle-Action Form

- Symplectic only if frequency & force are gradients

$$\Omega(y) = \nabla S(y)$$

Frequency Map

$$F(x) = -\nabla V(x)$$

Force

# Symplectic Maps

- A map $f(x,y) \rightarrow (x',y')$ is symplectic if

coordinates
momenta

$$Df^T J D f = J \qquad J = \begin{pmatrix} 0 & I \end{pmatrix}$$

insert "new" *y* here!

- Standard (Chirikov-Froeschlé) form on $\mathbb{T}^n \times \mathbb{R}^n$

$$x' = x + \Omega(y') \mod 1$$
$$y' = y + F(x)$$

Angle-Action Form

- Symplectic only if frequency & force are gradients

$$\Omega(y) = \nabla S(y)$$
$$F(x) = -\nabla V(x)$$

Frequency Map

Force

# Volume-Preserving Maps

- A map $f(x,y) \to (x',y')$ is volume preserving if

$$\det(Df) = 1$$

*m* actions

- Standard (angle-action) form on $\mathbb{T}^n \times \mathbb{R}^m$

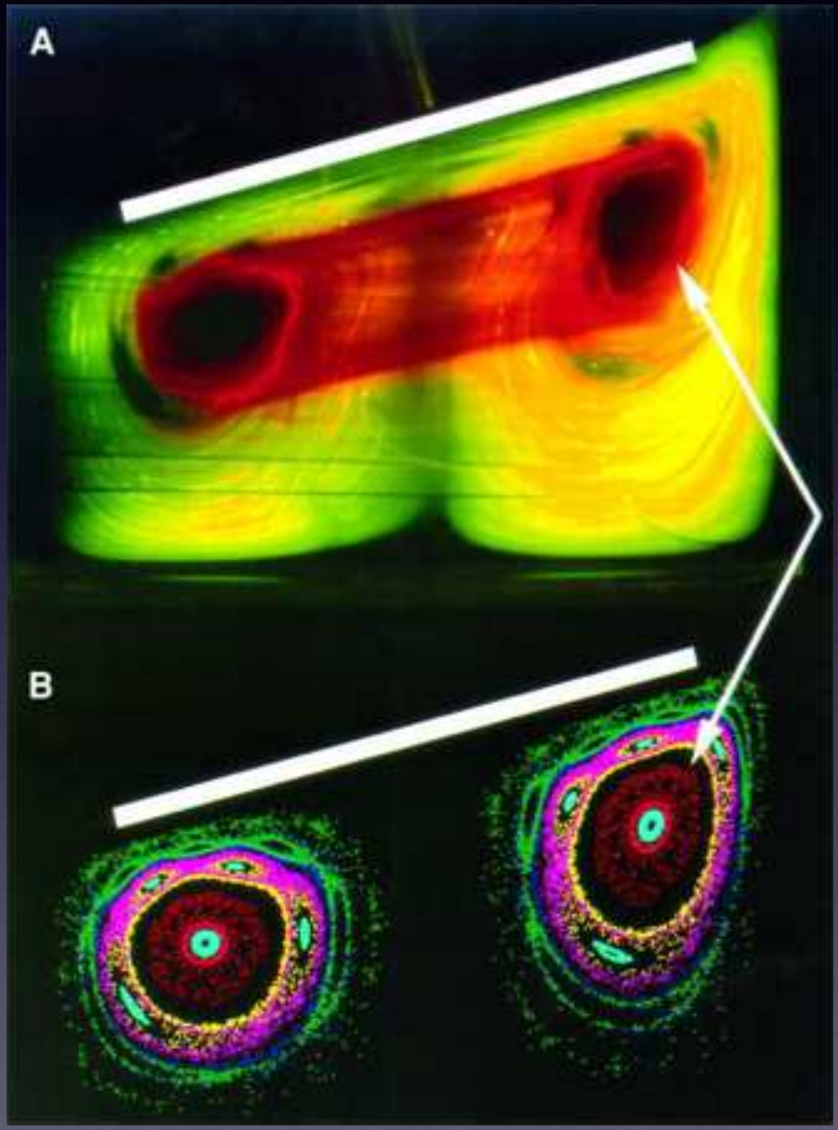
$$x' = x + \Omega(y') \mod 1$$
$$y' = y + F(x)$$

Angle-Action Form

- Note: number of angles and actions can be different, $n \neq m$
- Volume preserving for *any* frequency map $\Omega : \mathbb{R}^m \to \mathbb{T}^n$ and *any* force $F : \mathbb{T}^n \to \mathbb{R}^m$

# Why Volume-Preserving?

- Mixing (stirring) in incompressible fluids $\nabla \cdot v = 0$
  - Chaotic advection of dye $\dot{x} = v(x,t)$
- Magnetic fields $\nabla \cdot B = 0$
  - stellarators, solar flares, Earth's magnetotail
- 3D is simpler than 4D symplectic case!
- What is the impact of lack of symplecticity on dynamics?




Fountain, G. O., F. V. Khakhar and J. M. Ottino (1998). "Visualization of Three-Dimensional Chaos." Science 281: 683.

# Optics: Curl Forces

- Berry & Shukla note that odes on $\mathbb{R}^2 \times \mathbb{R}^2$ of the form

$$\dot{x} = y \qquad \nabla \times F \neq 0$$
$$\dot{y} = F(x)$$

arise in dynamics of polarizable particles, $p \propto E$, in a paraxial electric field $E(x, y, z) = e^{ikz}\psi(x, y)\hat{e}_p$

$$F(x) = -\nabla|\psi|^2 + a\,\mathrm{Im}[\psi^* \nabla\psi]$$

- Such systems are incompressible, but non-Hamiltonian.
- Poincaré Maps will be Volume-preserving!

Berry, M. V. and P. Shukla (2015). "Hamiltonian Curl Forces." Proc Roy Soc Lond A 471: 20150002.

# Volume-Preserving Maps

- Examples: (*n*,*m*)
  - Chirikov's Standard map (1,1) $(x', y') = (x + y', y - \varepsilon \sin(2\pi x))$
  - Froeschlé map (2,2) $(x', y') = (x + y', y - \varepsilon \nabla V(x))$
- Two Angle–One Action Normal form (Rank-One Resonance)

$\Omega = (\Omega_1(y), \Omega_2(y))$

$$x'_1 = x_1 + y' + \gamma$$
$$x'_2 = x_2 + \beta y'^2 - \delta$$
$$y' = y - \varepsilon[\sin(2\pi x_1) + \sin(2\pi x_2) + \sin(2\pi(x_1 - x_2))]$$

Dullin, H. R. and J. D. Meiss (2012). "Resonances and Twist in Volume-Preserving Mappings." SIAM J. Appl. Dyn. Sys. 11: 319-349.

# Invariant Tori

# Resonance Web

- Resonances

$$\mathcal{R} = \left\{\omega \in \mathbb{R}^n : m \cdot \omega = n, (m,n) \in \mathbb{Z}^{n+1} \setminus \{0\}\right\}$$

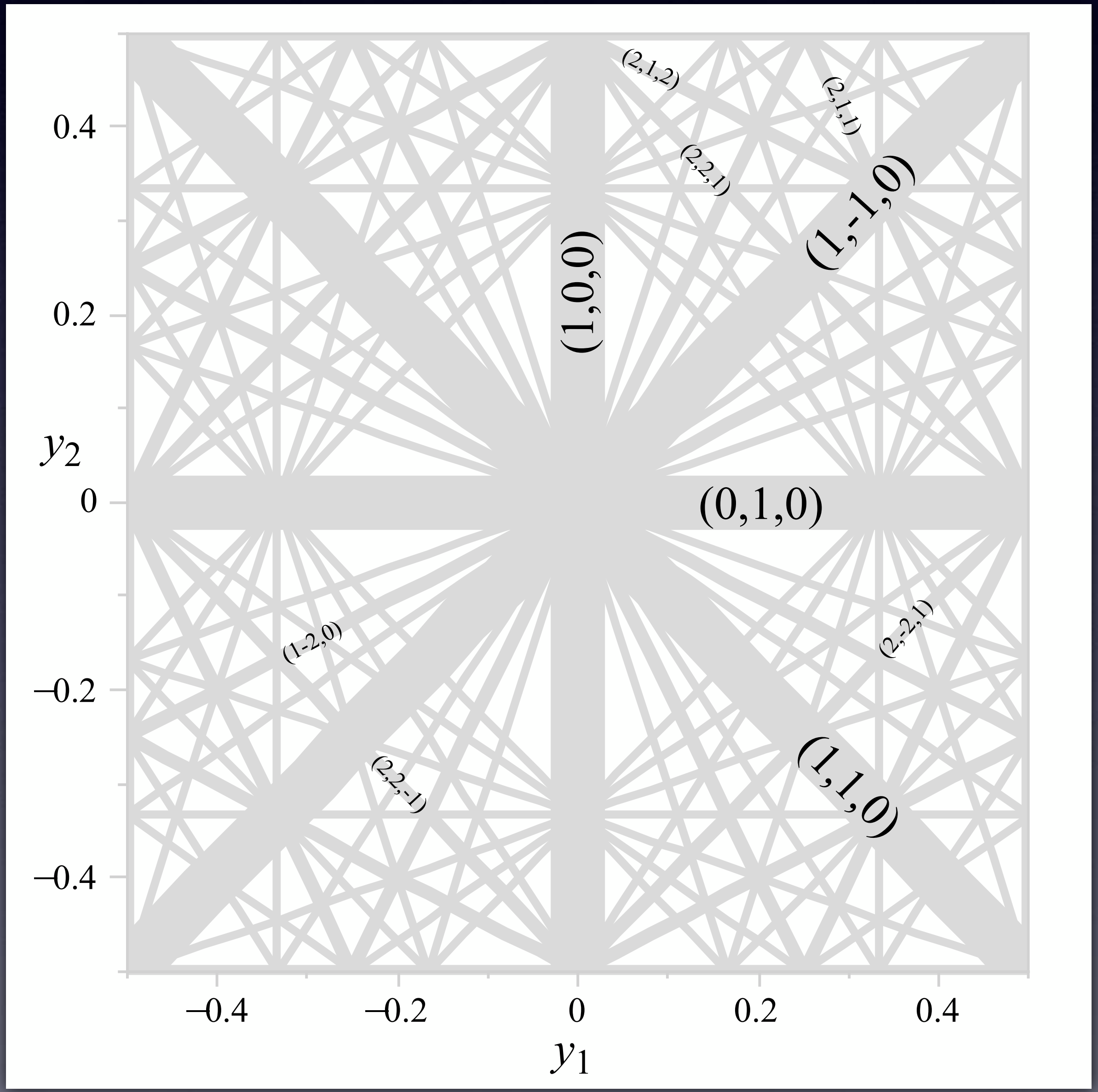
- For a resonance, the *rank* is the dimension of the module

$$\mathcal{L}_\omega = \{m \in \mathbb{Z}^n : m \cdot \omega \in \mathbb{Z}\}$$

- In Action space, resonances occur at

$$m \cdot \Omega(y) = n$$

- Nonlinearity “fattens resonances




$$|m \cdot \Omega - n| < \frac{c}{\|m\|^2}$$

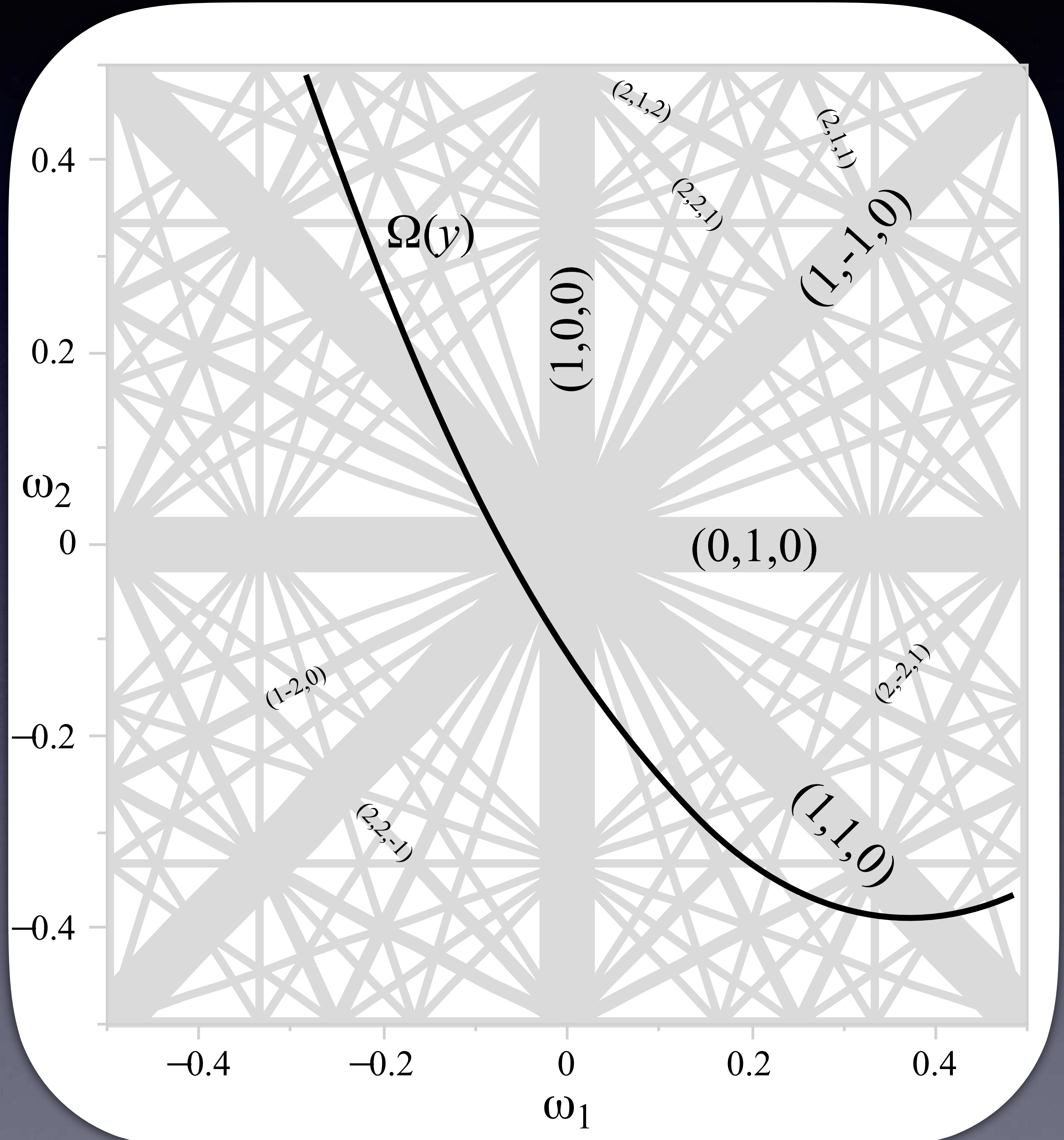
# Frequency Map

$(n,m) = (2,1)$

- Tori exist only where "true" frequency map crosses incommensurate frequency vectors
- To fix frequency: add a parameter:

  $$\Omega(y,\delta) \longrightarrow (\omega_1, \omega_2)$$

$$\Omega(y,\delta) = (y+\gamma, \beta y^2 - \delta)$$



# Volume-Preserving KAM

- Xia & Cheng/Sun: For one-action case: A Cantor set of codimension-one, Diophantine ($c > 0$, $s \geq 2$)

$$|p \cdot \omega - q| > \frac{c}{|p|^s} \quad p \in \mathbb{Z}^n \backslash 0,\ q \in \mathbb{Z}$$

tori persist in smooth, volume-preserving families, if $\Omega$ satisfies a non-degeneracy (twist) condition

$$\text{rank}(D\Omega, D^2\Omega, \ldots, D^n\Omega) = n$$

i.e., *Kolmogorov nondegeneracy*.

