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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Fraydoun Rezakhanlou

Talk Title: Hamiltonian ODE and Hamilton-Jacobi PDE with Stochastic Hamiltonian Function

Date: 11 / 26 / 2018 **Time:** 2 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Does the repeated action of a group action on a Hamiltonian converge to some Hamiltonian that it is invariant under the group action? Use ergodic theory on Hamilton-Jacobi equations to make the space compact. We can extend results (homogenization, Arnold conjectures) for almost all Hamiltonians.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

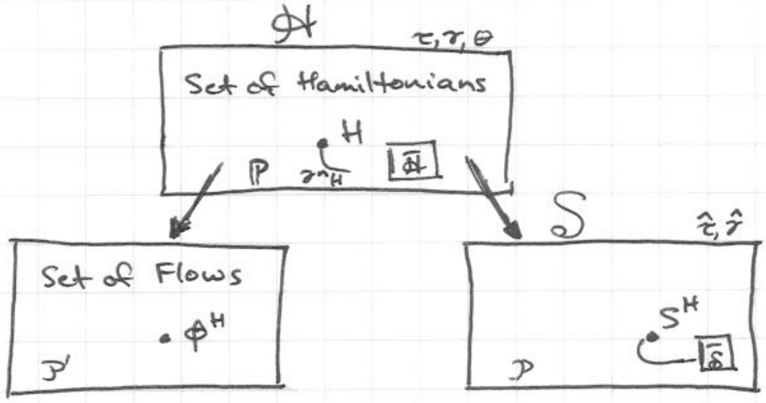
HAMILTONIAN ODE AND HAMILTON-JACOBI PDE WITH STOCHASTIC HAMILTONIAN FUNCTION

FRAYDOUN REZAKHANLOU

Notes by Jeffrey Heninger

Hamiltonian Functions

$H : \mathbb{R}^d \times \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$



Hamiltonian ODE

$x = (q,p) \in \mathbb{R}^d \times \mathbb{R}^d$

$$(1) \quad \begin{cases} \dot{x} = J\,\nabla H(x,t) \\ x(0) = a \end{cases} \quad \rightsquigarrow \quad x(t) = \phi_t^H(a)$$

Hamilton–Jacobi PDEs

$$(2) \quad \begin{cases} u_t + H(q, u_q, t) = 0 \\ u(q,0) = g(q) \end{cases}$$

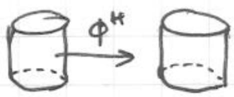
$u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$

$(S_t^H g)(q) = u(q,t)$

Three Classical Results

① $d = 1$, $H : \mathbb{R} \times [-1,1] \times [0,\infty) \to \mathbb{R}$ $\quad$ H is 1-periodic in q, t

$H : S^1 \times [-1,1] \times [0,\infty) \to \mathbb{R}$ $\quad$ $\phi^H = \phi_1^H$



flow on a cylinder

Poincaré – Birkhoff, assuming twist condition, then ϕ^H has at least two fixed points. (1) has at least two 1-periodic ~~points~~ orbits

② $H : \mathbb{R}^d \times \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is 1-periodic in all of its variables then regarding $\phi^H : \mathbb{T}^{2d} \to \mathbb{T}^{2d}$, ϕ^H has at least $2d+1$ (under some nondegeneracy condition 2^{2d}) many fixed points (1-periodic orbits). [Conley – Zehnder]

③ Consider viscosity solutions to (2) and rescale the solution as $u^n(q,t) = \frac{1}{n} u(nq, nt)$. Then if H is independent of t, and 1-periodic in q, then $\lim_{n\to\infty} u^n = \bar{u}$ exists, and solve the homogenized HJE

$\bar{u}_t + \bar{H}(\bar{u}_q) = 0$. $\quad$ $\bar{H}$ is some nontrivial average of H^n.

[Lions – Papanicolaou – Varadhan 1987]

c2

Extensions ① $H: T^*M \times [0,\infty) \to \mathbb{R}$

or even $H: \underset{\sim}{M} \times [0,\infty) \to \mathbb{R}$

symplectic

② In the Euclidean setting, put a probability measure on $\mathcal{H}$ and study $\lim_{n\to\infty} u^n$ with probability 1, or address Arnold's Conjectures for almost all H.

Hope Use ergodic theory to make up for the lack of compactness.

