MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Diego del-Castillo-Negrete

Talk Title: Plasma Physics Inspired Hamiltonian Dynamics Problems

Date: 11 / 27 / 2018 **Time:** 9 : 15 **am** / **pm (circle one)**

Please summarize the lecture in 5 or fewer sentences: Hamiltonian maps which violate the twist condition of KAM theory arise in magnetic confinement fusion (either motion along magnetic field lines or motion due to ExB drift) and in fluid systems. The nontwist curve breaks last and has different fractal structure for its residues. When we average the ExB drifts over the fast gyromotion, each particle sees a different Hamiltonian, possibly with a different topology. Gyroaveraging can either cause more chaos or it can suppress chaos. To get a self-consistent system, use a mean field more - e.g. a map where the parameters also have dynamics.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Plasma physics (and fluid dynamics) inspired Hamiltonian dynamical systems problems

D. del-Castillo-Negrete [1]
Oak Ridge National Laboratory
USA

Hamiltonian systems, from topology to applications through analysis II
MSRI
Berkeley, CA

Nov. 26-30, 2018

[1]delcastillod@ornl.gov

OUTLINE

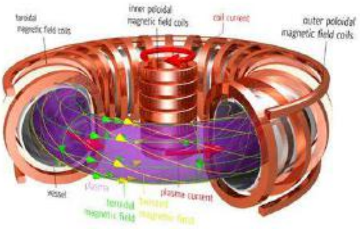
- Nontwist Hamiltonian systems
- Gyro-averaged Hamiltonian systems
- Mean field-coupled Hamiltonian systems

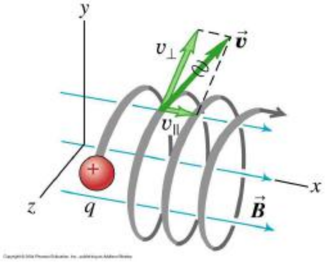
Work done in collaboration with
J. M. Greene and P.J. Morrison
Special thanks to
R. de la Llave, H. Swinney, and I. Caldas

References

- D. del-Castillo-Negrete, Ph.D. Thesis, Univ. Texas Austin (1994);
- D. del-Castillo-Negrete, and P.J. Morrison, Phys. Fluids A, 5, 948-965, (1993).
- D. del-Castillo-Negrete, and P.J. Morrison, Bull. Am. Phys. Soc., Serie II, 37:1543 (1992).
- D. del-Castillo-Negrete, J.M. Greene, and P.J. Morrison, Physica D., 91, 1-23, (1996).
- D. del-Castillo-Negrete, J.M. Greene, and P.J. Morrison, Physica D., 100, 311-329, (1997).
- D. del-Castillo-Negrete, Phys. Plasmas, 7, (5), 1702 (2000).

MAGNETIC CONFINEMENT





$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}, \qquad m\frac{d\boldsymbol{v}}{dt} = -e\left[\boldsymbol{E}(\boldsymbol{x}) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{x})\right]$$

The dynamics of RE spans a huge range of time scales, from the gyro-period $t \sim 10^{-11}$sec to the observational time scales $t \sim 10^{-3} \to 1$sec.

DISSECTING THE PARTICLE MOTION

- The particle position and velocity can be decomposed as:

$$\mathbf{r}(t) = \mathbf{r}_{||} + \mathbf{r}_{gyro} + \mathbf{r}_{gc}\,, \qquad \mathbf{v}(t) = \mathbf{v}_{||} + \mathbf{v}_{gyro} + \mathbf{v}_{gc}$$

- $\mathbf{r}_{||} = \ell\,\mathbf{b}$ and $\mathbf{v}_{||} = v_{||}\mathbf{b}$ denote the parallel motion along the magnetic field $\mathbf{B} = B\mathbf{b}$.
- $(\mathbf{r}_{gyro}, \mathbf{v}_{gyro})$ is the gyro-motion around the magnetic field.
- $(\mathbf{r}_{gc}, \mathbf{v}_{gc})$ is the position and velocity of the guiding center

$$\frac{d\mathbf{r}_{gc}}{dt} = \mathbf{v}_{gc}\,, \qquad \mathbf{v}_{gc} = \mathbf{V}_E + \mathbf{V}_{\nabla B} + \mathbf{V}_\kappa$$

  with velocity drifts given by

$$\mathbf{V}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \qquad \mathbf{V}_{\nabla B} = \mp\frac{v_\perp^2}{2\omega_c}\,\frac{\mathbf{B}\times\nabla B}{B^2}\,,$$

$$\mathbf{V}_\kappa = \mp\frac{v_\|^2}{\omega_c}\,\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2 B}\,.$$

- For parallel motion $d\ell/dt = v_\|$ and $m dv_\|/dt = qE_\| - \mu\partial_\ell B$

PURELY PARALLEL MOTION

Hamiltonian dynamics of magnetic field lines

- Neglecting gyro-motion and the velocity drifts, reduces the dynamics to parallel motion along the magnetic field $\mathbf{r} = \ell\, \mathbf{b}$ where $d\ell/dt = v_{||}$ and $m dv_{||}/dt = qE_{||} - \mu \partial_\ell B$.
- Neglecting $E_{||}$ and $\partial_\ell B$, the orbit is entirely determined by the dynamical system $d\mathbf{x}/ds = \mathbf{B}(\mathbf{x}(s))$.
- Modeling the tokamak as a periodic cylinder of length $2\pi R_0$

$$\frac{dr}{ds} = B_r\,, \qquad r\frac{d\theta}{ds} = B_\theta\,, \qquad \frac{dz}{ds} = B_z\,,$$

- Assuming B_z =constant, defining $\psi = r^2/2$, $\zeta = z/R_0$, and using $\nabla \cdot \mathbf{B} = 0 \;\rightarrow \mathbf{B} = \nabla \times \mathbf{A}$

$$\frac{d\theta}{d\zeta} = \frac{\partial H}{\partial \psi}\,, \qquad \frac{d\psi}{d\zeta} = -\frac{\partial H}{\partial \theta}$$

where $H = -A_z(\psi, \theta, \zeta) R_0/B_z$ is the Hamiltonian and $\zeta \rightarrow$'time", $(\theta, \psi) \rightarrow$ canonical conjugate variables.

HAMILTONIAN INTEGRABILITY AND MAGNETIC CHAOS

- When H is independent of z (or when the dependence on z can be removed by a change of coordinates), i.e. when the magnetic field is toroidally symmetric, the field-line trajectories are integrable.
- Simplest integrable case $H = H_0(\psi)$

$$\theta(\zeta) = \theta_0 + \Omega(\psi)\zeta \,, \qquad \psi(z) = \psi_0 \,, \qquad \Omega = \frac{\partial H_0}{\partial \psi}$$

- Magnetic perturbations, $\mathbf{B} = \mathbf{B}_0 + \epsilon \mathbf{B}_1$, and loss of integrability, "the fundamental problem of dynamics"

$$H = H_0(\psi) + \epsilon H_1(\psi, \theta, \zeta)$$

what is the fate of the invariant circles, $\psi = \psi_0$, (magnetic flux surfaces) under the perturbation ϵH_1 ?

- The answer to this question is critical for the understanding of magnetic confinement of fusion plasmas.

- Each invariant circle, $\psi = \psi_0$, has associated a rotation frequency, $\Omega(\psi_0)$.
- If Ω is rational the orbit is periodic and if Ω is irrational the orbit is quasiperiodic.
- (KAM theory) For sufficiently small ϵ, most of the quasiperiodic invariant circles persist and are only slightly deformed provided

$$\frac{\partial \Omega}{\partial \psi} = \frac{\partial^2 H_0}{\partial \psi^2} \neq 0$$

- This non-degeneracy condition is commonly satisfied in standard Hamiltonian problems of the form $H = K + V$ where K is the K denotes the kinetic energy and V the potential energy.
- Is it always the case that $\partial_\psi \Omega \neq 0$? In this condition general enough? Are we discarding interesting, physically relevant dynamical systems?

REVERSED SHEAR MAGNETIC FIELD CONFIGURATION AND DEGENERATE HAMILTONIAN PERTURBATION PROBLEMS

- In the plasma physics context

$$\Omega(r) = \frac{R_0}{r}\frac{B_\theta(r)}{B_z} = \frac{1}{q(r)},$$

where $q(r)$ is the safety factor and R_0 the major radius, and the non-degeneracy condition reduces to

$$\frac{dq(r)}{dr} \neq 0,$$

satisfied by many toroidal magnetic field configurations.

- However, there are important cases, known as reversed-shear magnetic field configuration, for which this is not the case.
- The study of magnetic perturbations in reversed-shear configuration lead to the study of perturbation Hamiltonian problems outside the standard KAM theory

AREA PRESERVING MAPS

- Area preserving maps

$$M(x^n, y^n) = (x^{n+1}, y^{n+1}), \qquad \frac{\partial\left(x^{n+1}, y^{n+1}\right)}{\partial\left(x^n, y^n\right)} = 1$$

- For the magnetic field lines problem, the stroboscopic Poincare map, $(\theta, \psi)(\zeta_0) \rightarrow (\theta, \psi)(\zeta_0 + 2\pi)$,is an area preserving map because the Hamiltonian evolution is a canonical transformation.
- In the integrable case

$$\psi^{n+1} = \psi^n, \qquad \theta^{n+1} = \theta^n + 2\pi\Omega(\psi^{n+1})$$

- Finding an analytical expression in the presence of a perturbation is in general not possible. But, insightful models capturing the fundamental aspects of the dynamics can be constructed.