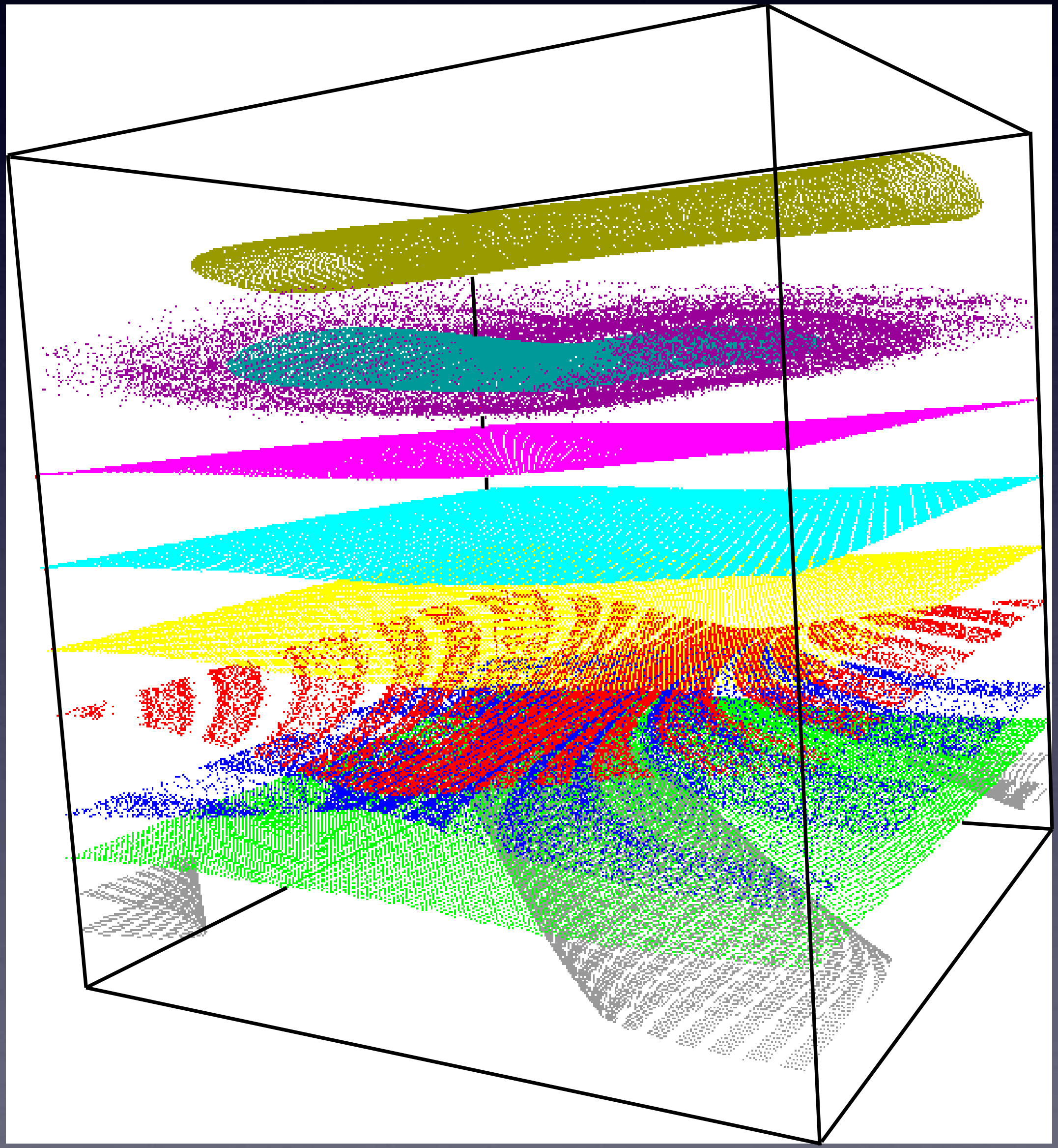

Cheng, C.-Q. and Y.-S. Sun (1989). "Invariant tori in 3D measure-preserving mappings." Celest. Mech. 47(3): 275-292.

Xia, Z. (1992). "Existence of invariant tori in volume-preserving diffeomorphisms." Erg. Th. Dyn. Sys. 12(3): 621-631.

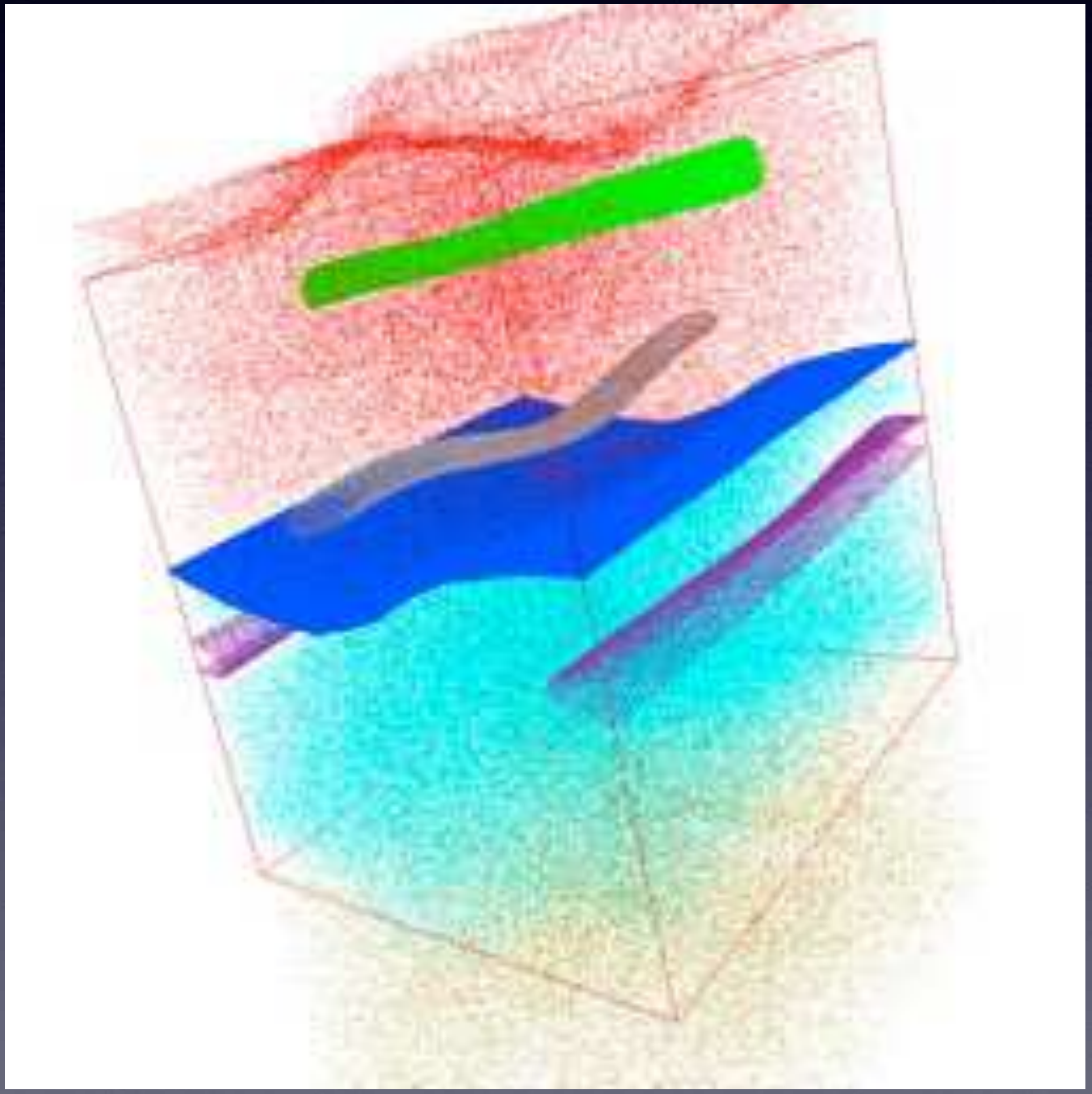
Blass, T. and de la Llave, R. (to be written) KAM theory for volume-preserving maps

# Tori act as Barriers

Many rotational tori



Only one rotational torus



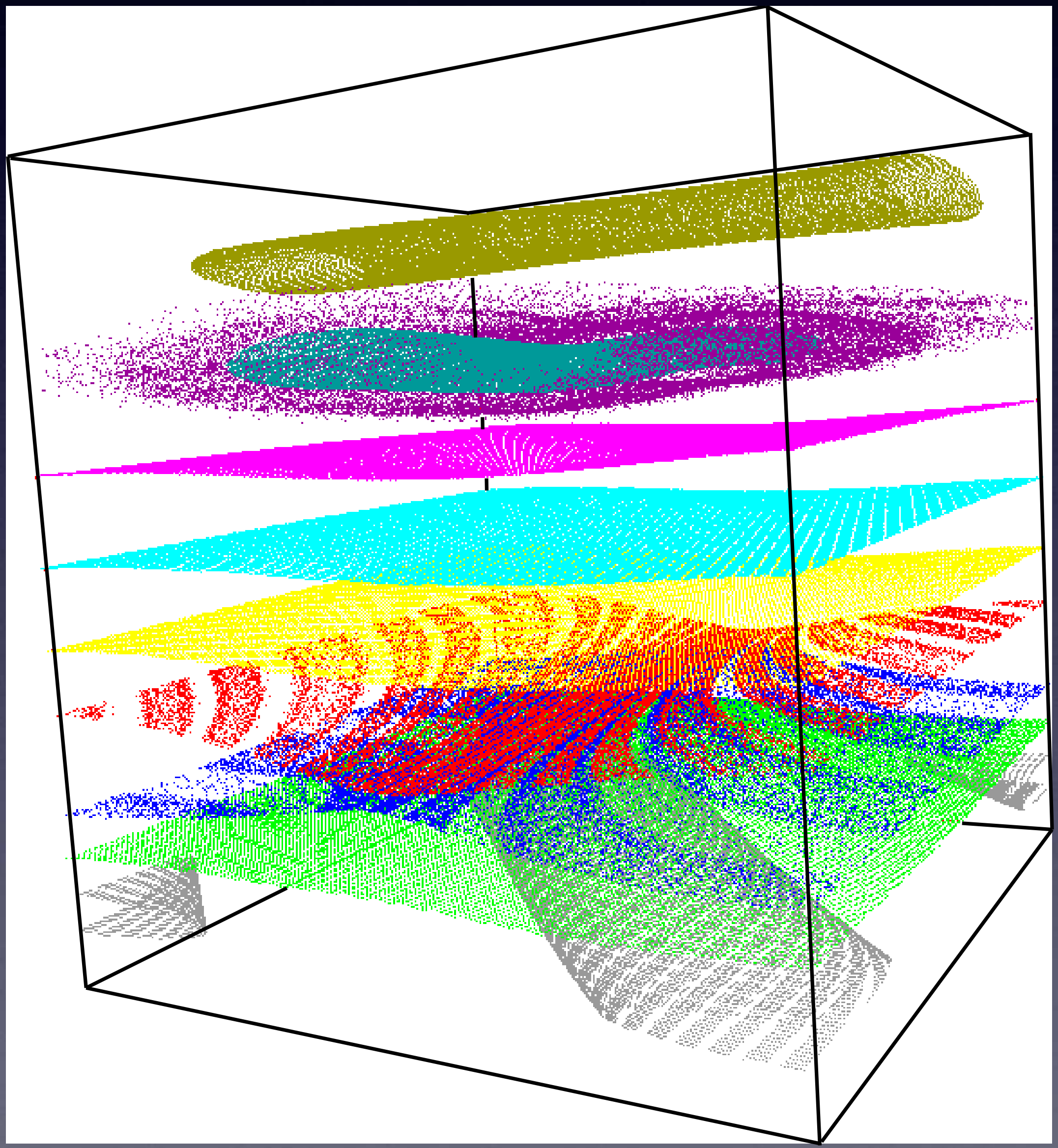
$\varepsilon = 0.005$

$\varepsilon = 0.02725$

$\beta = 2, \delta = 0.1, \gamma = \frac{1}{2}(1+\sqrt{5})$

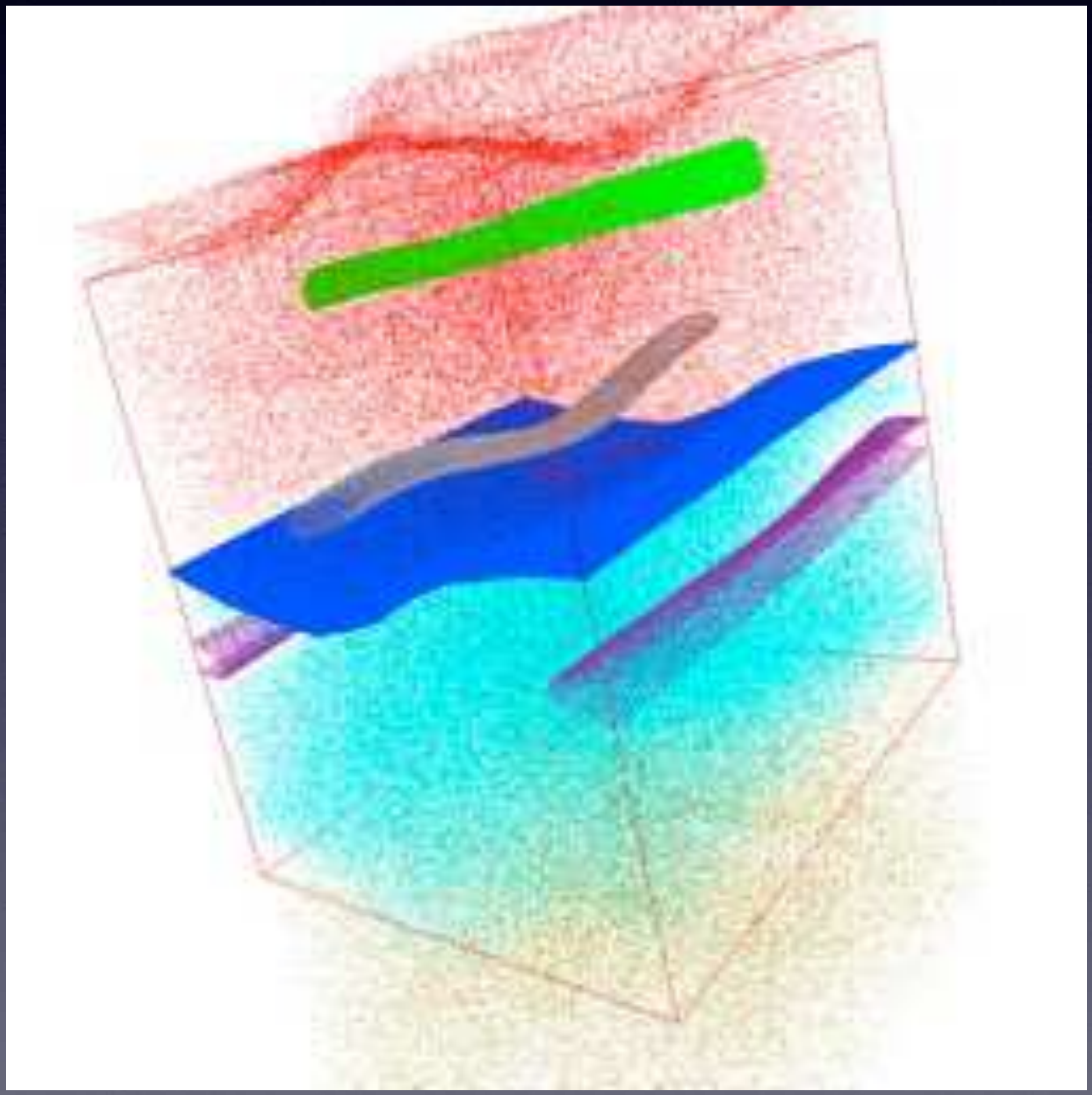
$$x_1' = x_1 + y' + \gamma$$
$$x_2' = x_2 + \beta y'^2 - \delta$$
$$y' = y - \varepsilon[\sin(2\pi x_1) + \sin(2\pi x_2) + \sin(2\pi(x_1 - x_2))]$$

Many rotational tori



$\varepsilon = 0.005$

Only one rotational torus



$\varepsilon = 0.02725$

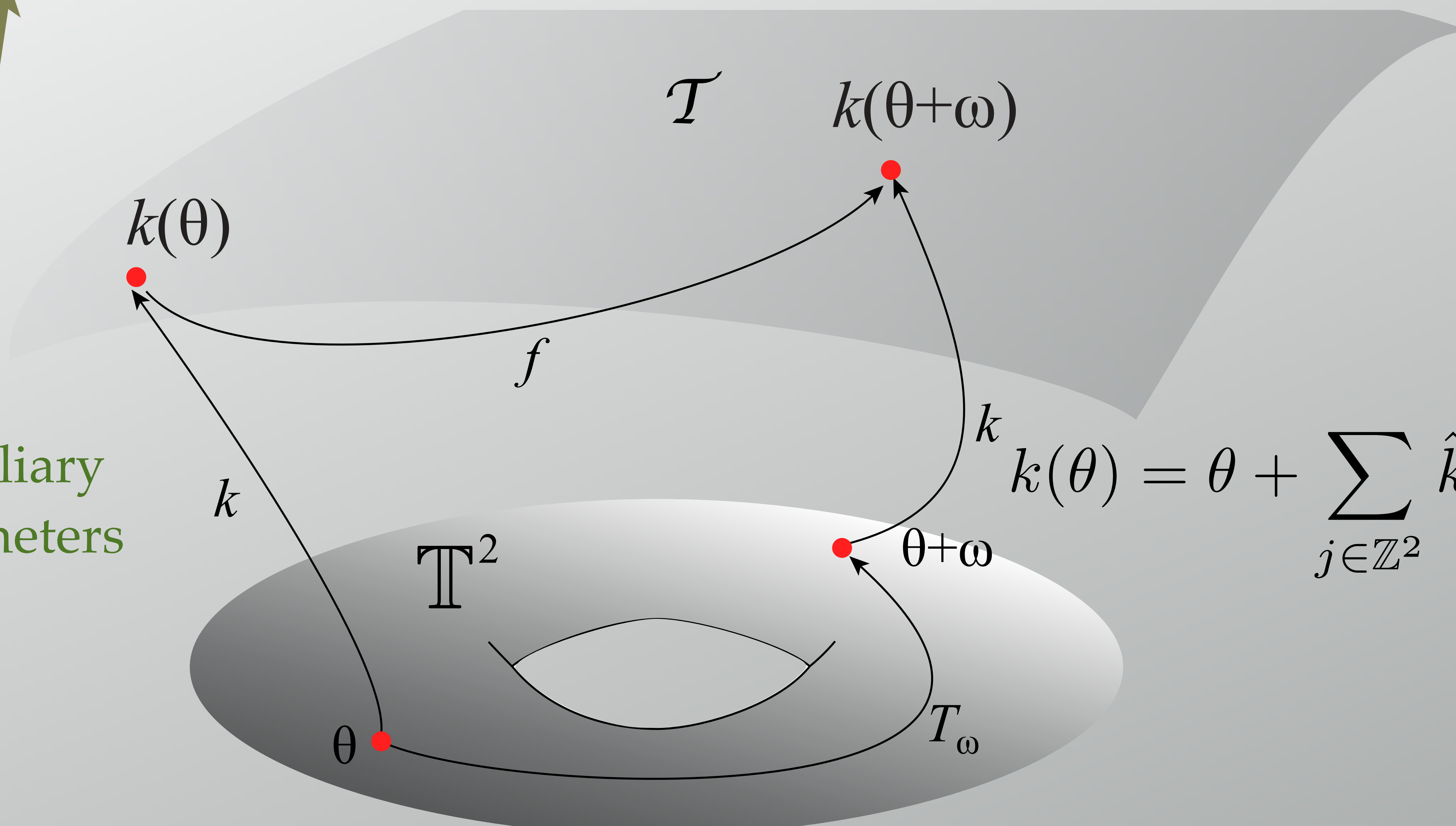
$\beta = 2$, $\delta = 0.1$, $\gamma = \frac{1}{2}(1+\sqrt{5})$

# Computing Tori: Parameterization

$$f_\lambda(k(\theta)) = k(\theta + \omega)$$

Fixed Rotation Vector

Auxiliary Parameters




$$k(\theta) = \theta + \sum_{j \in \mathbb{Z}^2} \hat{k}_j e^{2\pi i j \cdot \theta}$$

Haro, A., M. Canadell, et al. (2016). The Parameterization Method for Invariant Manifolds … Springer, 267pp.