Here are some results:

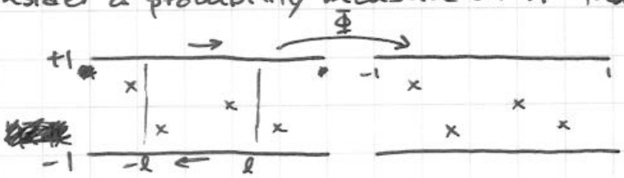
Here are some important group actions on $\mathcal{H}$

(i) $\tau_a(q,p) = (q+a,p)$ $(a \in \mathbb{R}^d, +)$ spatial translations

(ii) $\theta_{ab}(q,p) = (q+a, p+b)$ $(a,b \in \mathbb{R}^d, +)$ Galilean boosts

(iii) $\gamma_n(q,p) = (nq,p)$ $(n \in \mathbb{R}^d, \cdot)$ spatial dilation

① Consider a probability measure on $\mathcal{H}$ that is τ-invariant and ergodic.



map a strip forward in time

$H: \mathbb{R} \times [-1,1] \to \mathbb{R}$

H 1-periodic in time

Also assume twist condition

Question Does ϕ^H have fixed points (x)? How many?

Ideally, we put a condition to distinguish different fixed points.

$\lim_{\ell\to\infty} \frac{1}{2\ell}$ # fixed points of type in $[-\ell,\ell] \times [-1,1]$

The best possible scenario is true for a monotone twist map.

A weaker version is true in general. [FR & Pelayo]

② Consider $H: \mathbb{R}^d \times \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$

and take a probability measure $\mathbb{P}$ that is θ-invariant.

Assume "monotonicity," then under a nondegeneracy condition, one can show that ϕ^H has infinitely many fixed points for each Morse index. ($D^2\phi^H$ at fixed points)

Morse index = counting eigenvalues of $2d\times 2d$ matrix from generating function

③ Homogenization

$$u^n(q,t) = \frac{1}{n} u(nq, nt)$$

$$u^n_t + H(nq, u^n_q) = 0$$

$$H(q,p) \to H(nq,p) = (\gamma_n H)(q,p)$$

Take $\mathbb{P}$ such that $\mathbb{P}$ is τ-invariant.

We wish to show that $S^{\gamma_n H} \xrightarrow{?} S^{\bar{H}}$ with $\bar{H}(p)$, i.e. $\bar{H}$ is γ-invariant

does these semigroups converge to something (in a weak sense)?

The limit of H^n doesn't ~~[illegible]~~ exist in the original box.

Check if it makes sense in the other boxes.

$$\begin{cases} S_t^{\tau_a H} = \tau_a \circ S_t^H \circ \tau_{-a} = \hat{\tau}_a S \\ S^{\gamma_n H} = I_n \circ (S_t^H)^n \circ I_n^{-1} \quad \text{where } I_n(g)(a) = \frac{1}{n} g(na) \end{cases}$$

$\qquad\qquad\qquad\qquad\quad \| \; \hat{\gamma}_n S$ converts microscopic & ~~[illegible]~~ macroscopic functions from statistical mechanics

the actions $\hat{\tau}_a$ and $\hat{\gamma}_n$ can act on any semi-group not just if it's from the Hamilton-Jacobi equation

Homogenization $(\mathcal{S}, \hat{\tau}_a)$, $\mathcal{P}$ = probability measure which is $\hat{\tau}_a$-invariant

Homogenization really means that starting from $\hat{\tau}$-invariant prob. measure, we wish to show $\lim_{n\to\infty} (\gamma_n)^{\#} \mathcal{P} = \bar{\mathcal{P}}$ which is $\hat{\gamma}$-invariant.

then show that $\bar{\mathcal{P}}$ is concentrated on $\bar{\mathcal{H}} = \{H \text{ which depends on } p\text{-only}\}$

Using this language we can show that a weak homogenization is always possible.

<u>Idea</u> Build a metric on $\mathcal{S}$ so that $(\mathcal{S}, d)$ is compact!

We then have a group action on a compact space, so limit points are concentrated on an invariant set.

Then look at $\frac{1}{2\ell} \int_{-\ell}^{\ell} \delta_{\gamma_{e^m} S} \, dm$, any limit point is a probability measure.

$\hookrightarrow$ from $\mathcal{P}$