- Consider the perturbed area preserving map

$$\psi^{n+1} = \psi^n + g(\theta^n, \psi^{n+1}) , \qquad \theta^{n+1} = \theta^n + 2\pi\Omega(\psi^{n+1}) + f(\theta^n, \psi^{n+1})$$

$$\frac{\partial f}{\partial \theta^n} + \frac{\partial f}{\partial \psi^{n+1}} = 0$$

- Like in the case of flows, each invariant circle, $\psi^n = \psi_0$, of the integrable map has a rotation number $2\pi\Omega(\psi_0)$.
- (KAM theory) For sufficiently small ϵ, most of the quasiperiodic invariant circles persist and are only slightly deformed provided

$$\frac{d\Omega}{d\psi^{n+1}} \neq 0$$

- This non-degeneracy condition known as twist condition in typically satisfied by a large class of area preserving maps known as twist maps.
- Is the twist condition general enough? Are we discarding interesting, physically relevant dynamical systems?

THE STANDARD MAP

A prototype model for the transition to chaos in **twist maps**

- Motivated by the problem of magnetic confinement of fusion plasmas, Chirikov and Taylor proposed the standard-map for understanding the fundamental aspects of the transition to magnetic field line chaos.
- Based on the assumption of monotonicity of the q profile they propose $2\pi\Omega = \psi_{n+1}$
- Based on the observation that radial magnetic field perturbations are typically of the form $\delta B_r = \sum_{m,n} a_{mn}(\psi)\cos(m\theta - n\zeta)$, they proposed a simple harmonic perturbation of the form $\delta\psi \sim k\sin\theta^n$

$$\psi^{n+1} = \psi^n + k\sin\theta^n\,, \qquad \theta^{n+1} = \theta^n + \psi^{n+1}$$

THE STANDARD NONTWIST MAP

A prototype model for the transition to chaos in **nontwist maps**

- As mentioned before, reversed shear magnetic configurations exhibit a non-monotonic q profile.
- That is, the Hamiltonian describing fields lines in this case is in general degenerate, $d\Omega/d\psi \neq 0$, and the corresponding area preserving map violates the twist condition, $\Omega/\psi_{n+1} \neq 0$.
- The standard nontwist map

$$\psi^{n+1} = \psi^n + b \sin \theta^n \,, \qquad \theta^{n+1} = \theta^n + a[1 - (\psi^{n+1})^2]$$

was proposed to capture the fundamental aspects of the transition to chaos in systems that violate the twist condition.

TRANSITION TO CHAOS

- The transition to chaos, i.e. the destruction of invariant circles due to perturbations, is a fundamental problem in the theory and applications of dynamical systems.
- In the context of plasma physics this problem corresponds to the destruction of magnetic surfaces and the loss of confinement.
- In the fluid mechanics context (to be discussed later) this corresponds to the destruction of transport barriers and the onset of global fluid mixing.
- Some fundamental questions:
  - Given a Hamiltonian system depending on a set of parameters λ_i and an invariant circle with a rotation rotation number ω, what is the region in the parameter space for which the invariant circle exists?
  - What are the geometric properties of the invariant circle at criticality?
  - How universal is the transition to chaos?

TRANSITION TO CHAOS: CRITICALITY AND SCALING

The standard map universality class

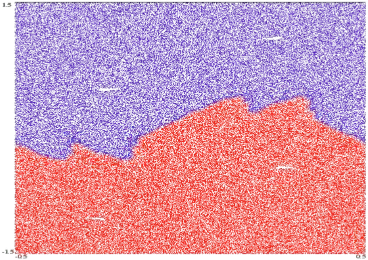
$$\psi^{n+1} = \psi^n + k \sin \theta^n \,, \qquad \theta^{n+1} = \theta^n + \psi^{n+1}$$

- The last invariant circle has $\omega = \gamma = (1 + \sqrt{5})/2$ (the golden mean) and the critical parameter is $k_c = 0.971635406\ldots$ [Grenne, 1979].
- The Residue criterion [Grenne, 1979] allows to determine the fate of a given invariant circle by looking at the stability (residue) of the nearby periodic orbits.
- At criticality, the residues (on the dominant symmetry line) converge to $R_c = 0.25\ldots$ [Greene, 1979] and the invariant circle exhibits fractal structure with scaling parameters $\alpha = 1.4148\ldots$ and $\beta = 3.0668\ldots$ [Kadanoff-Shenker 1981, 1982].
- Renormalization theory [MacKay, 1982] provides a framework to understands these results and explain why they are universal for a very large class of maps.

THE. STANDARD NONTWIST MAP

Shearless circles exhibit a remarkable resilience to perturbations

$$\psi^{n+1} = \psi^n + b\sin\theta^n\,, \qquad \theta^{n+1} = \theta^n + a[1-(\psi^{n+1})^2]$$

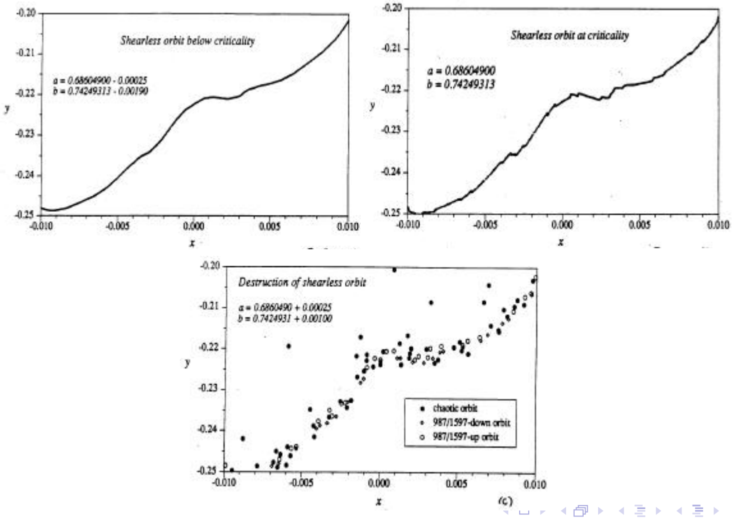


This explains the robustness of magnetic flux surfaces in reversed shear configurations, and the existences of transport barriers in non-monotonic shear flows in plasmas and fluids.

THE STANDARD NONTWIST MAP

Residue criterion gives critical parameter values for breakup of golden mean shearless circle

$$\psi^{n+1} = \psi^n + b \sin \theta^n \,, \qquad \theta^{n+1} = \theta^n + a[1 - (\psi^{n+1})^2]$$

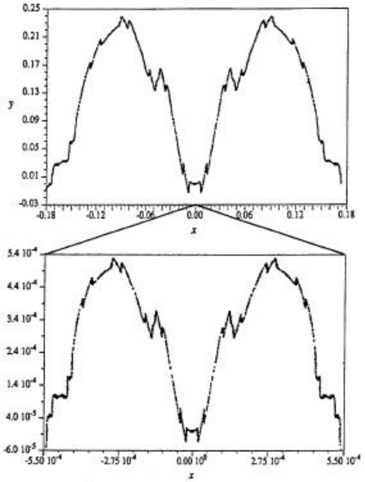


THE STANDARD NONTWIST MAP

Golden mean critical shearless invariant circle exhibits self-similar scaling different to the standard map universality class

$$(x, y) \to (\alpha x, \beta y), \qquad \alpha = 321.92 \qquad \beta = 463.82$$

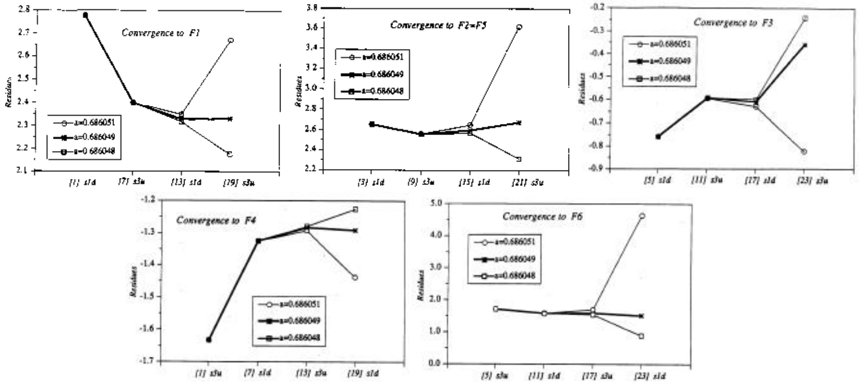
$$(a_c, b_c) = (0.686049,\ 0.742493131039)$$



THE STANDARD NONTWIST MAP

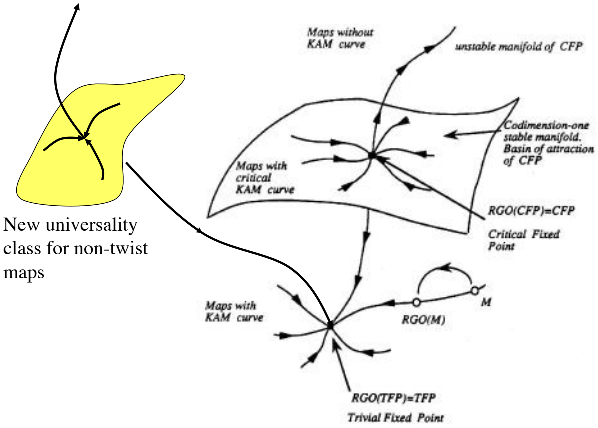
Golden mean critical shearless invariant circle exhibits residue convergence different to the standard map universality class

Residues converge to a 6-cycle $\{F_1, F_2, F_3, F_4, F_5 = F_2, F_6\}$



THE STANDARD NONTWIST MAP

The transition to chaos of the shearless invariant circle corresponds to a new universality class



BACK TO GENERAL PARTICLE MOTION IN MAGNETICALLY CONFINED PLASMAS

- The particle position and velocity can be decomposed as:

$$\mathbf{r}(t) = \mathbf{r}_{||} + \mathbf{r}_{gyro} + \mathbf{r}_{gc} \,, \qquad \mathbf{v}(t) = \mathbf{v}_{||} + \mathbf{v}_{gyro} + \mathbf{v}_{gc}$$