# Frequency Vectors

- Whereas integral bases of algebraic fields are Diophantine;*
- Whereas noble irrationals (related to the golden mean) give "robust" invariant circles 2D Maps;

- Be it sensibly resolved that elements of a cubic field should give "robust" two-tori.

- Some Possibilities:
  - Spiral Mean (Δ = -23):

$$\sigma^3 = \sigma + 1 \qquad \omega = (\sigma, \sigma^2)$$

$$\sigma \approx 1.324717957244746026$$

  - Totally Real Field (Δ= 49):

$$\tau^3 + \tau^2 - 2\tau - 1 = 0$$

$$\tau = 2\cos(2\pi/7) \qquad \omega = (\tau, \tau^2)$$

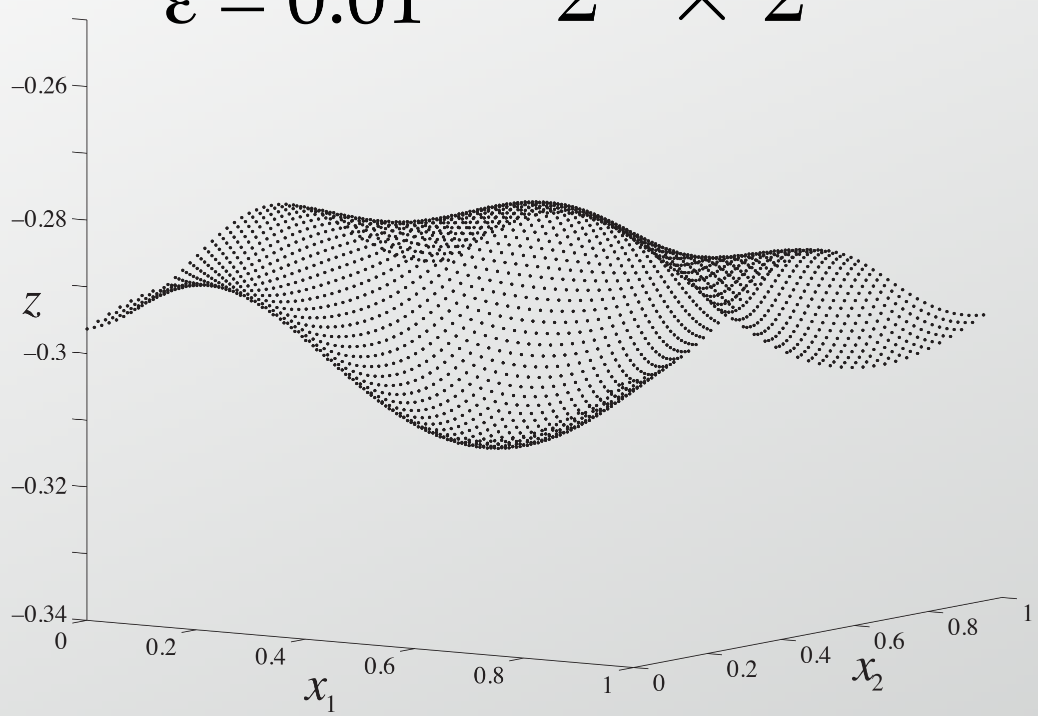
What class of frequencies plays the role of the nobles?

Cusick, T. W. (1974). "The 2D Diophantine Approximation Constant." Mon. für Math. 78: 297-304.

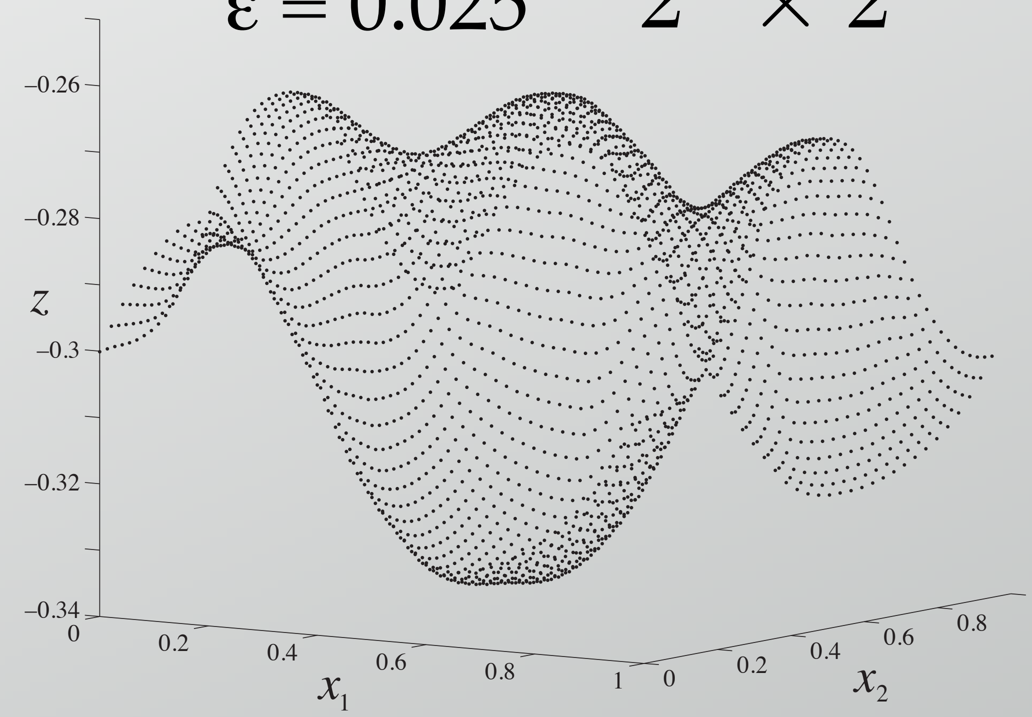
# Spectral Method

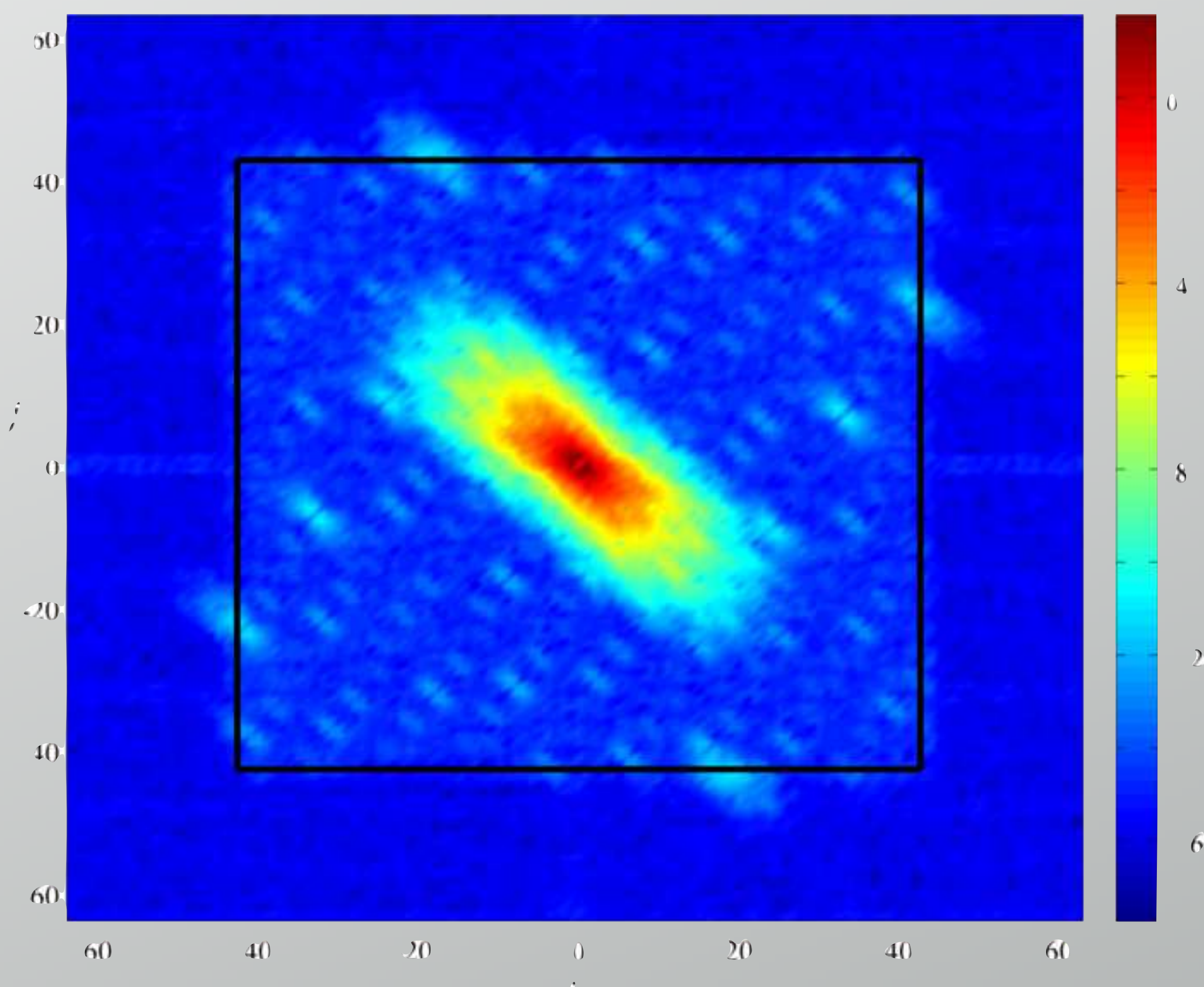
$$\omega = (\sigma - 1, \sigma^2 - 1)$$

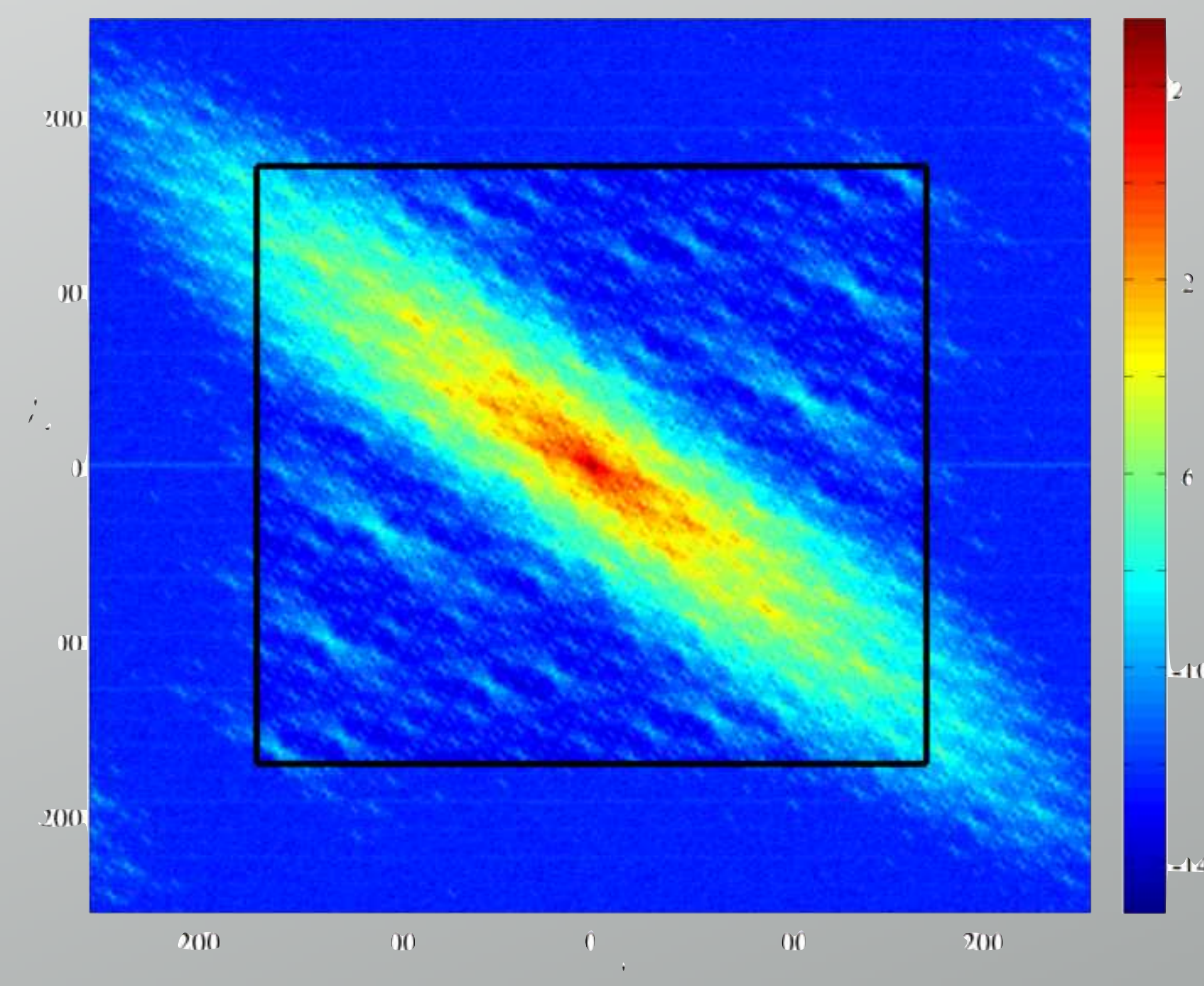
$\varepsilon = 0.01 \quad 2^7 \times 2^7$