- $\mathbf{r}_{||} = \ell\,\mathbf{b}$ and $\mathbf{v}_{||} = v_{||}\mathbf{b}$ denote the parallel motion along the magnetic field $\mathbf{B} = B\mathbf{b}$.
- $(\mathbf{r}_{gyro}, \mathbf{v}_{gyro})$ is the gyro-motion around the magnetic field.
- $(\mathbf{r}_{gc}, \mathbf{v}_{gc})$ is the position and velocity of the guiding center

$$\frac{d\mathbf{r}_{gc}}{dt} = \mathbf{v}_{gc} \,, \qquad \mathbf{v}_{gc} = \mathbf{V}_E + \mathbf{V}_{\nabla B} + \mathbf{V}_\kappa$$

  with velocity drifts given by

$$\mathbf{V}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \qquad \mathbf{V}_{\nabla B} = \mp\frac{v_\perp^2}{2\omega_c}\,\frac{\mathbf{B}\times\nabla B}{B^2} \,,$$

$$\mathbf{V}_\kappa = \mp\frac{v_\parallel^2}{\omega_c}\,\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2 B} \,.$$

- For parallel motion $d\ell/dt = v_\parallel$ and $m dv_\parallel/dt = qE_\parallel - \mu\partial_\ell B$

HAMILTONIAN DYNAMICS OF $\mathbf{E} \times \mathbf{B}$ MOTION

- Neglecting gyro-motion and the parallel motion along the magnetic field reduces the dynamics to the perpendicular drift motion.
- Neglecting the magnetic field gradient and curvature

$$\frac{d\mathbf{r}_{gc}}{dt} = \mathbf{v}_{gc} = \mathbf{V}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

- Assuming $\mathbf{B} = B_0 \hat{\mathbf{e}}_z$ with $B_0 = \mathit{constant}$ and writing $\mathbf{E} = -\nabla \phi$

$$\frac{dx}{dt} = -\frac{\partial H}{\partial y}, \qquad \frac{dy}{dt} = \frac{\partial H}{\partial x}$$

where $H = \phi(x, y, t)/B_0$ is the Hamiltonian and $(x, y) \rightarrow$ canonical conjugate variables.

HAMILTONIAN INTEGRABILITY AND $\mathbf{E} \times \mathbf{B}$ chaotic transport

- When$H$ is independent of t (or when the dependence on t can be removed by a change of coordinates), i.e. when the electric field is time independent, the $\mathbf{E} \times \mathbf{B}$ motion is integrable
- Simplest integrable case $\mathbf{E} = E_0(x)\,\hat{\mathbf{e}}_x$, i.e. $H = H_0(x)$

$$x(t) = x_0\,, \qquad y = y_0 + \Omega(x)t\,, \qquad \Omega = \frac{\partial H_0}{\partial x} = -\frac{E_0(x)}{B_0}$$

- Electrostatic perturbations, $\mathbf{E} = \mathbf{E}_0 + \epsilon \mathbf{E}_1$, and loss of integrability, "the fundamental problem of dynamics"

$$H = H_0(x) + \epsilon H_1(x, y, t)$$

  what is the fate of the invariant circles, $x = x_0$ under the perturbation ϵH_1 ?
- The answer to this question is critical for the understanding of $\mathbf{E} \times \mathbf{B}$ transport in plasmas.

$\mathbf{E} \times \mathbf{B}$ SHEAR AND DEGENERATE HAMILTONIAN PERTURBATION PROBLEMS

- In the $\mathbf{E} \times \mathbf{B}$ plasma physics context

$$\Omega = \frac{\partial H_0}{\partial x} = -\frac{E_0(x)}{B_0}$$

and the non-degeneracy condition reduces to $E_o'(x) \neq 0$ which is not a generic condition, since in general the electric field can have any dependence on x.

- In this case the $\mathbf{E} \times \mathbf{B}$ velocity, $\mathbf{V}_E = [E_0(x)/B_0]\hat{\mathbf{e}}_y$ and the nondegeneracy is equivalent to the non-vanishing of the shear

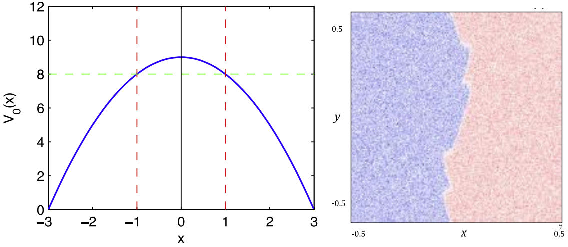
$$\sigma = \frac{d\mathbf{V}_E \cdot \hat{\mathbf{e}}_y}{dx} \neq 0$$

- The study of electrostatic perturbations in shearless $\mathbf{E} \times \mathbf{B}$ flows is degenerate perturbation Hamiltonian problems

SHEARLESS TRANSPORT BARRIERS IN $\mathbf{E} \times \mathbf{B}$ TRANSPORT

- Formally this problem is identically to the previously discussed reversed shear magnetic field line problem.
- Assuming a drift-wave electrostatic perturbation of the form $\phi_1(x, y, t) = \sum_j \epsilon_j \varphi_j(x) \cos k_k(y - c_j t)$ this problem can also be reduced to the standard nontwist map

$$x^{n+1} = x^n + b \sin y^n \,, \qquad y^{n+1} = y^n + a[1 - (x^{n+1})^2]$$



- Shearless $\mathbf{E} \times \mathbf{B}$ trajectories are very resilient to breakup due to perturbations.

A FLUID MECHANICS INTERLUDE

- The Hamiltonian description of $\mathbf{E} \times \mathbf{B}$ transport is equivalent of the description of transport in 2-D incompressible flows
- 2-D, $\mathbf{V} = V_x \hat{\mathbf{e}}_x + V_y \hat{\mathbf{e}}_y$, and incompressibility, $\nabla \cdot \mathbf{V} = 0$, implies $\mathbf{V} = \hat{\mathbf{e}}_z \times \nabla \psi$, where $\psi(x, y, t)$ is the streamfunction.
- The equations of motion of a passive tracer, $d\mathbf{r}/dt = \mathbf{V}(\mathbf{r})$ reduce to the Hamiltonian system

$$\frac{dx}{dt} = -\frac{\partial H}{\partial y}\,, \qquad \frac{dy}{dt} = \frac{\partial H}{\partial x}$$

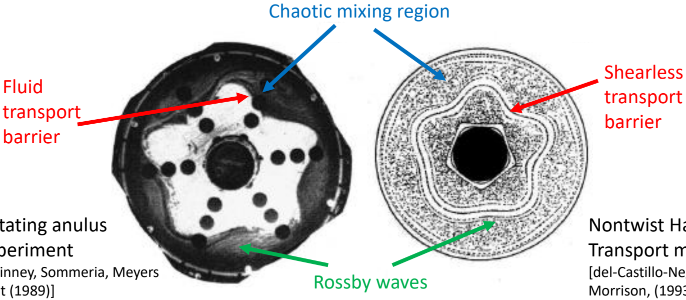
  where $H = \psi$ is the Hamiltonian and $(x, y) \rightarrow$ are canonical conjugate variables.
- In this case, the simplest integrable problem corresponds to transport in a parallel shear flow $\mathbf{V} = V_0 \hat{\mathbf{e}}_x$, and the non-degeneracy condition requires

$$\frac{d\Omega}{dy} = \frac{\partial^2 H_0}{\partial y^2} = -\frac{dV_0(y)}{dy} \neq 0$$

  which in general is not satisfied

TRANSPORT IN ZONAL FLOWS IN GEOPHYSICAL FLOWS

- The 2-D incompressibility assumption is a good approximation in the case of rapidly rotating fluids
- Non-monotonic zonal flows ("jets"), i.e. shear flows with regions of zero shear, $dV_0/dy = 0$, for some value(s) of y, are usually found in the atmosphere and the oceans
- Shearless transport barriers are very resilient to breakup due to perturbations.



Rotating anulus Experiment
[Swinney, Sommeria, Meyers et at (1989)]

Nontwist Hamiltonian Transport model
[del-Castillo-Negrete, Morrison, (1993)]

GYRO-AVERAGED HAMILTONIAN SYSTEMS

Work done in collaboration with

J. Martinell, J. Fonseca, I. Caldas, N Kryukov, I.M. Sokolov

References

- D. del-Castillo-Negrete and J. Martinell, Comm. in Nonlinear Science and Num. Simulation 17 pp. 2031-2044 (2012).
- J. Martinell and D. del-Castillo-Negrete, Phys. of Plasmas 20, 022303 (2013).
- J. Fonseca, D. del-Castillo-Negrete, and I. Caldas, Physics of Plasmas 21, 092310 (2014).
- J. Fonseca, D. del-Castillo-Negrete, I.M. Sokolov and I.L. Caldas, Physics of Plasmas 23, 082308 (2016).
- N. Kryukov, J. Martinell, D. del-Castillo-Negrete, Journal of Plasma Physics 84, 3 (2018).

BACK TO GENERAL PARTICLE MOTION IN MAGNETICALLY CONFINED PLASMAS

- The particle position and velocity can be decomposed as:

$$\mathbf{r}(t) = \mathbf{r}_{||} + \mathbf{r}_{gyro} + \mathbf{r}_{gc} \,, \qquad \mathbf{v}(t) = \mathbf{v}_{||} + \mathbf{v}_{gyro} + \mathbf{v}_{gc}$$

- $\mathbf{r}_{||} = \ell\,\mathbf{b}$ and $\mathbf{v}_{||} = v_{||}\mathbf{b}$ denote the parallel motion along the magnetic field $\mathbf{B} = B\mathbf{b}$.
- $(\mathbf{r}_{gyro}, \mathbf{v}_{gyro})$ is the gyro-motion around the magnetic field.
- $(\mathbf{r}_{gc}, \mathbf{v}_{gc})$ is the position and velocity of the guiding center

$$\frac{d\mathbf{r}_{gc}}{dt} = \mathbf{v}_{gc} \,, \qquad \mathbf{v}_{gc} = \mathbf{V}_E + \mathbf{V}_{\nabla B} + \mathbf{V}_\kappa$$

with velocity drifts given by

$$\mathbf{V}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \qquad \mathbf{V}_{\nabla B} = \mp\frac{v_\perp^2}{2\omega_c}\frac{\mathbf{B}\times\nabla B}{B^2} \,,$$

$$\mathbf{V}_\kappa = \mp\frac{v_\|^2}{\omega_c}\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2 B} \,.$$

- For parallel motion $d\ell/dt = v_\|$ and $m dv_\|/dt = qE_\| - \mu\partial_\ell B$

GYRO-MOTION EFFECTS ON $\mathbf{E} \times \mathbf{B}$ TRANSPORT

- In the previous discussion we neglected gyro-motion, parallel motion, magnetic field gradient and curvature, and reduced the dynamics to

$$\frac{dx}{dt} = -\frac{\partial H}{\partial y}, \qquad \frac{dy}{dt} = \frac{\partial H}{\partial x}$$

where $H = \phi(x, y, t)/B_0$ is the Hamiltonian and $(x, y) \to$ canonical conjugate variables.

- One way to approximately incorporate the dependence on gyro-motion due to finite Larmor radius effects is to to substitute the $\mathbf{E} \times \mathbf{B}$ flow by its value averaged over a ring of radius ρ, where ρ is the Larmor radius

$$\frac{dx}{dt} = -\left\langle \frac{\partial \phi}{\partial y} \right\rangle_\theta, \qquad \frac{dy}{dt} = \left\langle \frac{\partial \phi}{\partial x} \right\rangle_\theta$$

where the gyroaverage, $\langle\,\rangle_\theta$, is defined as

$$\langle \Psi \rangle_\theta \equiv \frac{1}{2\pi} \int_0^{2\pi} \Psi\left(x + \rho\cos\theta, y + \rho\sin\theta\right) d\theta\,.$$

GYO-AVERAGED MODEL

- Gyro-averaging of the Hamiltonian:

  $$\phi = \tanh x - \eta x + \epsilon_1 \mathrm{sech}^2 x \, \cos(k_1 y) + \epsilon_2 \mathrm{sech}^2 x \, \cos(k_2 y - \omega t)$$

  leads to

$$\frac{dx}{dt} = \epsilon_1 k_1 I_{k_1,\rho}(x) \sin k_1 y + \epsilon_2 k_2 I_{k_2,\rho}(x) \sin(k_2 y - \omega t) \, ,$$

$$\frac{dy}{dt} = I_{0,\rho}(x) - \eta - 2\epsilon_1 K_{k_1,\rho}(x) \cos k_1 y - 2\epsilon_2 K_{k_2,\rho}(x) \cos(k_2 y - \omega t) \, .$$

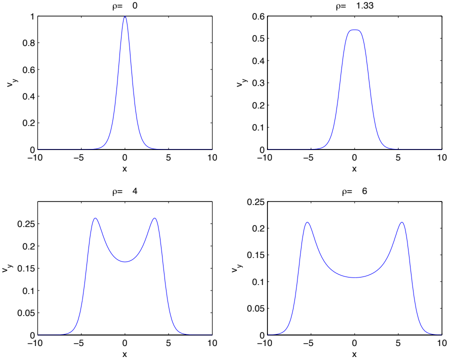
where

$$I_{k,\rho}(x) = \frac{1}{\pi} \int_0^{\pi} \mathrm{sech}^2(x - \rho\cos\theta) \cos(k\rho\sin\theta) \, d\theta \, ,$$

$$K_{k,\rho}(x) = \frac{1}{\pi} \int_0^{\pi} \mathrm{sech}^2(x - \rho\cos\theta) \tanh(x - \rho\cos\theta) \cos(k\rho\sin\theta) \, d\theta \, .$$

ZONAL SHEAR FLOW BIFURCATION

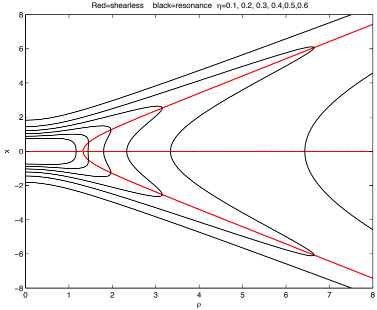
$$v_0(x) = \frac{\partial \langle \phi_0 \rangle_\theta}{\partial x} = I_{0\rho}(x)$$



For $\rho = 0$ the zonal flow exhibits a maximum at $x = 0$. However for $\rho > 1.33\ldots$ there is a bifurcation: a velocity minimum forms at $x = 0$ along with two symmetrically located velocity maxima.

SHEARLESS AND RESONANCES ZONES

$$\sigma_0(x) = \frac{\partial^2 \langle \phi_0 \rangle_\theta}{\partial x^2} = -2K_{0\rho}(x) \qquad R(x;\rho,\eta) = \frac{\partial \langle \phi_0 \rangle_\theta}{\partial x} - \eta = I_{0\rho}(x) - \eta$$

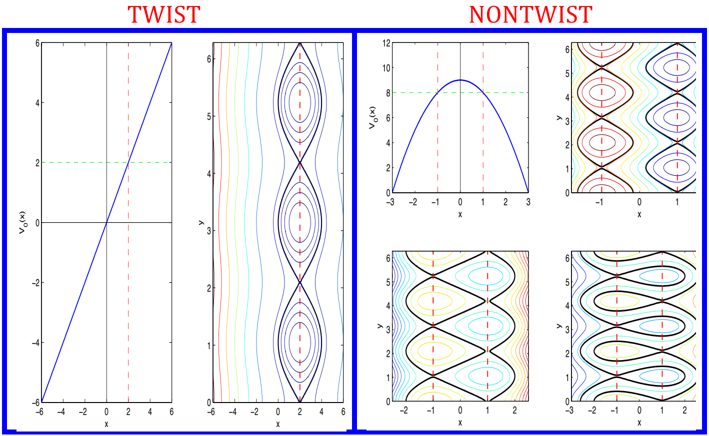


The red curves correspond to $\sigma_0(x;\eta,\rho) = 0$. The black curves correspond to $R(x;\eta,\rho) = 0$ from left to right $\eta =$0.6, 0.5 0.4, 0.3, 0.2 and 0.1.

This introduces highly nontrivial dependences of the phase space topology on the gyro-motion.

RESONANCE TOPOLOGY

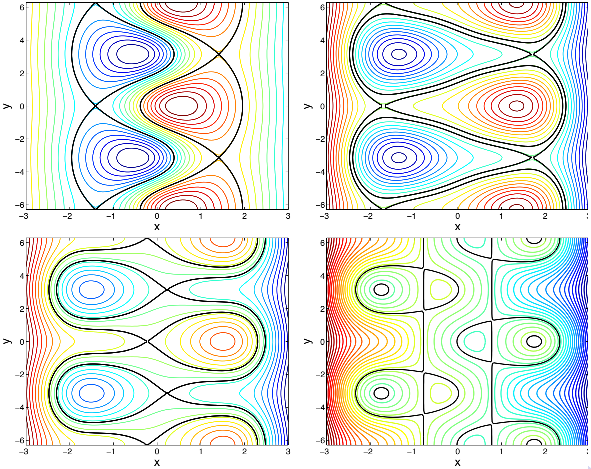
In nontwist maps, each mode creates two resonances and there is separatrix reconnection



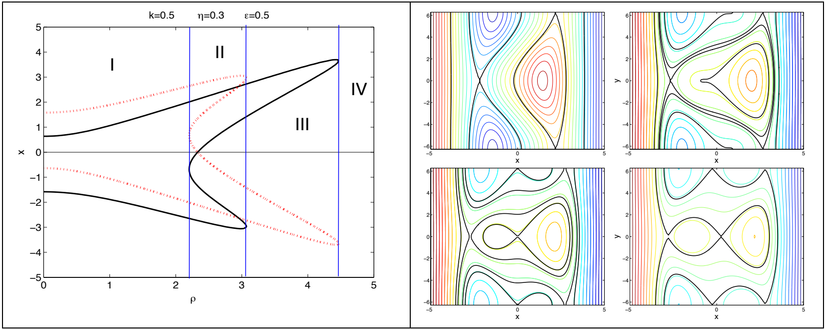
Separatrix reconnection has been extensively studied in nontwist systems, here we discuss its dependence on gyro-averaging

HETEROCLINIC-HOMOCLINIC RECONNECTION AND DIPOLE TOPOLOGY

Contour plots of gyro-averaged Hamiltonian with $\rho = 0$ (top left), $\rho = 1.5$ (top right), $\rho = 1.7$ (bottom left) and $\rho = 2$ (bottom right). The bold black line is the separatrix.



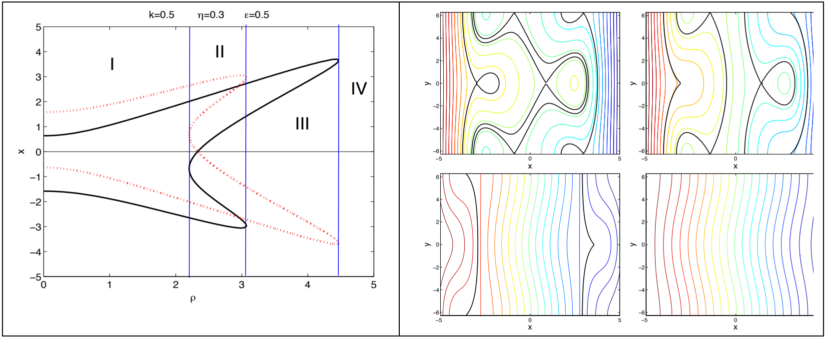
FIXED POINTS CREATION AND RECONNECTION



Left figure: $x = x_*$ fixed points as function of ρ. The solid-black (dashed-red) curve tracks x_* for $y_* = 0$ ($y_* = \pi/k_1$).