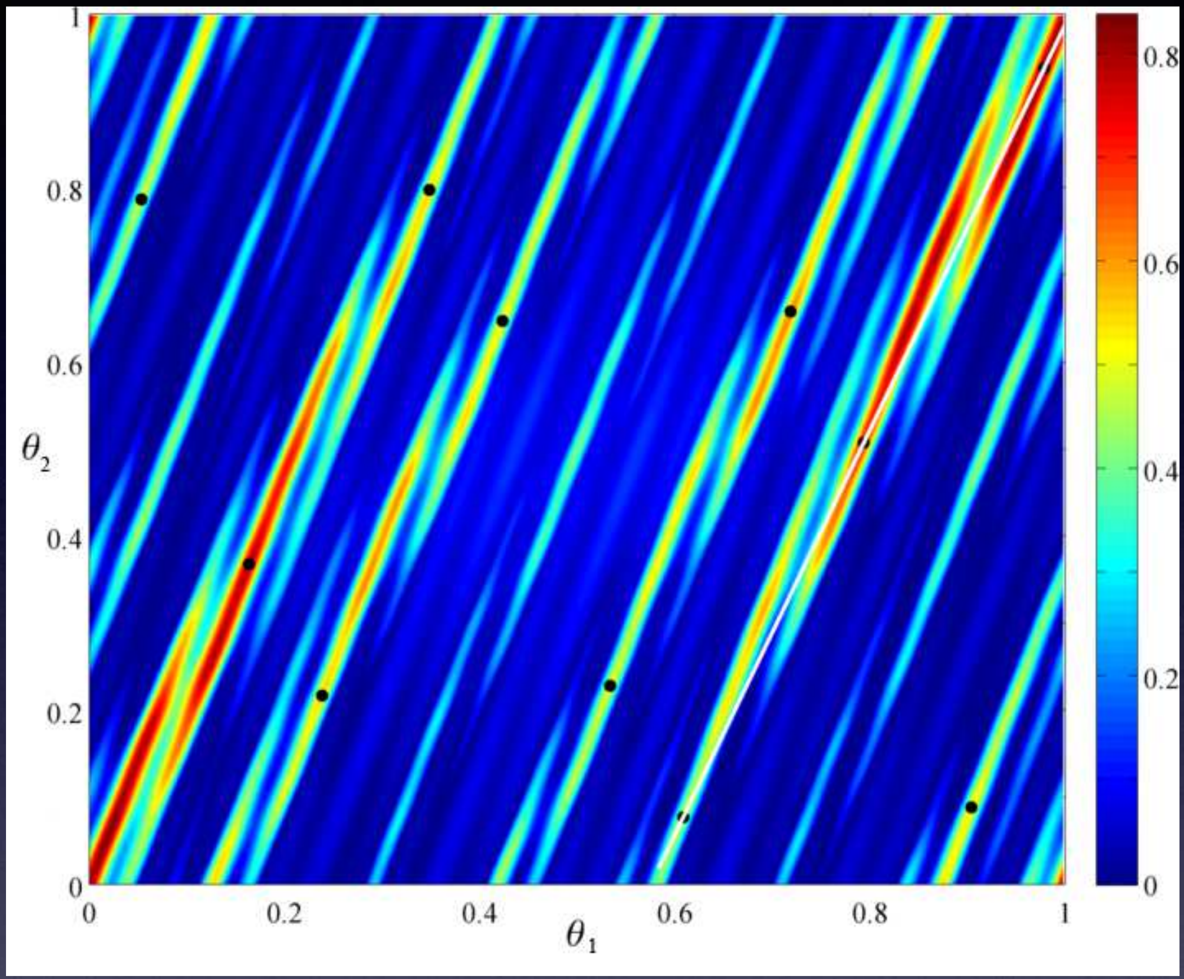

$\varepsilon = 0.025 \quad 2^9 \times 2^9$



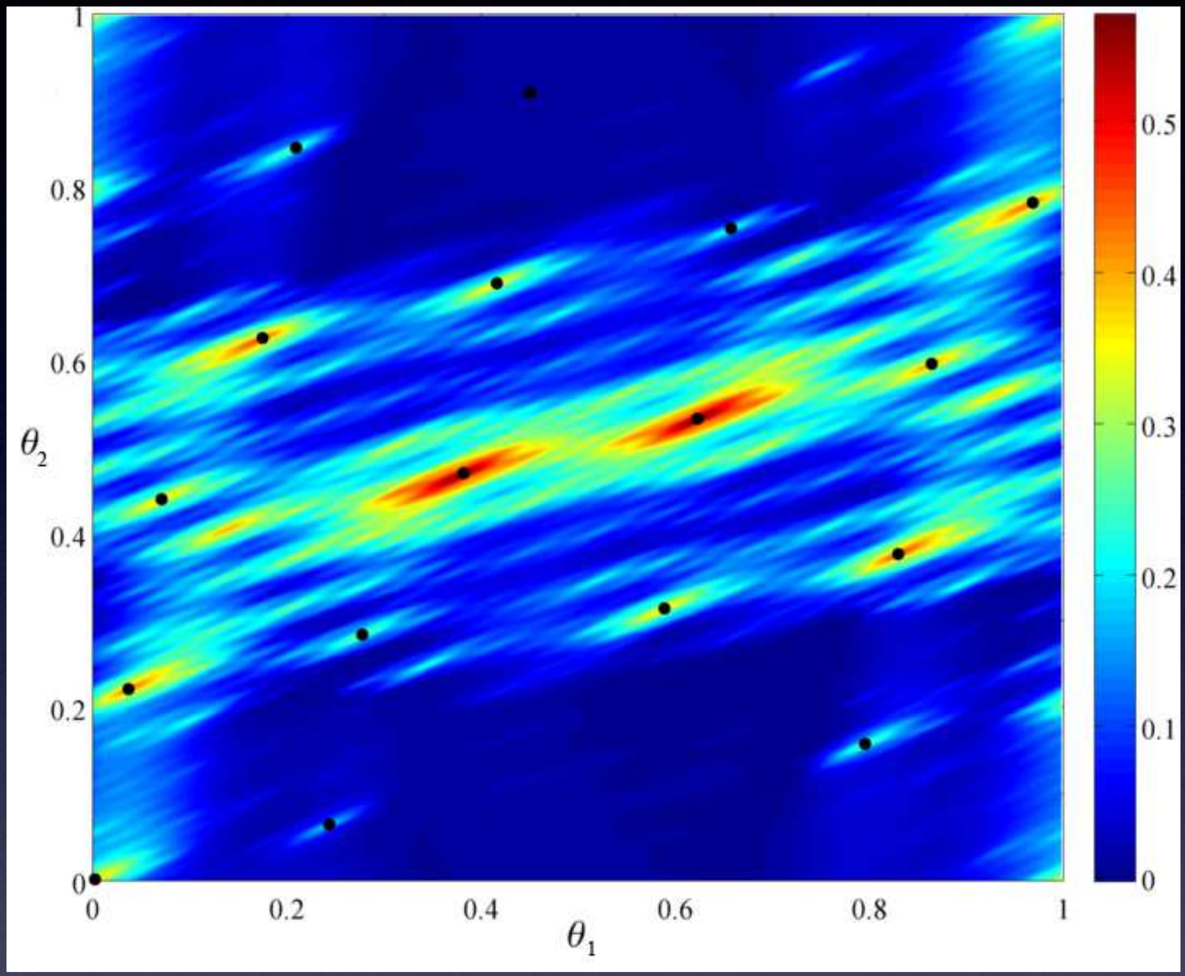



# Critical Tori




A spiral mean torus *llrl̲* ε = 0.0117

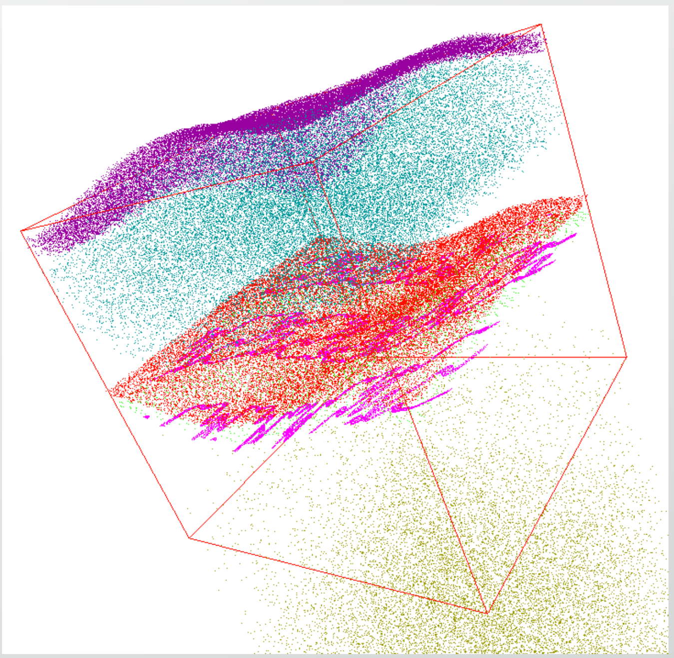



A spiral mean torus *llrrlr̲* ε = 0.0168125

- Compute largest singular value of the Jacobian *Dk* of the conjugacy
  $k: (\theta_1, \theta_2) \rightarrow (x_1, x_2, y)$
  - Stripes vs spires:

Are there remnant tori: Sierpinski or Cantor?

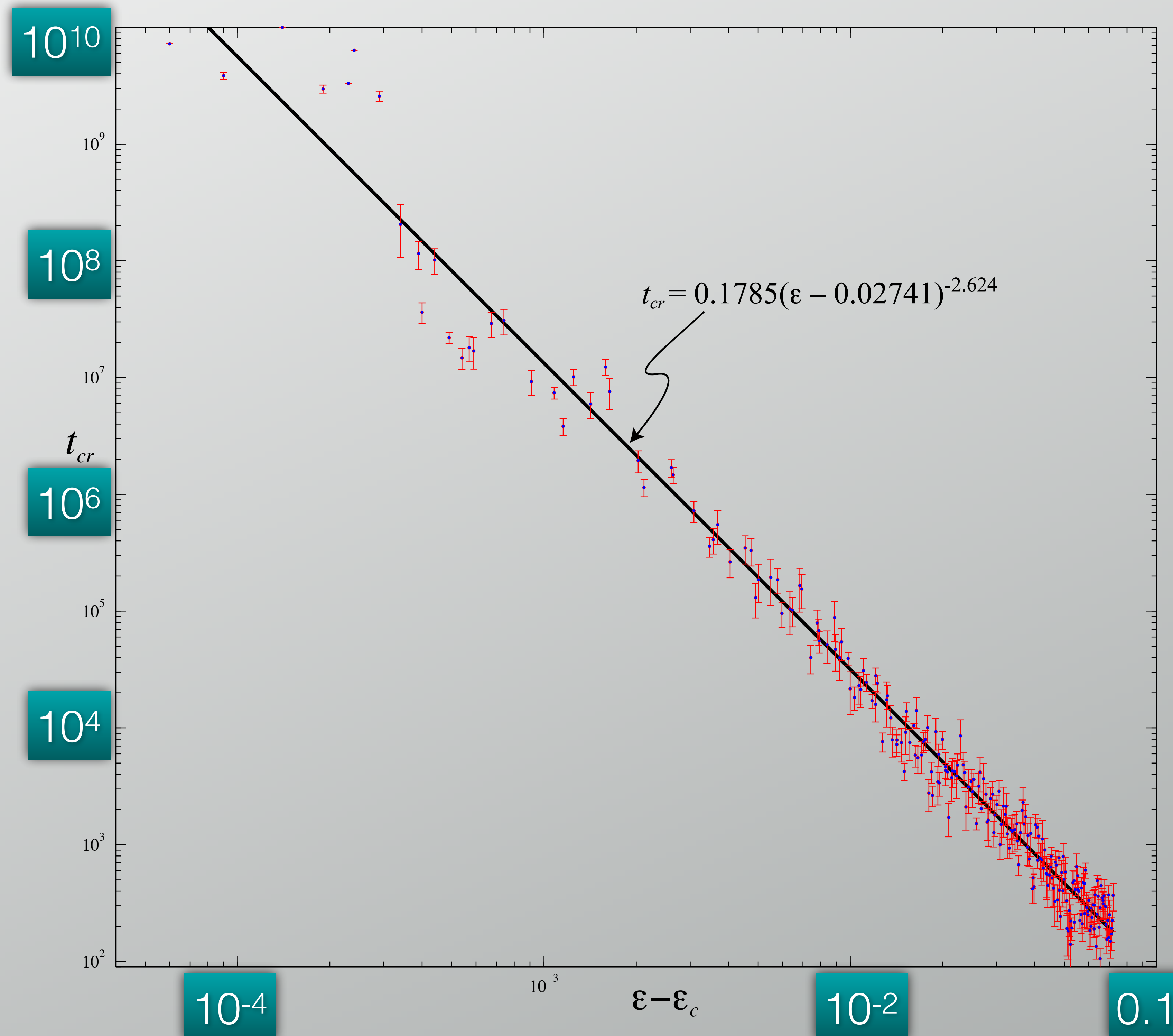
# Destruction ⇒ Transport



Crossing Time vs. $\varepsilon - \varepsilon_{cr}$

$\delta = 0.1$

$\varepsilon_{cr} \approx 0.02741$




Meiss, J. D. (2012). “The Destruction of Tori in Volume-Preserving Maps.” Comm. Nonl. Sci and Num. Sim. 17: 2108-2121.

# Fight of the Century: Drift vs. Nekhoroshev

# Nekhoroshev Theory

- Near integrable Symplectic Map ($\varepsilon << 1$)

$$x' = x + \nabla S(y') \mod 1$$
$$y' = y - \epsilon \nabla V(x)$$

  - *S, V* analytic, *S* convex (though *steep* is sufficient)

- the actions do not drift far in exponentially long times:

$$\|y_t - y_0\| \leq c\varepsilon^{\alpha} \qquad t \leq T \exp{(c/\varepsilon)}^{\beta}$$

Guzzo, M. (2004). “Nekhoroshev Theorem for Symplectic Maps.”
Ann. Poincaré 5(6): 1013-1039.
Lochak, P. (1992). “Canonical Perturbation Theory via Simultaneous Approximation.”
Rus. Math. Sur. 47(6): 59-140.

# Is there Nekhoroshev for Volume-Preserving Maps?

Guillery, N. and J. D. Meiss (2017). "Diffusion and Drift in Volume-Preserving Maps." Reg. and Chaotic Dyn. 22(6): 700-720.

# Near-Integrable 4D Map

$$x' = x + y' \mod 1$$
$$y' = y + F(x)$$

two angles

two actions

$m = n = 2$

- Froeschlé-like forces

$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) & + & c\sin(2\pi(x_1+x_2)) \\ b\sin(2\pi x_2) & + & c\sin(2\pi(x_1+x_2+\varphi)) \end{pmatrix}$$

  - $\varphi = 0$: Symplectic since $F = -\nabla V$
  - $\varphi = ½$: "maximally non-symplectic" coupling

- Full Spectrum Force:

$$F_{fs} = -\frac{1}{2\pi}\frac{d}{(2.1+\cos(2\pi x_1)+\cos(2\pi x_2))^2}\begin{pmatrix} \sin(2\pi x_1) \\ \sin(2\pi(x_2+\varphi)) \end{pmatrix}$$

# Near-Integrable 4D Map

$$x' = x + y' \mod 1$$
$$y' = y + F(x)$$

*two angles*

*two actions*

*m = n = 2*

- Froeschlé-like forces

$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) & + & c\sin(2\pi(x_1+x_2)) \\ b\sin(2\pi x_2) & + & c\sin(2\pi(x_1+x_2+\varphi)) \end{pmatrix}$$

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# Visualizing the Dynamics

# Fast Lyapunov Indicator

- Iterate arbitrarily chosen initial deviation $v_0$
- Compute the supremum up to time $T$

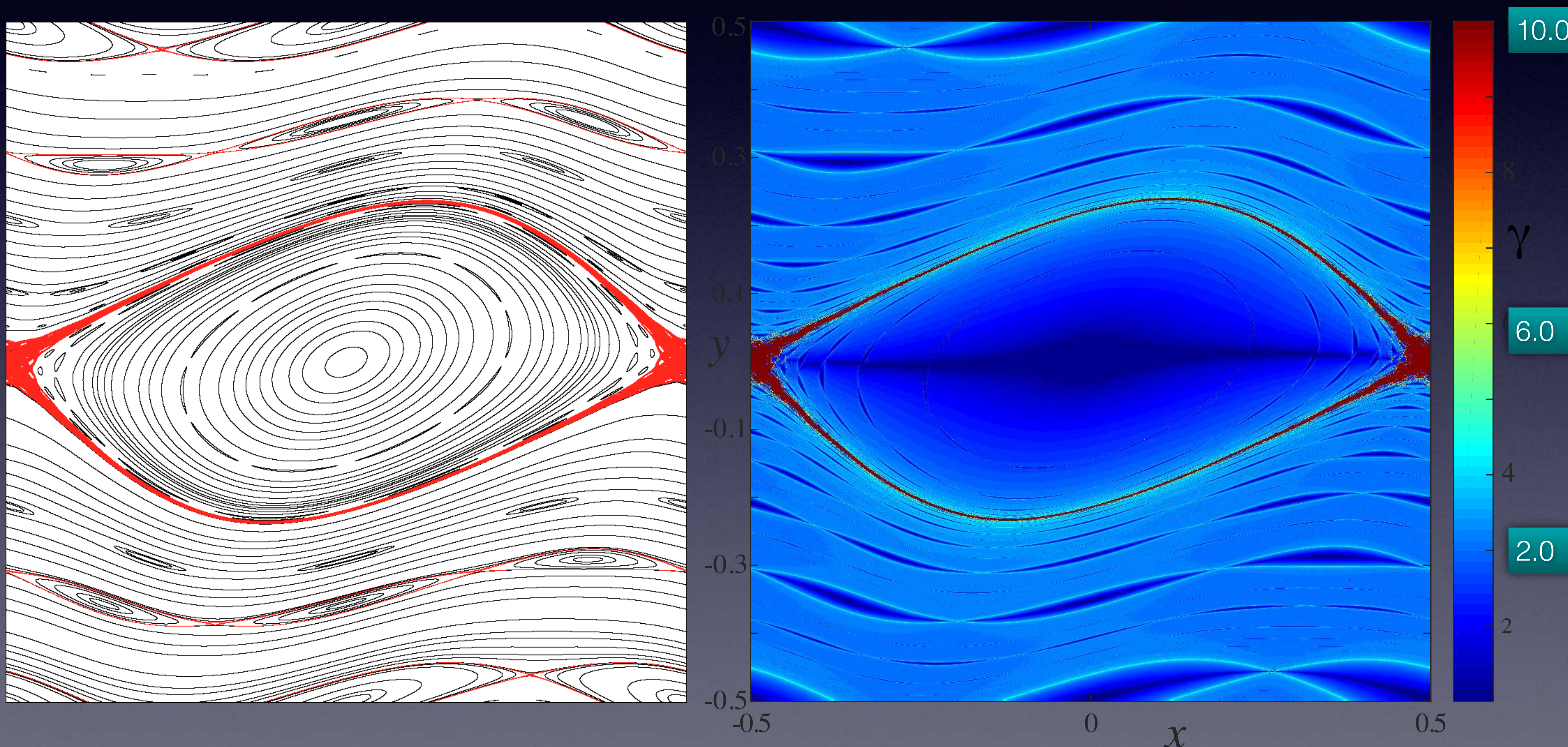
$$FLI = \sup_{t<T} \left( \log_{10} \| Df^t(x_0) v_0 \| \right)$$

  - similar to FTLE, but supremum reduces oscillations

Froeschle, C., R. Gonczi and E. Lega (1997). "The fast Lyapunov indicator: a simple tool to detect weak chaos. Planetary and Space Science 45(7): 881-886.