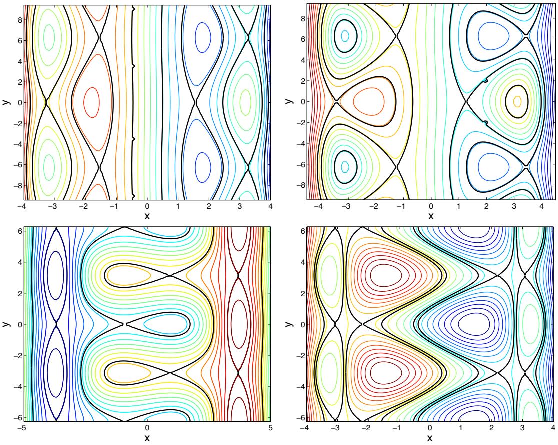
Right figure: Top-left $\rho = 1.5$ (region I); Top-right $\rho = 2.204$ (boundary between region I and region II) Bottom left $\rho = 2.3312$ (region II). Bottom right $\rho = 2.43$(region II)

FIXED POINTS ANIHILATION AND FLOW RECTIFICATION



Left figure: $x = x_*$ fixed points as function of ρ. The solid-black (dashed-red) curve tracks x_* for $y_* = 0$ ($y_* = \pi/k_1$).
Right figure: Top-left $\rho = 2.75$ (region II). Top-right $\rho = 3.0748$ (boundary between region II and region III). Bottom left $\rho = 4.464$ (boundary between region III and IV). Bottom right $\rho = 6$ (region IV).

DOUBLE SEPARATRIX RECONNECTION

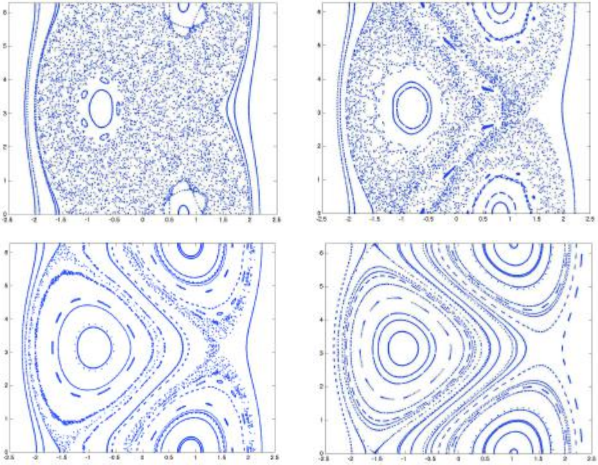


Left figure: $x = x_*$ fixed points as function of ρ.

Right figure: Top-left panel: double heteroclinic topology. Top-right panel: double homoclinic topology. Bottom-left panel: double heteroclinic-homoclinic topology. Bottom-right panel: double dipole topology.

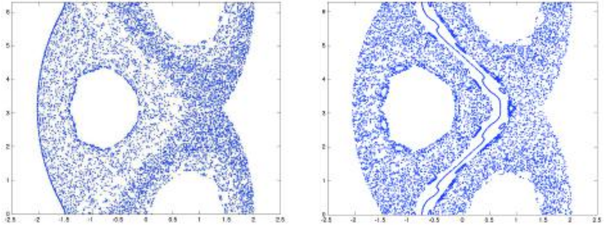
GYRO-AVERAGE INDUCED CHAOS SUPPRESSION

Poincare plots for: Top-left panel, $\rho = 0$. Top-right panel, $\rho = 0.5$. Bottom-left panel, $\rho = 0.75$. Bottom-right panel, $\rho = 1$.

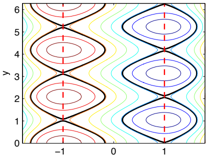


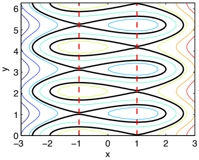
SHEARLESS CURVE RECOVERY DUE TO GYROAVERAGING

Poincare plots showing how the increase of the Larmor radius leads to the recovery the shearless curve going through $(x, y) \approx (-0.75, 0)$, and the suppression of global transport across the resonances.

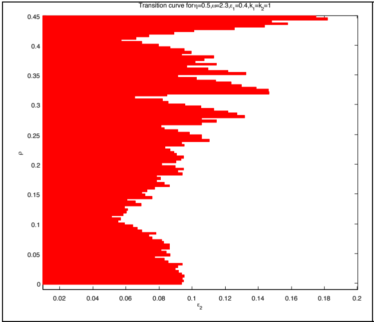


THRESHOLD FOR SHEARLESS BARRIER DESTRUCTION IN (ρ, ϵ) PLANE

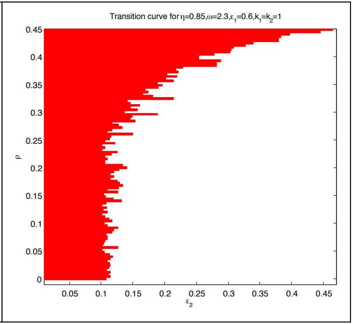




HETEROCLINIC



HOMOCLINIC



GYRO-AVERAGED AREA PRESERVING MAPS

- Gyro-averaged maps can be constructed starting from the "kicked-rotor" type Hamiltonian

$$\phi = \phi_0(x) + \hat{A} \sum_{m=-\infty}^{\infty} \cos\left(\kappa y - m\omega_0 t\right)$$

- Applying the gyro-averaged operator

$$\langle \phi \rangle = \langle \phi_0(x) \rangle_\theta + 2\pi \hat{A} \mathrm{J}_0(\kappa\rho) \cos \kappa y \sum_{m=-\infty}^{\infty} \delta\left(\omega_0 t - 2\pi m\right)$$

  where J_0 is the zeroth-order Bessel function.
- As in the standard case, the equations of motion can be formally integrated over a period of the perturbation to get the discrete area preserving map:

$$J^{n+1} = J^n + A\mathrm{J}_0(\hat{\rho}) \sin \theta^n \,, \qquad \theta^{n+1} = \theta^n + \Omega\left(J^{n+1}\right)$$

  where $J \sim x$, $\theta \sim y$, and $\Omega \sim d\left\langle \phi_0(x) \right\rangle / dx$.

GYRO-AVERAGED STANDARD MAP

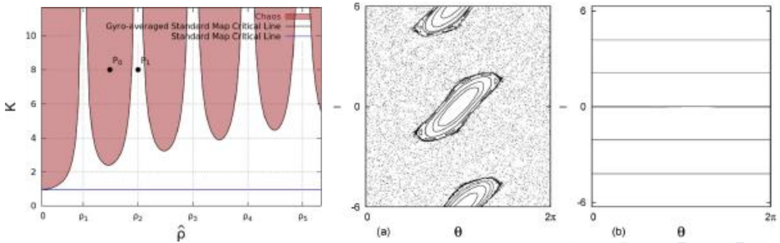
- In the standard-map twist case $\Omega(x) \sim x$

$$J^{n+1} = J^n + k_{eff} \sin\theta^n \,, \qquad \theta^{n+1} = \theta^n + J^{n+1}$$

where k_{eff} depends on the Larmor radius $\hat{\rho}$ according to

$$k_{eff} = k\mathrm{J}_0(\hat{\rho}) \,.$$

- When the Larmor radius can be neglected, $k_{eff} = k\mathrm{J}_0(0) = k$. However, in the general, each particle, "sees" a different perturbation amplitude, k_{eff}, which vanishes at the zeros of J_0



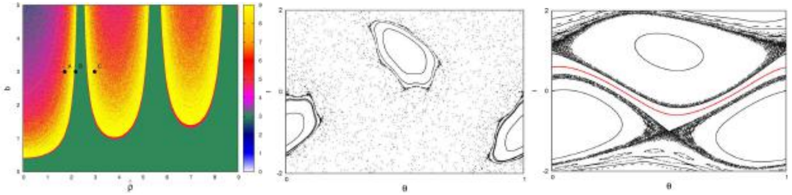
GYRO-AVERAGED STANDARD NONTWIST MAP

- In the standard nontwist case $\Omega(x) \sim x^2$

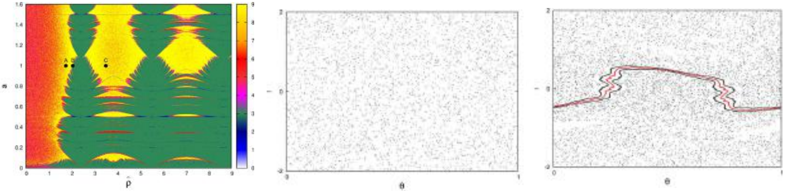
$$
\begin{aligned}
J^{n+1} &= J^n + b\mathrm{J}_0(\hat{\rho}) \sin\theta^n \\
\theta^{n+1} &= \theta^n + a\left[\left(1 - \frac{\bar{\rho}^2}{2}\right) - \left(J^{n+1}\right)^2\right]
\end{aligned}
$$

where $\hat{\rho} = \rho k$, and in the $\bar{\rho} = \rho/L$.

GYRO-AVERAGED STANDARD NONTWIST MAP



Left panel, break-up diagram in $(\hat{\rho}, b)$ plane for $\bar{\rho} = 0$ and $a = 0.1$. Middle and right panels, Poincare plots for cases A and B.



Left panel, break-up diagram in $(\hat{\rho}, a)$ plane for $\bar{\rho} = 0$ and $b = 1.5$. Middle and right panels, Poincare plots for cases A and B.

GYRO-AVERAGED STANDARD NONTWIST MAP

- In a plasma the particles exhibit a statistical distribution of Larmor radii, e.g. a Maxwellian distribution

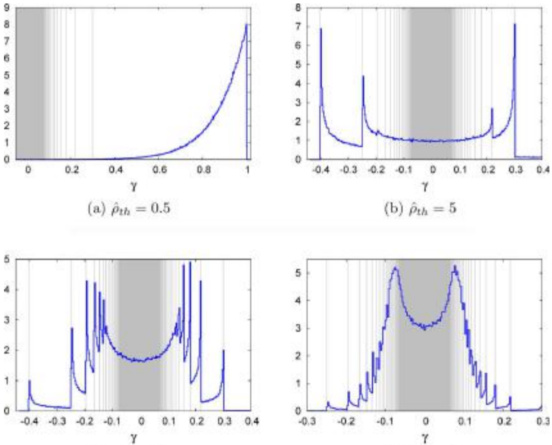
$$f(\hat{\rho}) = \frac{\hat{\rho}}{\hat{\rho}_{th}^2} \exp\left[-\frac{1}{2}\left(\frac{\hat{\rho}}{\hat{\rho}_{th}}\right)^2\right]$$

- The previous results determining the fate of the shearless curve for a given value of $\hat{\rho}$ need to be extended to a statistical distribution of Larmor radii.
- Given a distribution of Larmor radii, $f(\hat{\rho})$, the probability distribution of the effective perturbation parameter $b\mathrm{J}_0(\hat{\rho}) = b\gamma$ is

$$g(\gamma) = \frac{1}{\hat{\rho}_{th}^2} \sum_{\hat{\rho}_i \in \Gamma_\gamma} \frac{\hat{\rho}_i}{|\mathrm{J}_0'(\hat{\rho}_i)|} \exp\left[-\frac{1}{2}\left(\frac{\hat{\rho}}{\hat{\rho}_{th}}\right)^2\right] ,$$

where $\Gamma_\gamma = \{\hat{\rho}_0, \hat{\rho}_1, \ldots\}$ is the set of non-negative solutions of $\gamma = \mathrm{J}_0(\hat{\rho}_i)$

GYRO-AVERAGED STANDARD NONTWIST MAP



(a) $\hat{\rho}_{th} = 0.5$ (b) $\hat{\rho}_{th} = 5$ (c) $\hat{\rho}_{th} = 15$ (d) $\hat{\rho}_{th} = 50$

This probability distribution of γ provides the basis to study the probability distribution of confinement.

MEAN-FIELD COUPLED HAMILTONIAN SYSTEMS

Work done in collaboration with
M.C. Firpo, A. Olvera and R. Calleja, D. Martinez-del-Rio, L. Carbajal, A. Vulpiani, G. Boffetta, J. Martinell
References

- del-Castillo-Negrete, Phys. Plasmas, 5, (11), 3886, (1998).
- del-Castillo-Negrete, Phys. Letters A, 241, 99-104, (1998).
- del-Castillo-Negrete, CHAOS, 10, (1), 75-88, (2000).
- del-Castillo-Negrete, Physica A, 280, 10-21,(2000).
- del-Castillo-Negrete, "Dynamics and self-consistent chaos in a mean field Hamiltonian model". Lecture Notes in Physics Vol. 602, Springer (2002).
- del-Castillo-Negrete, Firpo, CHAOS, 12, 496-507, (2002).
- Boffetta, D. del-Castillo-Negrete, C. Lopez, G. Pucacco, and A. Vulpiani. Phys. Rev. E, 67, 026224 (2003).

References(cont.)

- del-Castillo-Negrete, Plasma Phys. Contr. Fusion 47 (2005).
- Carbajal, D. del-Castillo-Negrete, and J. J. Martinell, Chaos, 22 013137 (2012).
- Martinez, and D. del-Castillo-Negrete, A. Olvera and R. Calleja, Qualitative Theory of Dynamical Systems 14 (2), 313-335 (2015).
- Calleja, D. del-Castillo-Negrete, D. Martinez-del-Rio and A. Olvera, Commun. Nonlinear Sci. Numer. Simulat. 51 198-215 (2017).

LOW DEGREES-OF-FREEDOM HAMILTONIAN SYSTEMS

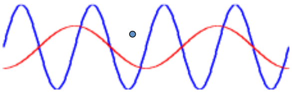
- The simplest Hamiltonian systems with nontrivial (chaotic) dynamics are the well-understood 1-1/2 degrees-of-freedom systems

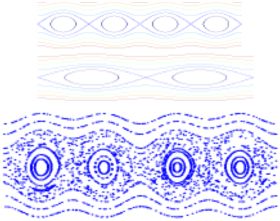
$$H(q,p,t) = \frac{p^2}{2m} + \phi(q,t)\,.$$

- A canonical example is a charged particle in 1-d in a time-dependent external electrostatic field

Chaotic motion in a two-waves field

$$\phi = \cos(k_1 x - \omega_1\, t) + \cos(k_2 x - \omega_2\, t)$$





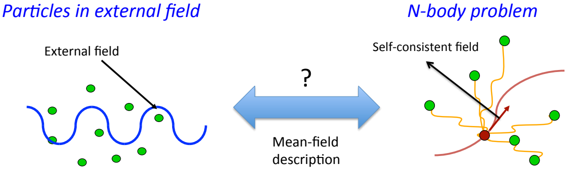
- When the spatial dimensionality increases, $d = 2\,,3$, this single particle problem complicates but relatively speaking (i.e., compared with what comes next) is a tractable problem.

VERY LARGE NUMBER OF DEGREES-OF-FREEDOM

- A canonical example is the (extremely difficult) N-body problem in which each particle interacts with each other, e.g.

$$H(q_i, p_i, t) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i<j} \phi(|q_i - q_j|)\,.$$

- The main motivation underlying mean-field models is to find a tractable description of intermediate complexity between the N-body problem and the dynamics in an external field.



- Among the key problems we would like to study is chaos and integrability in very large d.o.f. systems.

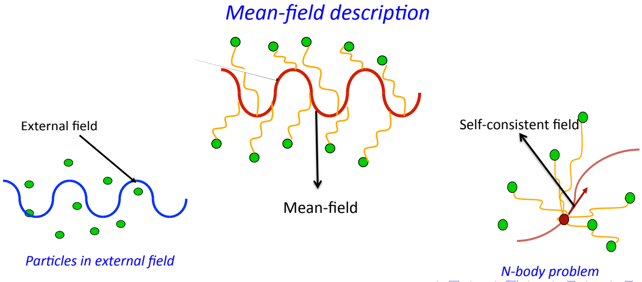
MEAN-FIELD MODELS

- Like in the external field problem, in the mean-field description all the particles "see" the same field

$$H(q_i, p_i, t) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_i \phi(q_i; \lambda) \, .$$

- But, like in the N-body problem there is a coupling between the particles that feeds-back onto the mean-field

$$\lambda = \mathcal{D}\left(q_1, q_2, \ldots q_N\right) \, .$$



THE SINGLE WAVE MODEL

- The mean-field model of interest here is the so-called Single-Wave-Model (SWM) which is a Hamiltonian system consisting of an ensemble of N-particles in one-dimension

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial u_j}, \qquad \frac{du_j}{dt} = -\frac{\partial H}{\partial x_j},$$

with a single-wave potential Hamiltonian

$$H(q_i, p_i, t) = \sum_{k=1}^{N} \left[\frac{u_k^2}{2} - a(t) e^{ix_k} - a^*(t) e^{-ix_k} \right].$$

- In this model the mean-field coupling determines the time evolution of the single-wave potential amplitude from

$$\frac{da}{dt} - iUa = \frac{i}{N} \sum_{k=1}^{N} \Gamma_k e^{-ix_k}.$$

where U and Γ_k, $k = 1, 2, \ldots N$ are constants.

[Onischenko, et al. (1970), O'Neil, Wilfrey and Malberg (1971)]

THE SINGLE WAVE MODEL: DERIVATION AND GENERALIZATION

- Weakly nonlinear theory provides a systematic derivation of the previously stated SWM:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0\,, \qquad \phi = a(t)e^{ix} + a^*(t)e^{-ix}$$

$$\frac{da}{dt} - iUa = i\frac{1}{2\pi}\int_0^{2\pi} dx \int_{-\infty}^{\infty} du\, f(x,u,t)\,.$$

and the corresponding discrete particle formulation

$$\frac{dx_k}{dt} = u_k\,, \qquad \frac{du_k}{dt} = -\frac{\partial \phi}{\partial x}\,, \qquad \frac{da}{dt} - iU = \frac{i}{N}\sum_{k=1}^{N} \Gamma_k e^{-ix_k}\,.$$

as a universal model for marginal stable systems.
- Most importantly, going beyond the original formulation, the theory extends the SWM to $f > 0$ (clumps) and $f < 0$ (holes). In the discrete case this corresponds to $\Gamma_k > 0$ and $\Gamma_k < 0$. [dCN, Phys. Plasmas, 5 (1998); dCN, CHAOS, 10 (2000)]

THE SINGLE WAVE MODEL: $N+1$ HAMILTONIAN FORMULATION

- Defining

$$a = \sqrt{J} e^{-i\theta} \,, \qquad p_k = \Gamma_k y_k \,,$$

the SWM can be equivalently written as an $N+1$, particles+field, Hamiltonian system

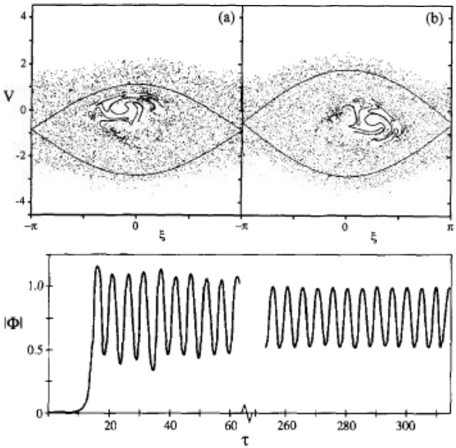
$$\frac{dx_k}{dt} = \frac{\partial \mathcal{H}}{\partial p_k} \,, \qquad \frac{dp_k}{dt} = -\frac{\partial \mathcal{H}}{\partial x_k} \,,$$

$$\frac{d\theta}{dt} = \frac{\partial \mathcal{H}}{\partial J} \,, \qquad \frac{dJ}{dt} = -\frac{\partial \mathcal{H}}{\partial \theta} \,,$$

in which (x_k, p_k) are the canonical coordinates of the N particles, (θ, J) are the canonical coordinates of the mean-field, and

$$\mathcal{H} = \sum_{j=1}^{N} \left[\frac{1}{2\Gamma_j} \frac{p_j^2}{2} - 2\Gamma_j \sqrt{J} \cos(x_j - \theta) \right] - UJ \,.$$

MACROPARTICLE VORTEX FORMATION IN THE SINGLE WAVE MODEL

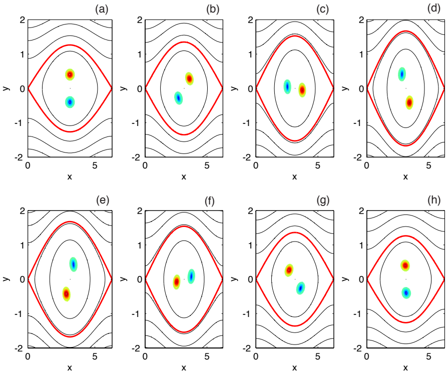


[Tennyson, Meiss and Morrison, Physica D 1994.]