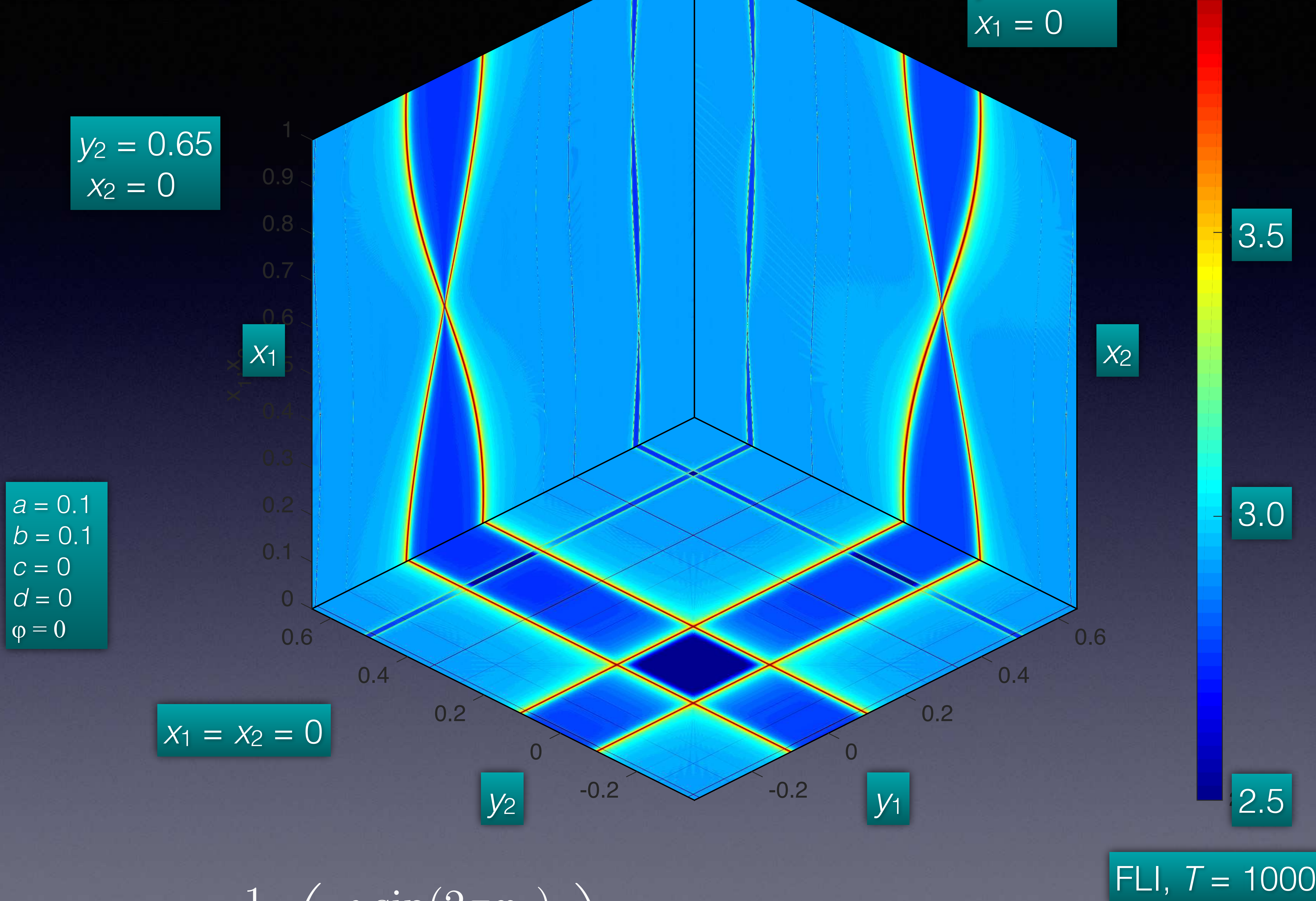
# FLI: Standard Map

$$FLI = \sup_{t<T} \left( \log_{10} \| Df^t(x_0)v_0 \| \right)$$



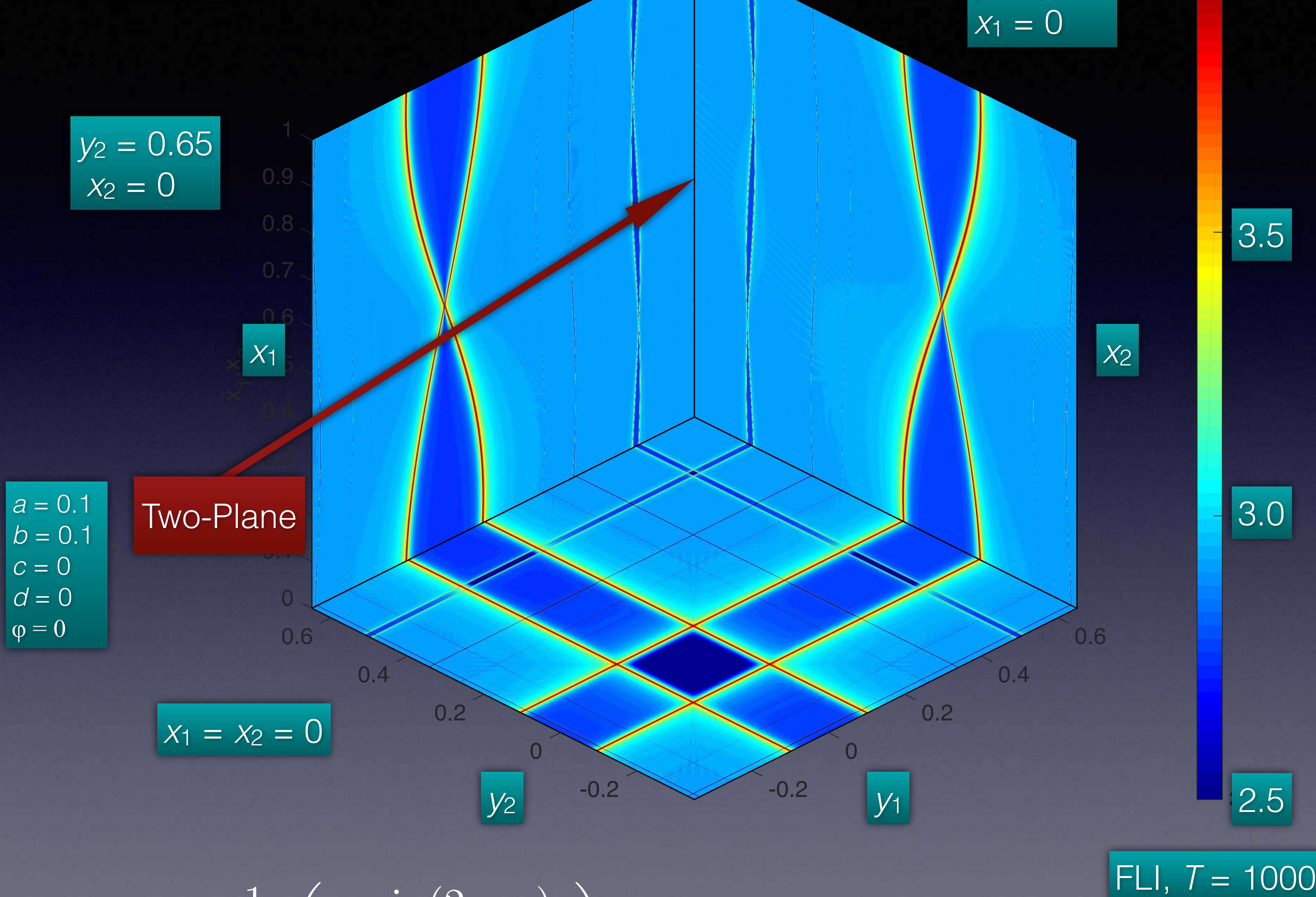

- $a = 0.52$, $T = 10^3$, grid of $10^6$ points

# Uncoupled Slices



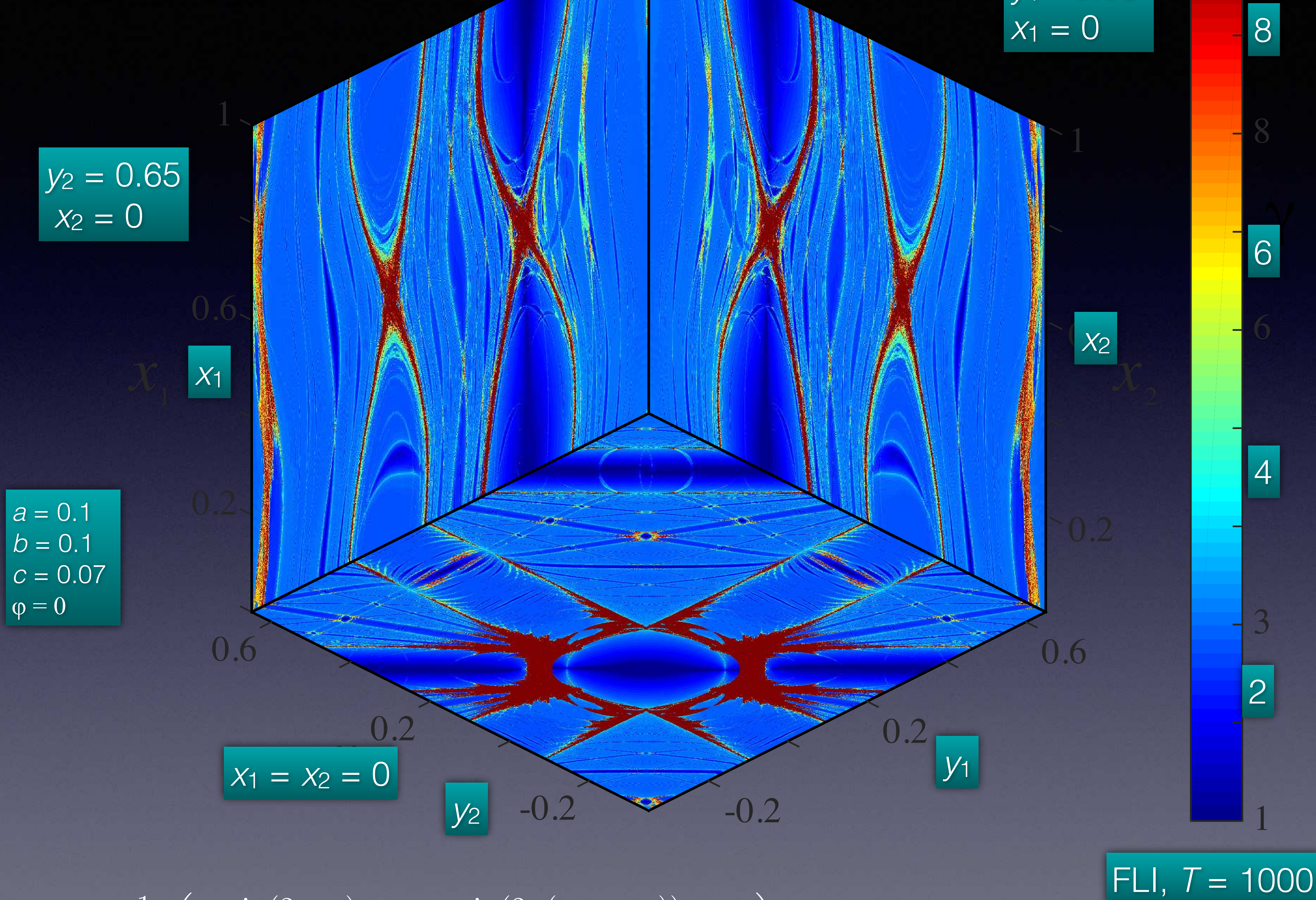

$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) \\ b\sin(2\pi x_2) \end{pmatrix}$$

# Uncoupled Slices



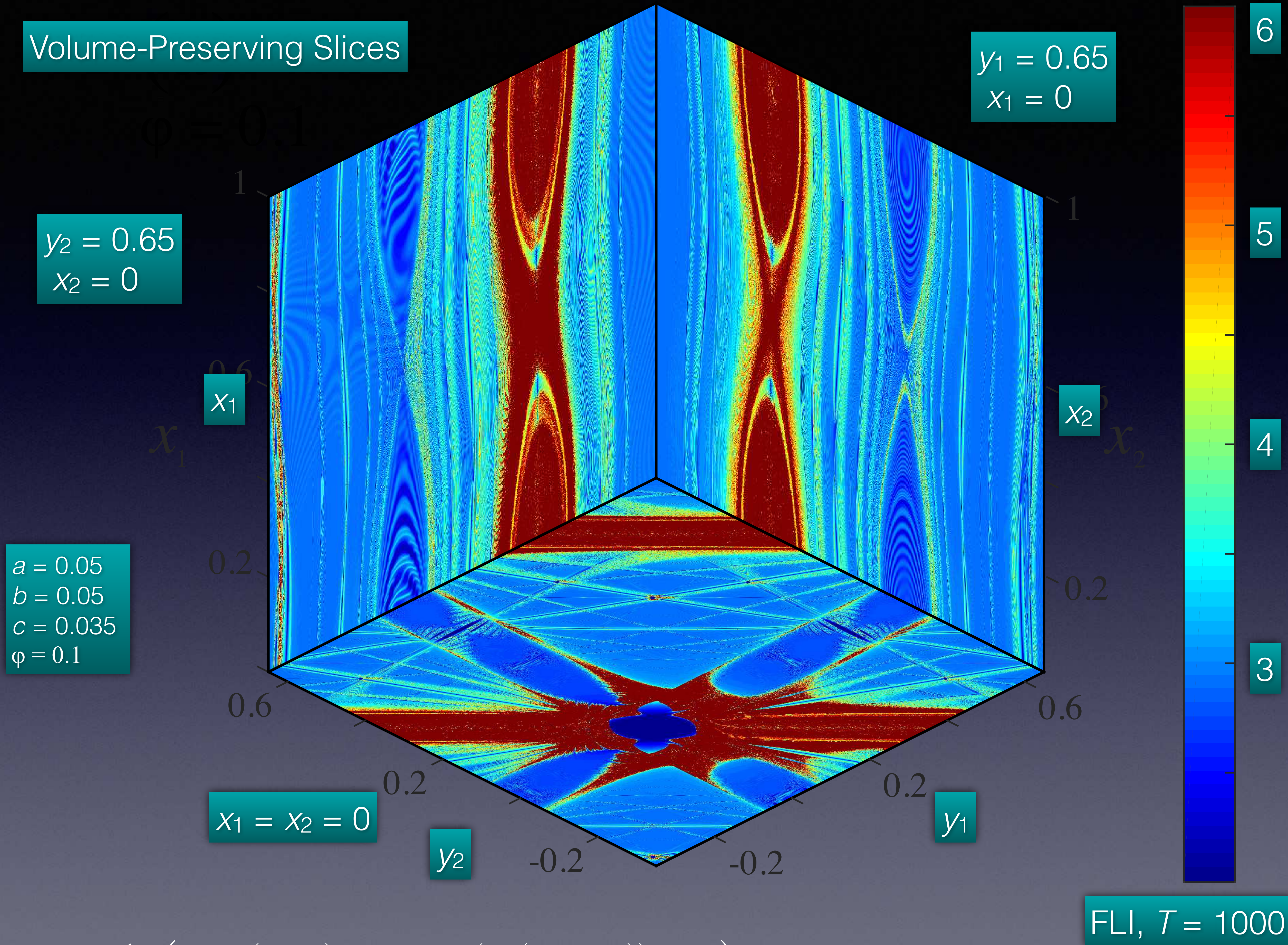

$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) \\ b\sin(2\pi x_2) \end{pmatrix}$$

# Symplectic Slices




$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) & + & c\sin(2\pi(x_1+x_2)) \\ b\sin(2\pi x_2) & + & c\sin(2\pi(x_1+x_2+\varphi)) \end{pmatrix}$$

# Volume-Preserving Slices




$$F = -\frac{1}{2\pi}\begin{pmatrix} a\sin(2\pi x_1) & + & c\sin(2\pi(x_1+x_2)) \\ b\sin(2\pi x_2) & + & c\sin(2\pi(x_1+x_2+\varphi)) \end{pmatrix}$$

# Resonance Web

- Near resonance $\Omega(y) = y$

$$y = y^* + \delta y \qquad m \cdot y^* = n$$

$$F(x) = F_R(m \cdot x) + F_{NR}(x)$$

$$F(x) = \sum_{j \in \mathbb{Z}^n} \hat{F}_j e^{2\pi i j \cdot x}$$

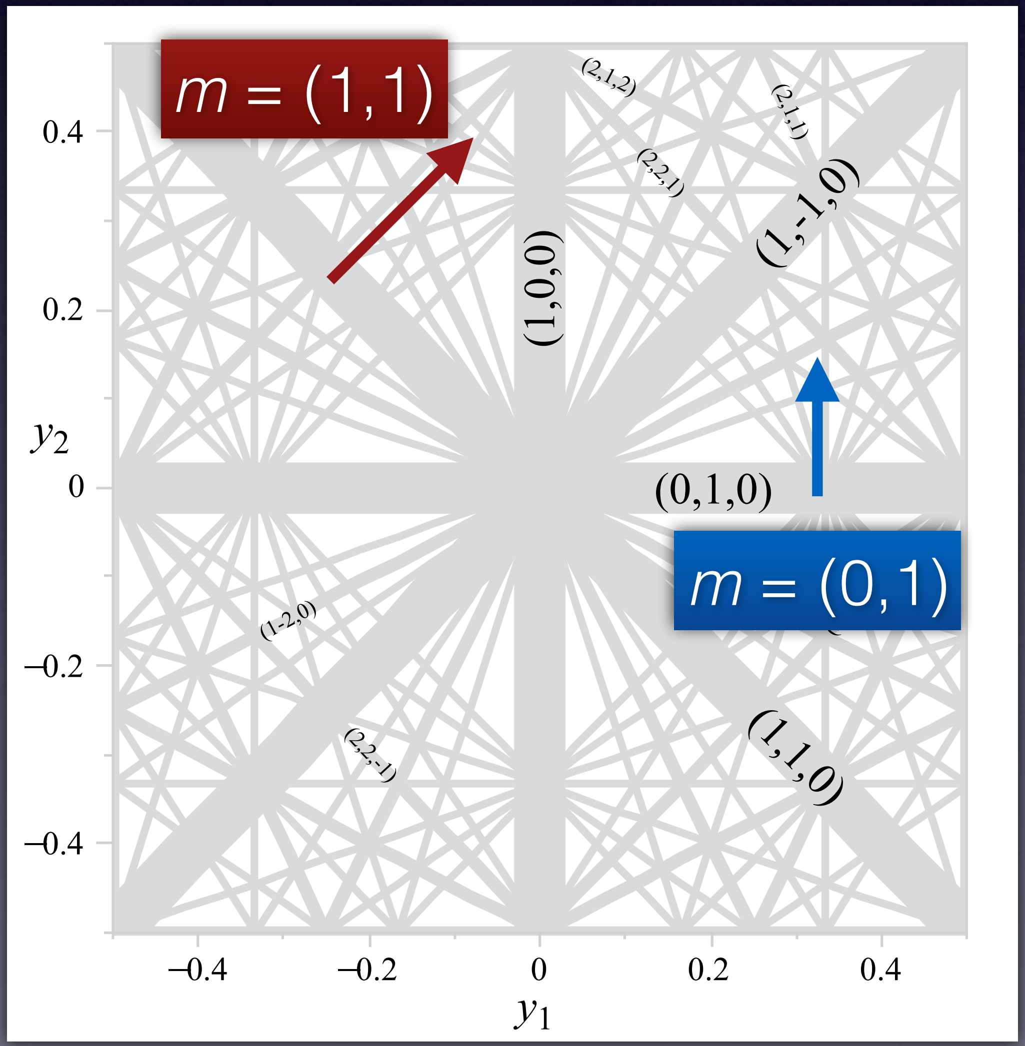
  - Note: $m$ orthogonal to resonance channel

- Resonant Phase and action

$$\psi = m \cdot x$$

$$J_R = m \cdot \delta y$$

$$J_{||} = m_\perp \cdot \delta y$$




$$|m \cdot \Omega - n| < \frac{c}{\|m\|^2}$$

# Resonance Web

$$\psi = m \cdot x$$
$$J_R = m \cdot \delta y$$
$$J_{\|} = m_{\perp} \cdot \delta y$$

- Average away nonresonant forces $F_{NR}$

$$\left.\begin{aligned} \psi' &= \psi + n + J_R' \\ J_R' &= J_R + m \cdot F_R(\psi) \end{aligned}\right\} \text{ 2D Area-preserving map}$$
$$J_{\|}' = J_{\|} + m_{\perp} \cdot F_R(\psi)$$

- For Symplectic case

$$F_R = -\nabla V(m \cdot x) = -mV'(\psi)$$

- Action along channel is approximate invariant!

$$J_{\|}' = J_{\|}$$

# Resonance Web

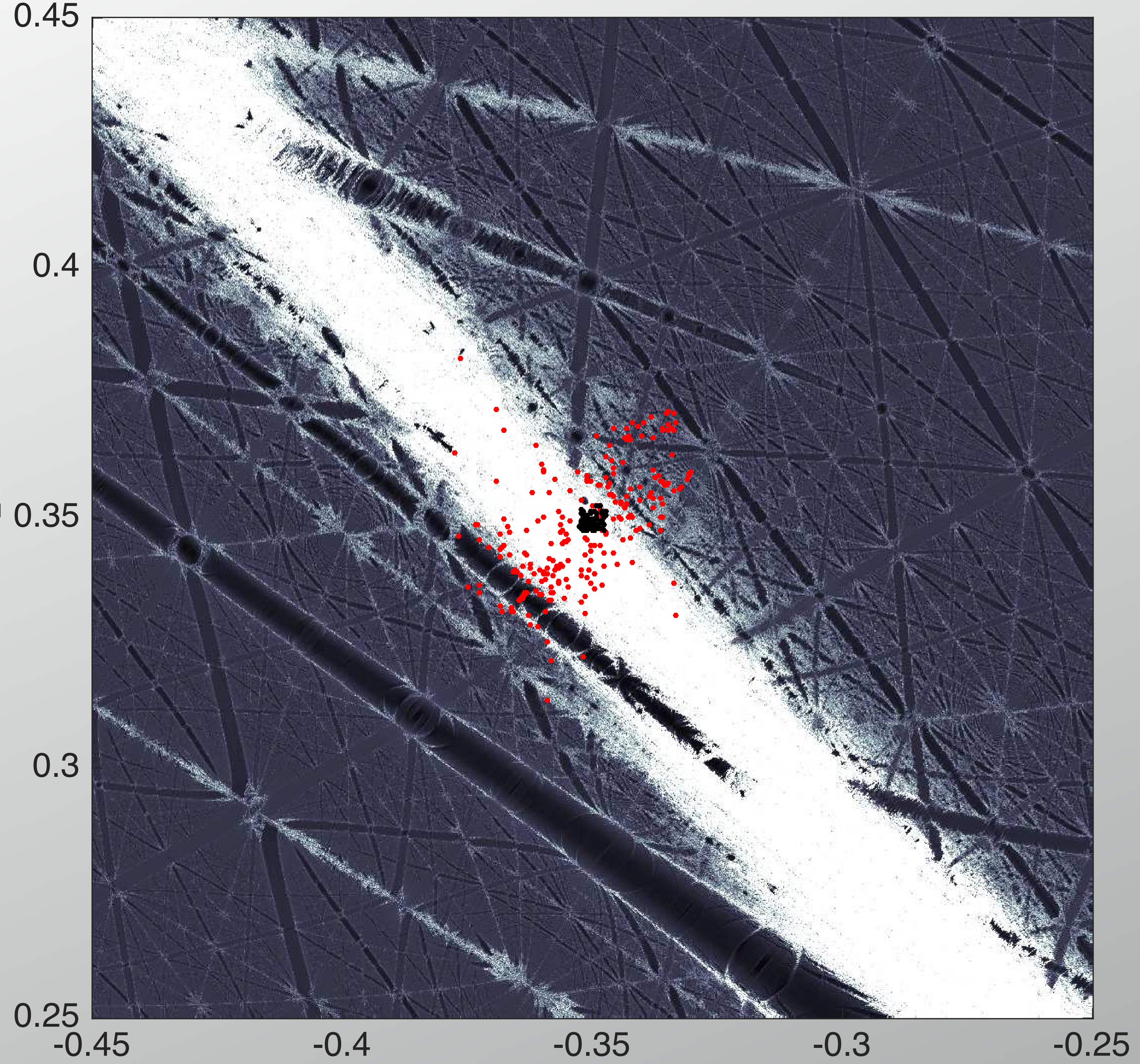
- But for a Volume-Preserving map, $J_{||}$ can be driven!

$$\psi' = \psi + n + J'_R$$
$$J'_R = J_R + m \cdot F_R(\psi)$$
$$J'_{||} = J_{||} + m_\perp \cdot F_R(\psi)$$

# Drifting Orbits

Symplectic: $(a,b,c,d) = (0,0.1,0.07,0.0001)$

$T = 10^6$




$y_2$

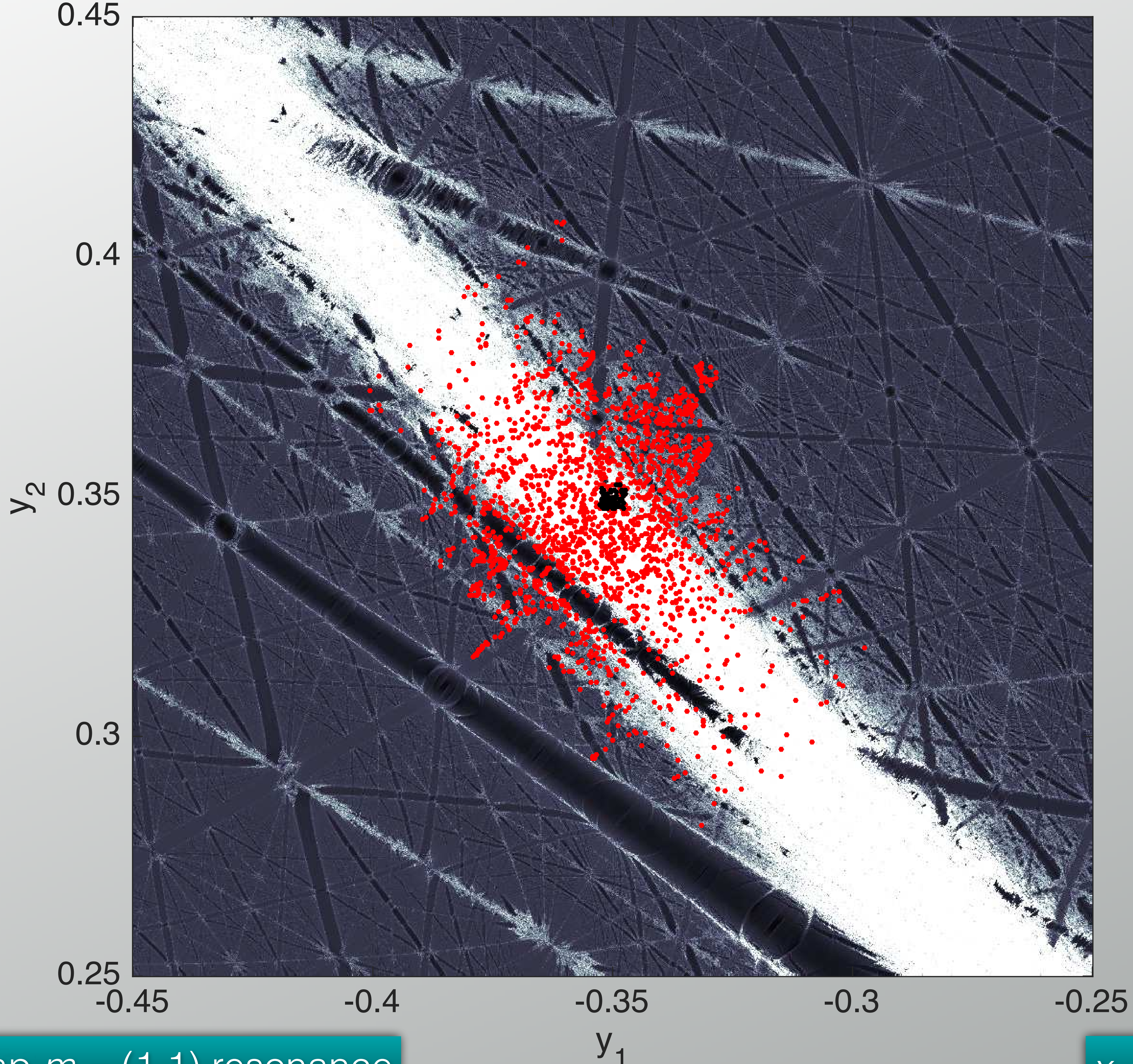
$y_1$

Symplectic Map $m = (1,1)$ resonance

$x = (0,0.5)$ slice

Symplectic: $(a,b,c,d) = (0,0.1,0.07,0.0001)$

$T = 10^7$



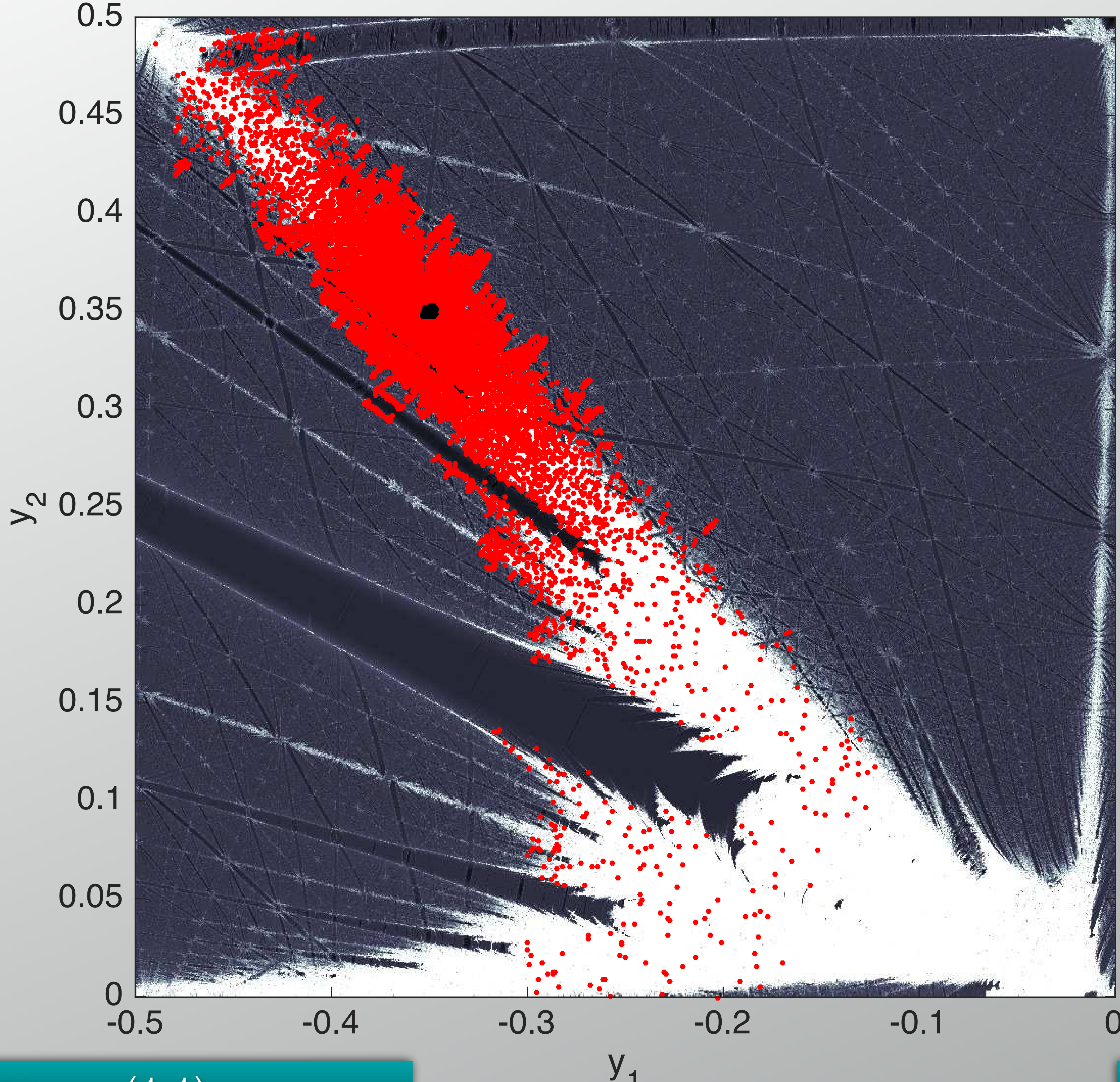

Symplectic Map $m = (1,1)$ resonance

$x = (0,0.5)$ slice

Zooming Out!