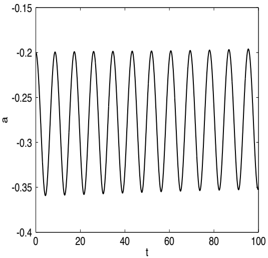
The system relaxes into a time asymptotic periodic state where only few collective degrees of freedom are active.

DIPOLE COHERENT STRUCTURES IN THE SINGLE WAVE MODEL

Numerical simulation of the continuum single wave model



Self-consistent periodic evolution of wave mean field

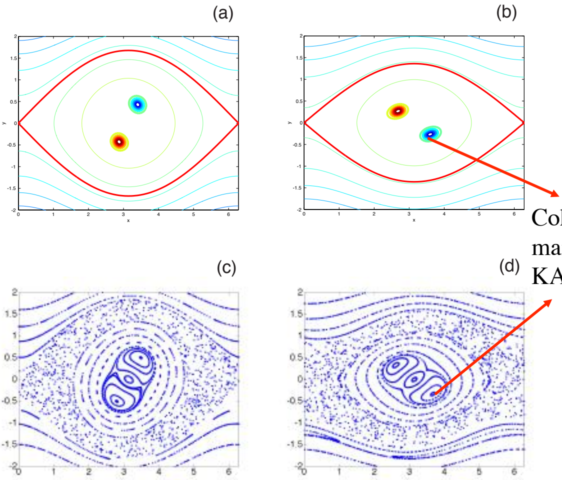


Separatrix "breathing" due to self-consistent wave-particle interaction

ROTATING DIPOLE COHERENT STRUCTURES AND SELF-CONSISTENT CHAOS

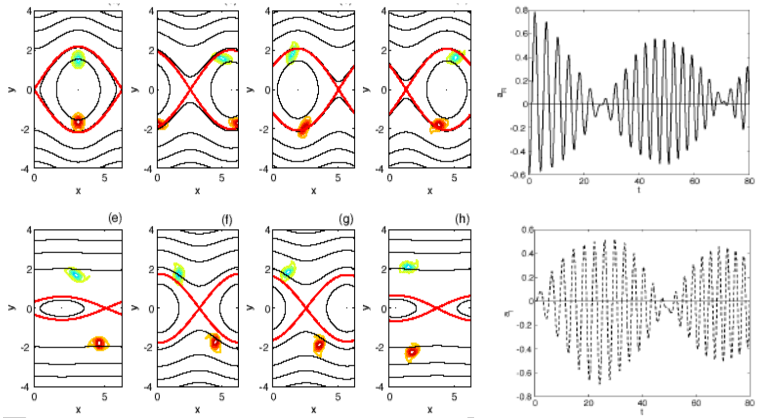
Rotating coherent dipole

Poincare section of time periodic self-consistent mean-field



Coherence maintained by KAM surfaces

ASYMMETRIC DIPOLE STATE



• D. del-Castillo-Negrete, Plasma Physics and Controlled Fusion **47**, 1-11 (2005).

STANDARD MEAN FIELD MAP

$$\frac{d x_j}{dt} = y_j \qquad j = 1,2,\ldots\ldots N$$

$$\frac{dy_j}{dt^2} = -2\rho(t)\sin\left[x_j - \theta(t)\right]$$

particles

$$x_k^{n+1} = x_k^n + y_k^{n+1}$$

$$y_k^{n+1} = y_k^n - \kappa^{n+1}\sin\left(x_k^n - \theta^n\right)$$

$$\frac{d}{dt}\left(\rho\, e^{-i\theta}\right) + i\,U\rho\, e^{-i\theta} = i\sum_k \Gamma_k\, e^{-ix_k}$$

mean field

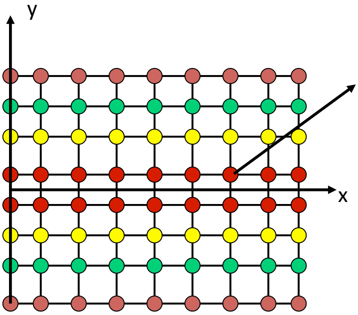
$$\kappa^{n+1} = \sqrt{\left(\kappa^n\right)^2 + \left(\eta^n\right)^2} + \eta^n$$

$$\theta^{n+1} = \theta^n + \frac{1}{\kappa^{n+1}}\frac{\partial\eta^n}{\partial\theta^n}$$

$$\eta^n = \sum_{j=1}^{N}\gamma_j \sin\left(x_j^n - \theta^n\right)$$

- D. del-Castillo-Negrete: CHAOS, 10, 75, (2000).

STANDARD MEAN FIELD MAP



$\left(x_k^0 , y_k^0\right)$ Particles initial positions

$$k = 1,2, ...2 \times 10^5$$

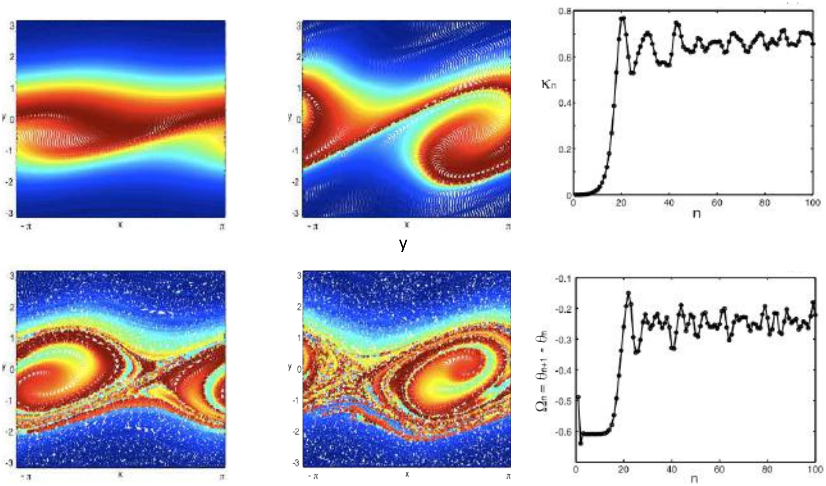
Particle's "charge"

$$\gamma_k = \frac{\tau^3}{\pi} \exp\left(\frac{-y_k^2}{2}\right)$$

Mean field initial condition

$$\theta^0 = 0 \qquad \kappa^0 = 0.001$$

BEAM-PLASMA INSTABILITY AND COHERENT STRUCTURE FORMATION IN THE STANDARD MEAN-FIELD MAP



- D. del-Castillo-Negrete: CHAOS, 10, 75, (2000).

NONTWIST MEAN FIELD MAP

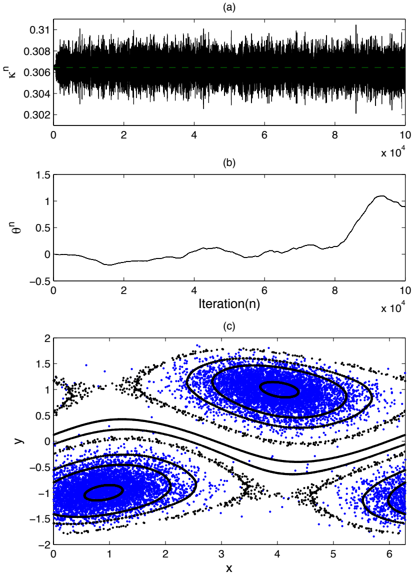
$k = 1, 2, \ldots N$

$$\begin{aligned} x_k^{n+1} &= x_k^n + a\left[1 - \left(\frac{\tau}{\Gamma_k} p_k^{n+1}\right)^2\right], \\ p_k^{n+1} &= p_k^n - 2\tau\Gamma_k\sqrt{J^{n+1}}\sin\left(x_k^n - \theta^n\right), \\ \theta^{n+1} &= \theta^n - U\tau - \frac{\tau}{\sqrt{J^{n+1}}}\sum_{k=1}^{N}\Gamma_k\cos\left(x_k^n - \theta^n\right), \\ J^{n+1} &= J^n + 2\tau\sqrt{J^{n+1}}\sum_{k=1}^{N}\Gamma_k\sin\left(x_k^n - \theta^n\right), \end{aligned} \tag{1}$$

- L. Carbajal, D. del-Castillo-Negrete, and J. J. Martinell, Chaos, 22 013137 (2012).