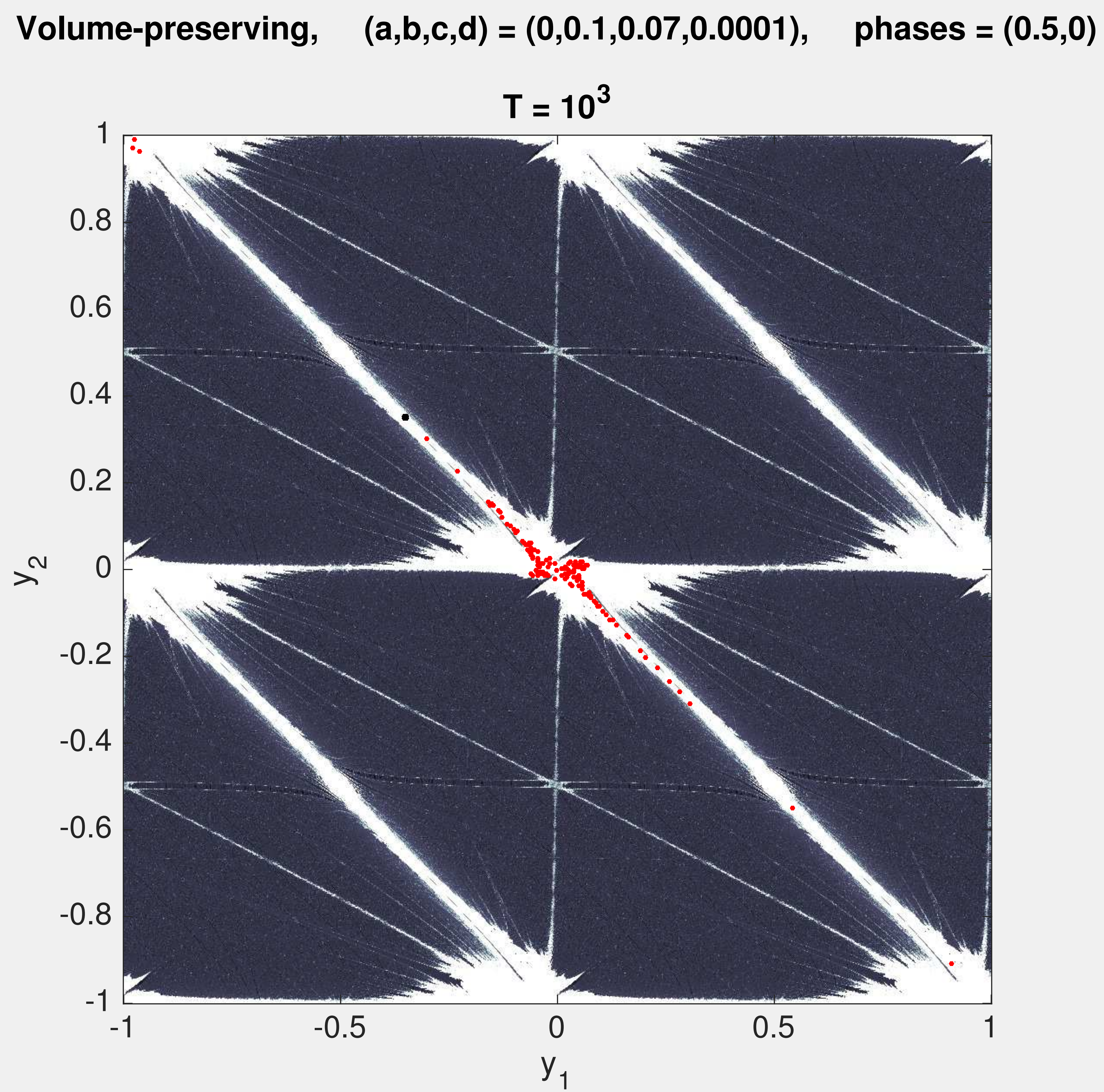
Symplectic: (a,b,c,d) = (0,0.1,0.07,0.0001)

$T = 10^8$




Symplectic Map $m$ = (1,1) resonance

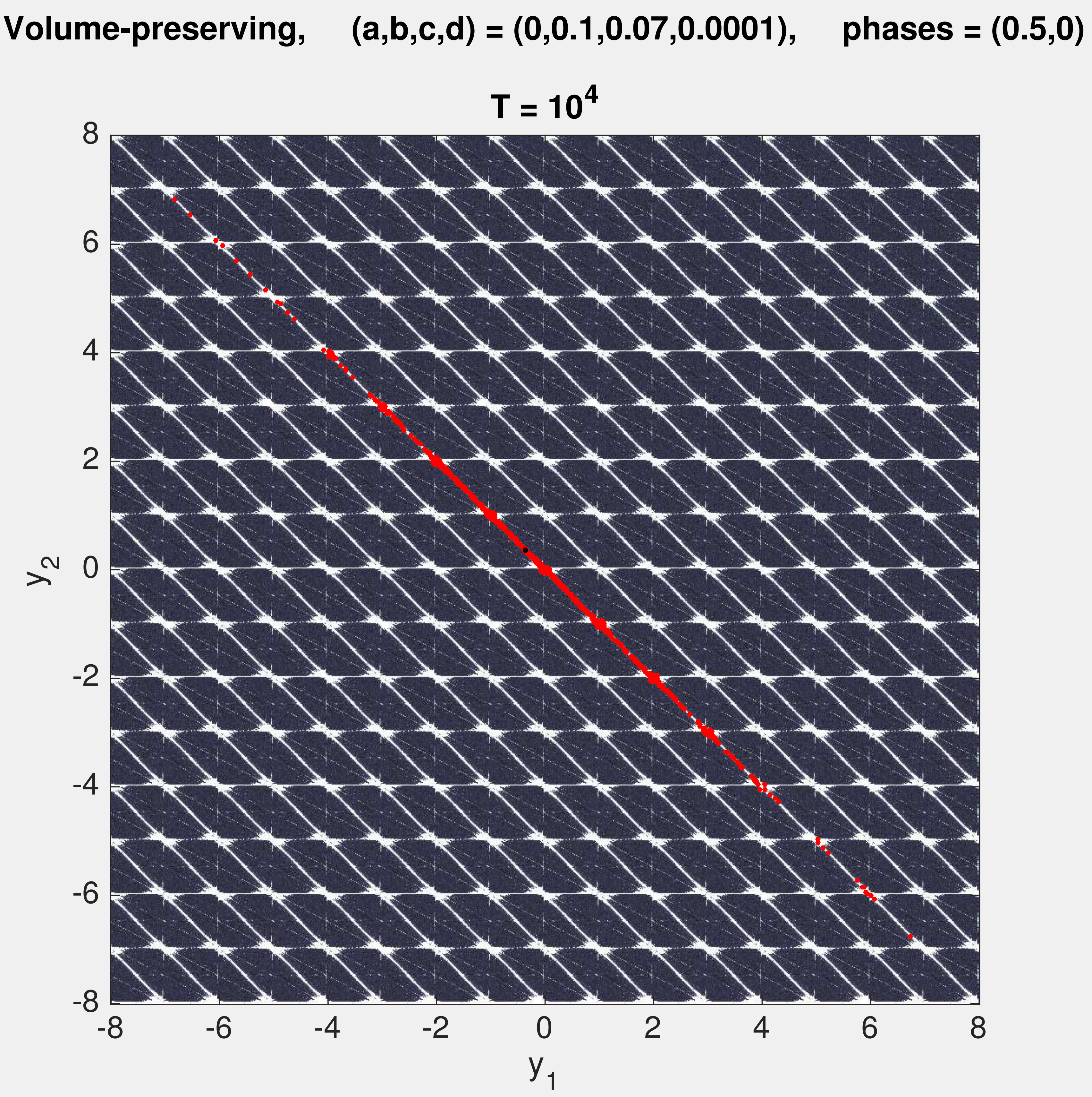
$x$ = (0,0.5) slice




Volume-Preserving Map $m$ = (1,1) resonance

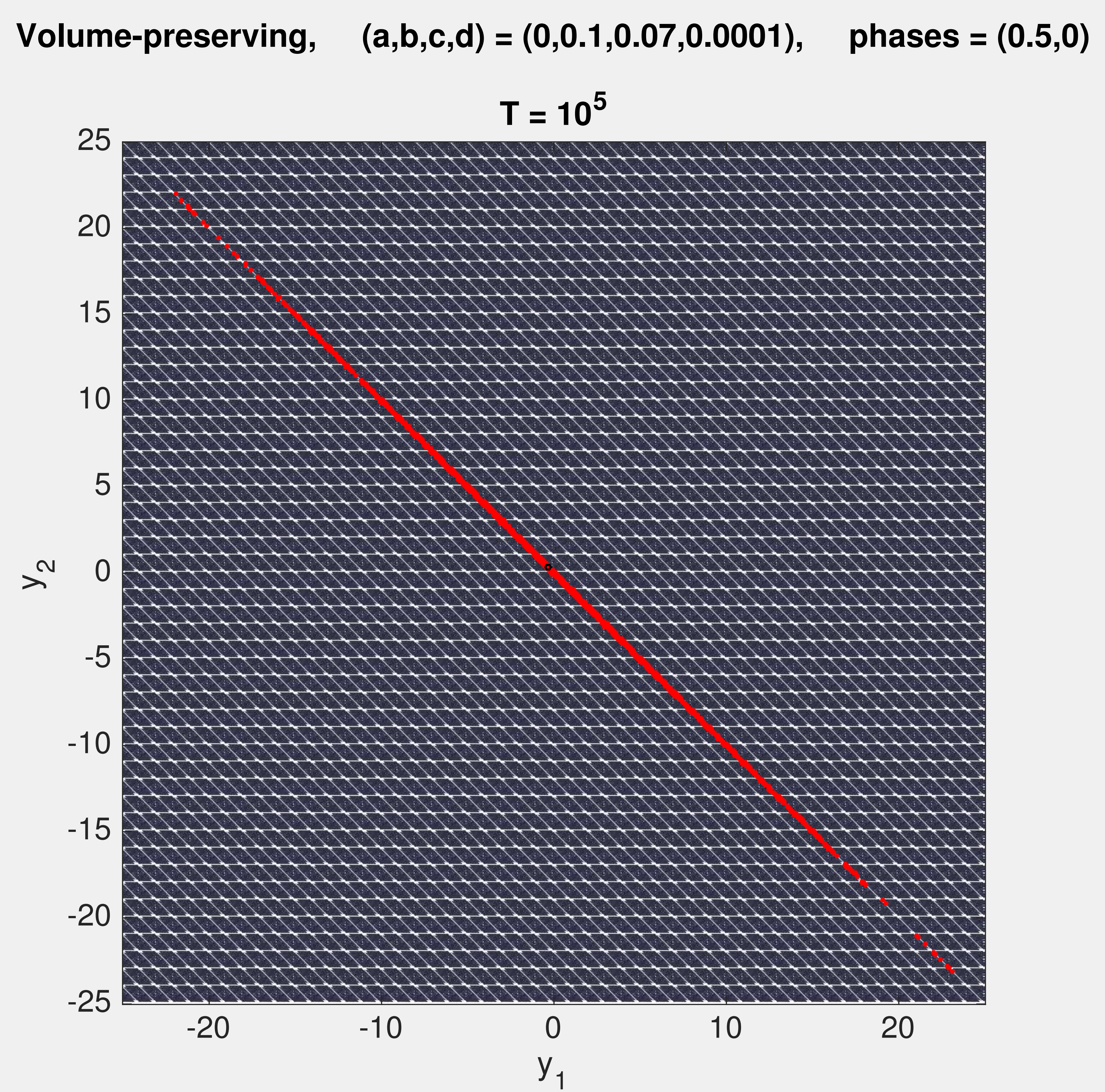
$x$ = (0,0.5) slice

# Zooming Out!




Volume-Preserving Map $m = (1,1)$ resonance

# Zooming Way Out!

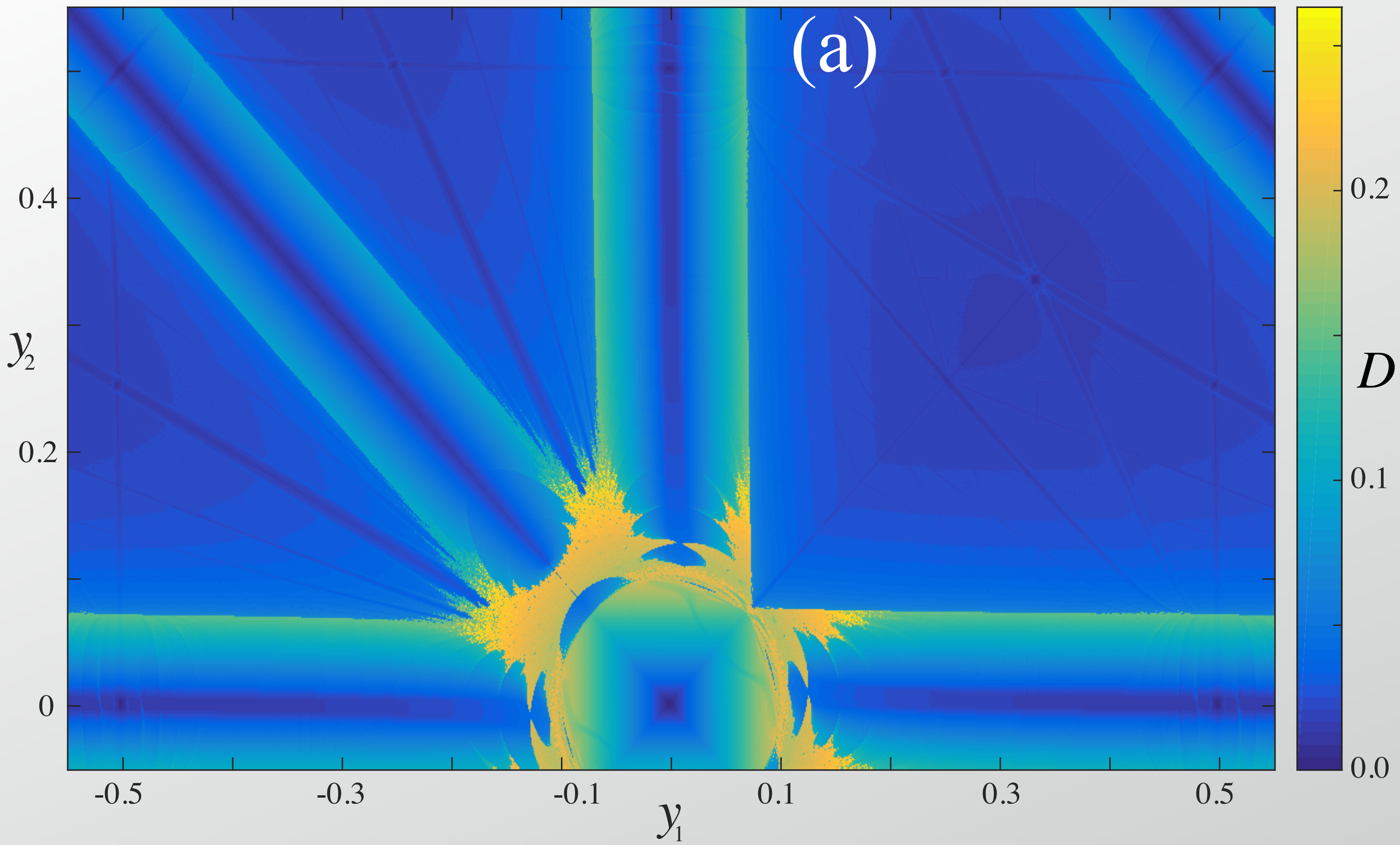



Volume-Preserving Map $m = (1,1)$ resonance

$x = (0,0.5)$ slice

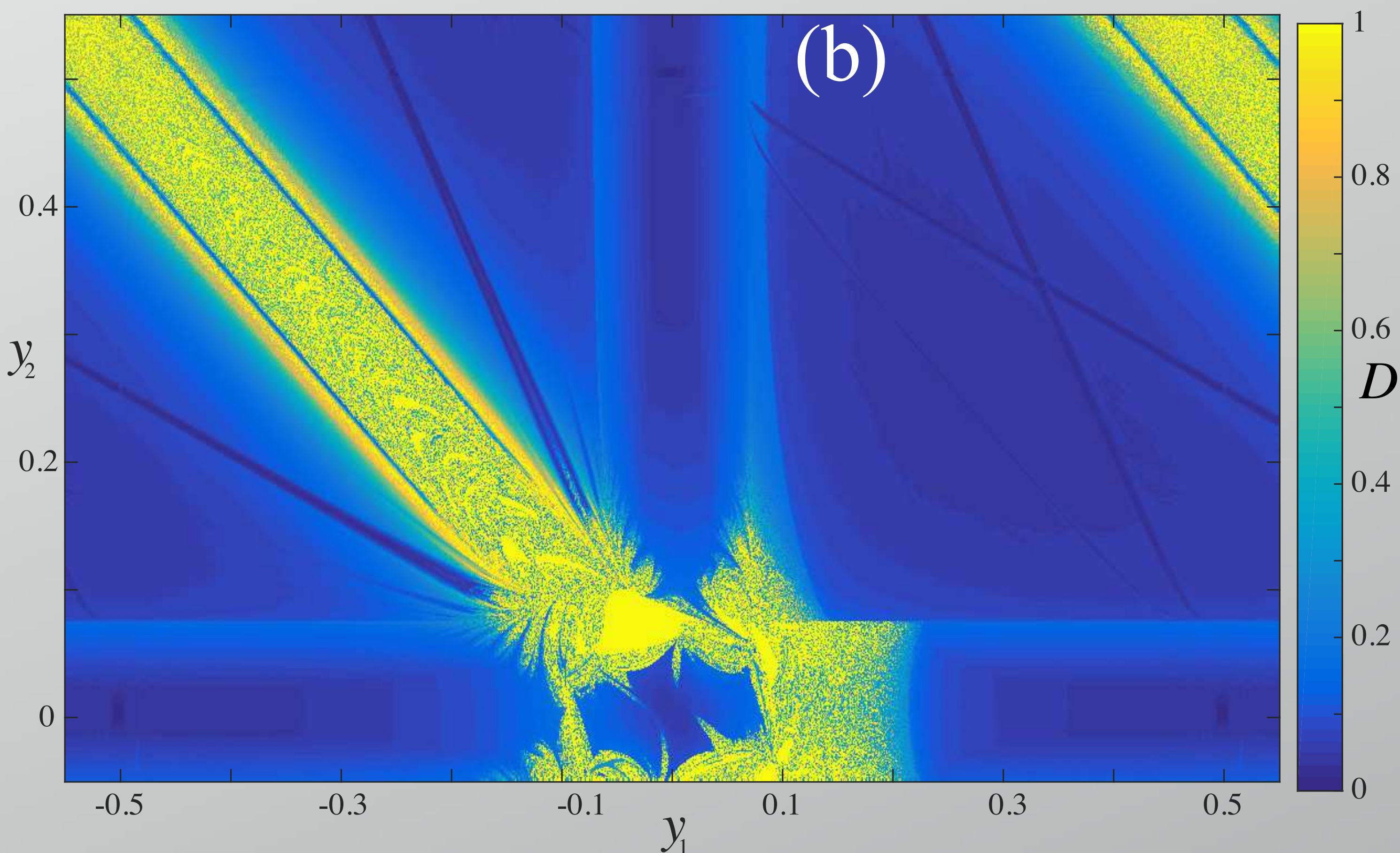
# Action Diameter

(a,b,c) = (0.05, 0.05, 0.035)
T = $10^3$



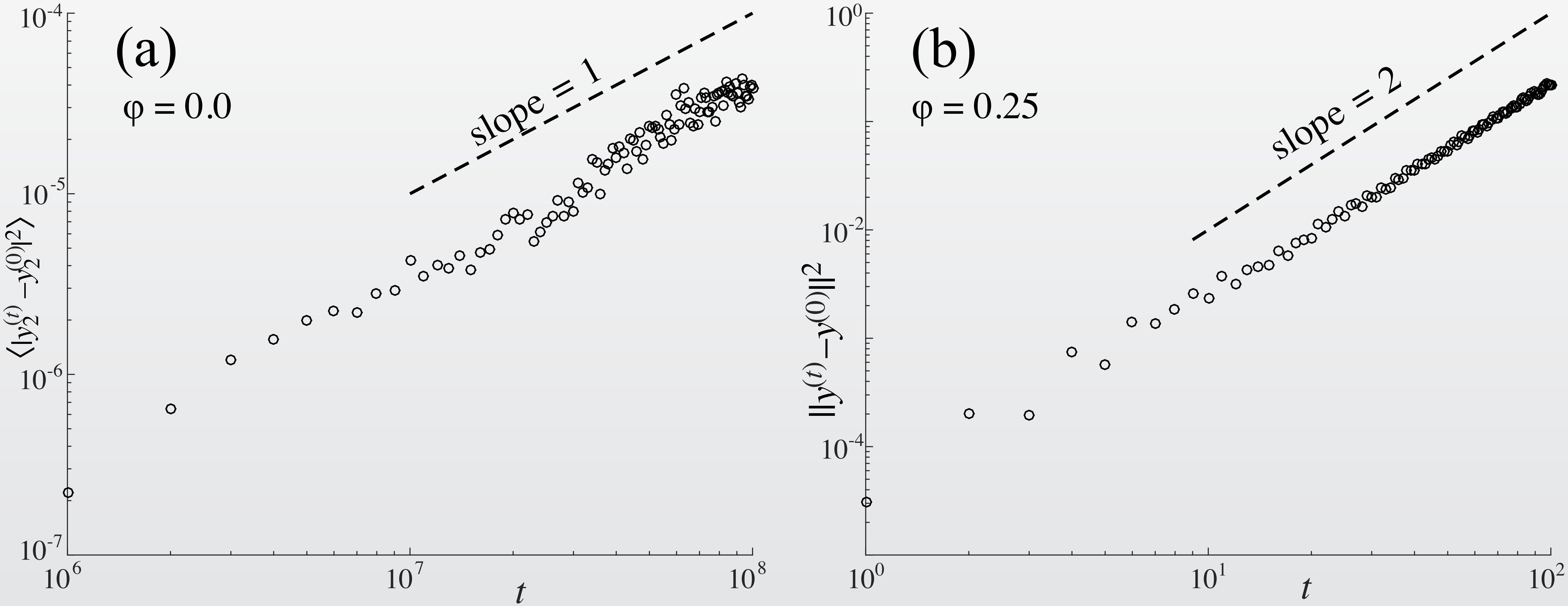

$\varphi = 0.0$

$$D = \sup_{0<t,s,<T} \|y^{(t)} - y^{(s)}\|$$




$\varphi = 0.25$

# Mean Square Deviation: Diffusion vs Ballistic Drift

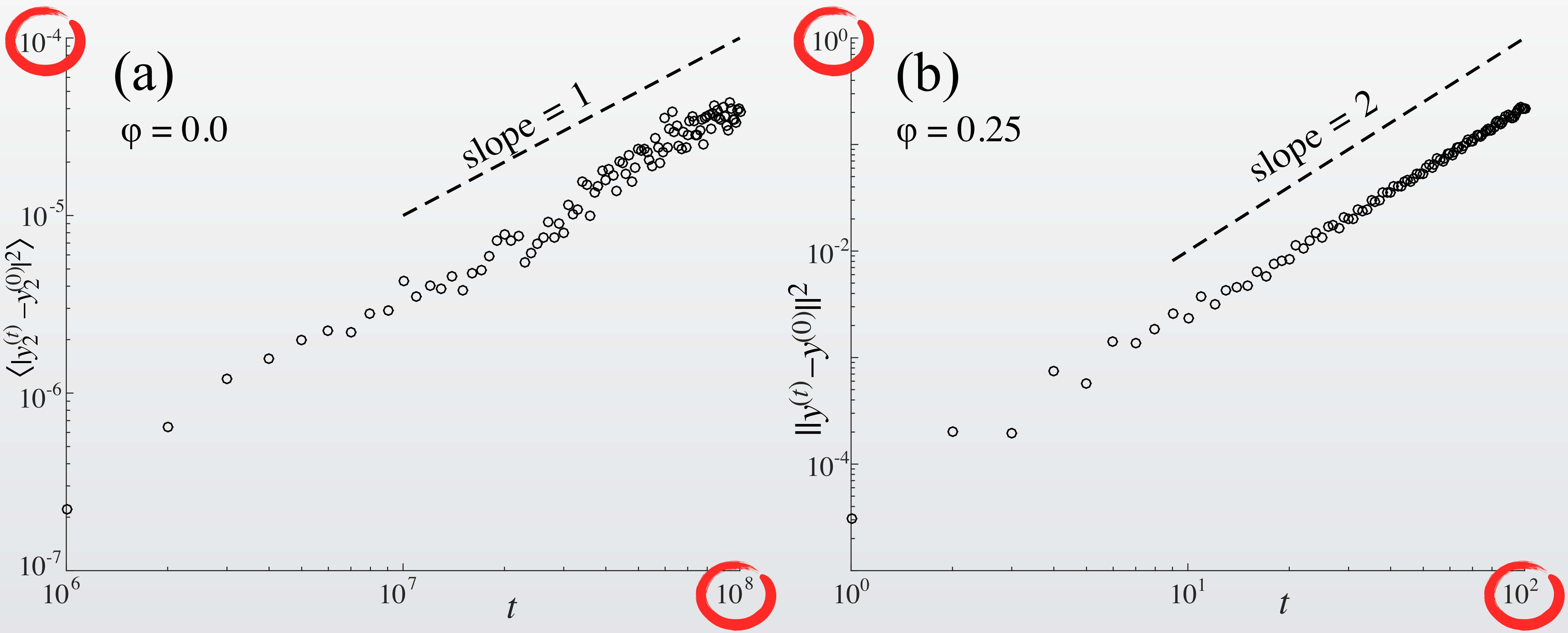



$(x_0, y_0) = (0.5, 0, 0, 0.35)$

$(x_0, y_0) = (0, 0, -0.45, 0.45)$

$(a, b, c) = (0.05, 0.05, 0.035)$

# Mean Square Deviation: Diffusion vs Ballistic Drift



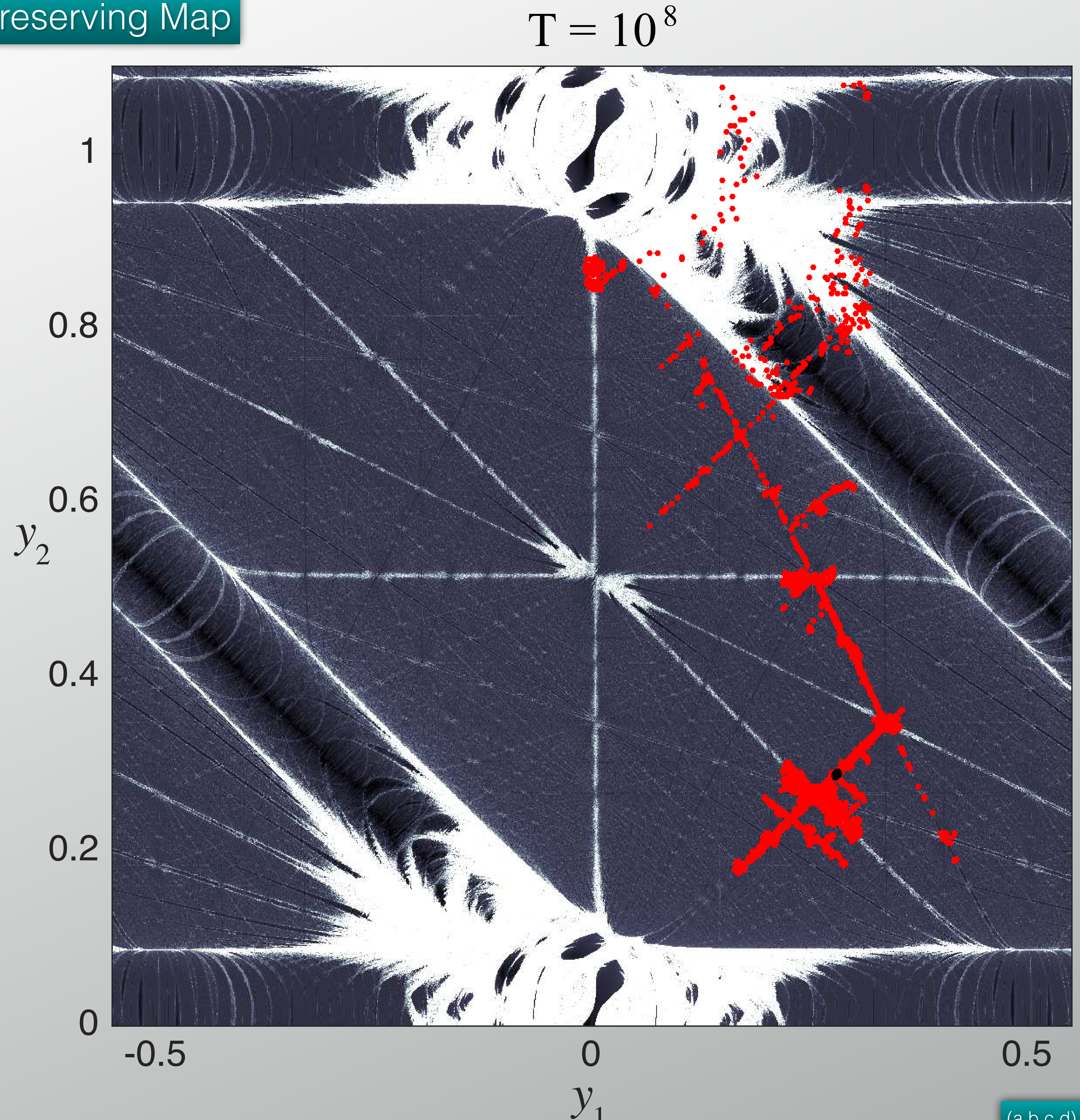

$(x_0, y_0) = (0.5, 0, 0, 0.35)$

$(x_0, y_0) = (0, 0, -0.45, 0.45)$

$(a, b, c) = (0.05, 0.05, 0.035)$

# Crossing Resonances

## Volume-Preserving Map




$x$=(0,0.25) slice

(a,b,c,d) = 0.0,0.1,0.07, 0.0001, ϕ = (0, 0..5)

This round goes to Drift!

**Thanks for your attention**

# More about VP Maps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L1

# Drift and Diffusion in Symplectic and Volume-Preserving Maps

James Meiss    Notes by Jeffrey Heninger

Hamiltonian flow is symplectic – convert this to a symplectic map
  preserve symplectic form / Poincaré loop action

Standard (Chirikov – Froeschlé) form
  x – coordinates, y – momenta, $\Omega$ – frequency map, F – force
  $x' = x + \Omega(y')$ new momentum   ← these have to be gradient   $\Omega = \nabla S(y)$
  $y' = y + F(x)$   $F = -\nabla V(x)$

volume-preserving maps
  number of angles and actions can be different
  angles – $x \in \mathbb{T}^n$   actions – $y \in \mathbb{R}^m$
why look at these?
  mixing in incompressible fluids $\nabla \cdot v = 0$ – look at advection of dye
  magnetic fields $\nabla \cdot B = 0$

what are differences between area and volume preserving maps?
  3D volume preserving is simpler than 4D symplectic

curl forces in optics with polarizable media
  Poincaré maps are volume preserving, but not symplectic

Two Angle – One Action Normal Form:   (near a rank 1 resonance)
  $x_1' = x_1 + y' + \gamma$   $\Omega = (\Omega_1(y), \Omega_2(y))$
  $x_2' = x_2 + \beta y'^2 - \delta$
  $y' = y - \varepsilon\left(\sin(2\pi x_1) + \sin(2\pi x_2) + \sin(2\pi(x_1 - x_2))\right)$
The limit $\varepsilon \to 0$ is an integrable limit

## Invariant Tori

Resonances: frequency vector $\omega \in \mathbb{R}^n$, resonant if $m \cdot \omega = n$, $(m,n) \in \mathbb{Z}^{n+1} \setminus \{0\}$
  rank 1 – along one of lines
  rank 2 – intersection of 2 resonant lines
  nonlinearity "fattens" resonances
  Diophantine condition to stay away from fattened resonances

Frequency Map
  tori map exists when frequency map goes through incommensurate values
  use parameter to move this frequency map around

Volume - Preserving KAM

need Diophantine condition, nondegeneracy condition (like twist)

→ thick Cantor set of invariant tori for small perturbation

pictures of phase space

invariant 2-dim tori ← barrier for dynamics

tubes - rank 1 resonances

chaotic regions

finding invariant tori - using parametrization method

fixed rotation vector

additional parameters needed

how do we choose a good frequency vector?

should be Diophantine integral basis of algebraic fields

noble irrationals (golden mean) → most robust two torus

look at cubic fields - which ones?

spiral mean $\sigma^3 = \sigma + 1$ $\omega = (\sigma, \sigma^2)$

totally real field $\tau^3 + \tau^2 - 2\tau - 1 = 0$ $\omega = (\tau, \tau^2)$

we don't know which ones generate most robust tori

use spectral method to check robustness of tori with different frequency vectors

no clear results yet

as tori get closer to break up, they get more stretched

high stretching only along stripes for some tori - after breakup, Cantor × circles ?

high stretching only in spots for other tori - after breakup, Sierpinski carpet ?

could later how more topological changes later - no Aubrey-Mather Theory here.

destruction of tori → transport

compare crossing time as we approach critical perturbation

power law

<u>Drift vs. Nekhoroshev</u>

Nekhoroshev theory:

for symplectic maps (near-integrable) - need some conditions on S, V

actions do not drift far for exponentially long times

Is there Nekhoroshev for volume-preserving case?

consider a map with 2 actions and 2 angles

Froeschlé-like forces - add $\varphi$ to make it not a gradient of a scalar

Visualizing Dynamics

fast Lyapunov indicator

(Lyapunov exponents converge slowly

this is similar, but converges must faster

use it to check for regions of chaos - can also see shearing

↳ rotational invariant tori vs. resonances

look at 2-dim slices through 4-D space

2 canonical places $(x_1, y_1)$ and $(x_2, y_2)$ and momenta $(y_1, y_2)$

can see 2 rank 1 resonances - when they cross, there's a rank 2 resonance

1-1 resonance from coupling term

make $\varphi \neq 0$ - not longer symplectic

lots more stretching in 1-1 resonance (where force stopped being a gradient)

Resonance Web

expand near a resonance

average over nonresonant forces

→ system with one phase (and 2 actions)

one angle, one action don't depend on the other action

symplectic case - action along the resonance is approx. invariant ⇒ Nekhoroshev

if force is not a gradient, we can get a force along the resonance direction

in symplectic case, drift is diffusive along resonance channel

in non-symplectic case, drift is ballistic along resonance channel

Crossing Resonances

the previous example only has a simple non-gradient force

for more general forces, drift can switch between different resonance channels