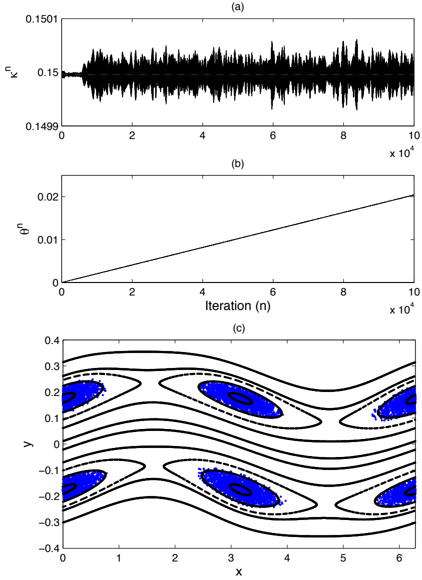
NONTWIST MEAN FIELD MAP

Period-one coherent structures



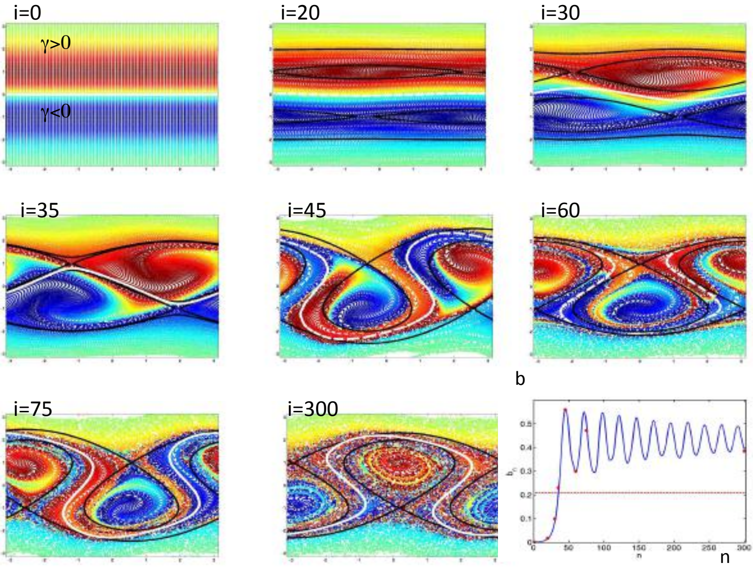
NONTWIST MEAN FIELD MAP

Period-two coherent structures



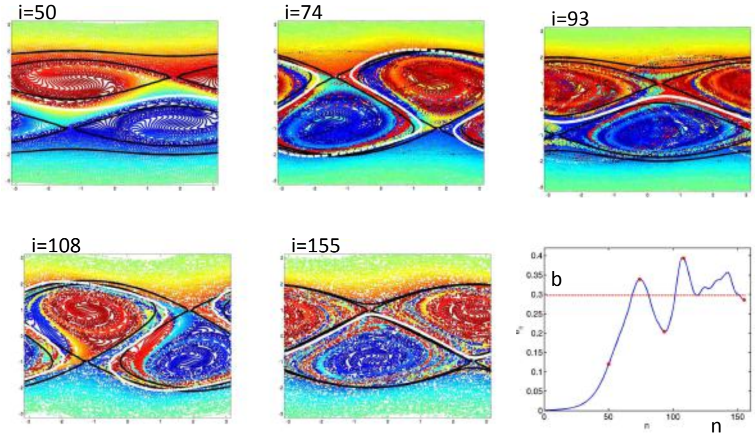
NONTWIST MEAN FIELD MAP

Separatrix reconnection and coherent structure formation



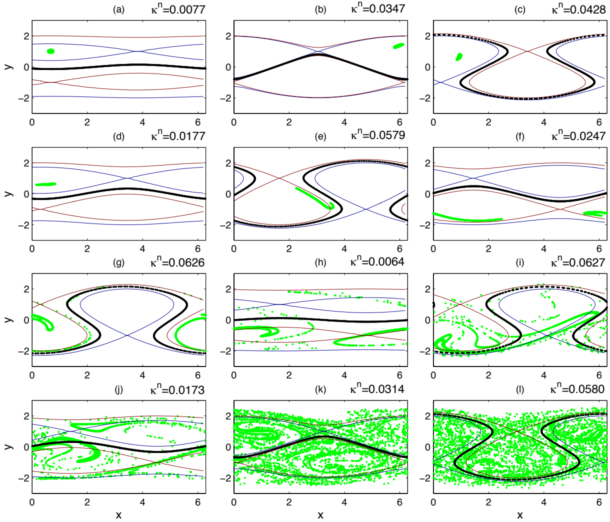
NONTWIST MEAN FIELD MAP

Separatrix reconnection and coherent structure formation



NONTWIST MEAN FIELD MAP

Self-consistent separatrix reconnection in the mean-field map



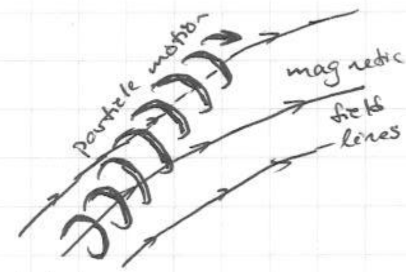
- L. Carbajal, D. del-Castillo-Negrete, and J. J. Martinell, Chaos, 22

PLASMA PHYSICS (AND FLUID DYNAMICS) INSPIRED HAMILTONIAN DYNAMICAL SYSTEMS PROBLEMS

DIEGO DEL-CASTILLO-NEGRETE Notes by Jeffrey Heninger

problems in plasma physics → interesting mathematical problems

Nontwist Hamiltonian Systems



magnetic confinement fusion
- huge span of time scales - gyro time vs. confinement time
- dissect particle motion
- parallel velocity + gyration around field lines + drifts
- simplest approx for particle motion = move along magnetic field lines

write magnetic field as a Hamiltonian system
- simple - integrable - invariant circles
- what happens due to a perturbation?
- KAM - perturbation doesn't qualitatively change things for irrational tori (Diophantine)
  - assumes a nondegeneracy / twist condition

there are important plasma physics problems where the twist condition fails

most of this talk will be about area-preserving maps
finding an exact Poincaré map for a continuous Hamiltonian system is typically impossible
but we can write simple model maps that well approximate this behavior

Standard Map
- Chirikov & Taylor - modeling breaking up magnetic flux surfaces

Standard Nontwist Map
- obvious modification of standard map to allow a nontwist line

transition to chaos → breakup magnetic flux surfaces → loss of confinement
↳ (in fluids) → mixing

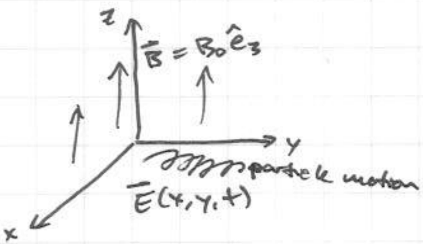
if we replace one of these with another map, do we get similar results? yes

Greene residue criterion
- determine fate of a torus by looking at residues of periodic orbits
- tells us when the last torus breaks up

shearless curve has remarkable robustness - last torus to break fractal structure of residues for golden mean torus in standard map vs. shearless curve in standard nontwist map have different scaling

if residues → 0, still integrable
→ ∞, torus is broken
→ 0.25, critical - at breakup (standard golden mean torus)
→ different value for shearless curve - different universality class

$\vec{E} \times \vec{B}$ drift



what happens ⊥ to the magnetic field?
look at one drift due to an electric field.
if $\vec{E}$ constant, drift is simple
otherwise, we have an interesting dynamical system
no reason that the shear would never vanish
electric field is arbitrary → nontwist systems

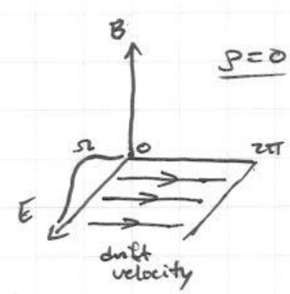
Fluid Mechanics
nontwist condition imposes something on the equilibrium flow
often not held - e.g. in jets.
experiments by Swinney

Gyro-Averaged Hamiltonian Systems

What happens to $\vec{E} \times \vec{B}$ drifts when we also deal with gyromotion?

Where do you evaluate the electric field?
center of gyromotion or current location of particle or average location of particle

sometimes, you can do gyroaverage analytically - often numerically



each particle gets a *label* based on its gyromotion
each particle sees a slightly different Hamiltonians

2 frequencies: gyrofrequency & frequency associated with drift velocity crossing the system size

Maxwellian distribution of gyroradii
different particles can have Hamiltonians with different topologies
for large gyroradii, Hamiltonian is averaged over a larger region
↑ ρ. (also includes scale of turbulent variation of electrostatic turbulance)

gyroaveraged induced chaos suppression
take a chaotic system for $\rho = 0$
larger gyroradius can make the Hamiltonian regular - recreates invariant curves

Gyro-Averaged Area Preserving Maps

use simple models to get similar results

"kicked-rotor" Hamiltonian (→ standard map)
gyroaverage it first → Bessel function of ρ in coefficient for standard maps
for which values of ρ have chaos? do you have a shearless curve?
if ρ is at a zero of the Bessel function, it will be integrable regardless of other parameters

two gyroradii - normalized by different things
$\hat{\rho} = \rho k$ (scale length of electrostatic fluctuations)
$\bar{\rho} = \rho / L$ (scale length of equilibrium flow)

a real system has particles with multiple Larmor radii (ρ) - e.g. Maxwellian.
before, we've been looking at ρ as a parameter for a single particle
there is also a distribution for the strength of the perturbation.
↳ probability distribution for confinement

quasilinear theory - can convert it to a diffusion problem
$D(\rho) = $ "$\sqrt{K(\rho)}$" different parts of distribution function diffuse at
$f = f(\rho)\ G(\Delta)$ different rates → no longer Gaussian

<u>Mean-Field Coupled Hamiltonian Systems</u>

self-consist problem is much more difficult
can we make simple models for this?

N body problem ($N \gg 1$) - each particle feels fields of each other particle

mean field models - one mean field that interacts with all the particles
single wave model - all particles see same Hamiltonian, which is determined
↳ by the behavior of the particles
↳ this is simplest high degree of freedom Hamiltonian system to study.
use weak nonlinear system - gives you a parameter
system can relax to a state with only a few degrees of freedom
can turn this into a map - parameters also have dynamics