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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Richard Moeckel

Talk Title: Two Dimensional Examples of the Jacobi-Maupertuis Metric

Date: 11 / 27 / 2018 **Time:** 10 : 30 **am** / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Looks at the surface corresponding to configuration space of a 2 degree of freedom system with the Jacobi-Maupertuis metric. Can we embed this as a surface of revolution in R^3 ? Problems occur near the Hill boundary. Also looks at geodesic motion on the collision manifold.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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Two-Dimensional Examples of the Jacobi Maupertuis Metric

MSRI, November 2018

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Longstanding Goal: Understand the geometry of the Jacobi-Mauptertuis metric arising from problems in celestial mechanics. Start with low dimensional cases where we can draw pictures.

Lagrangian system of two degrees of freedom with fixed energy, h

↓

Planar Kepler problem
Collinear three-body problem
Isosceles three-body problem

Riemannian metric on two-dimensional configuration space — a surface

- Get intuition about the geometry for its own sake
- Visualize the surfaces, e.g., by embedding in R^3
- Use known properties about the orbits in phase space to understand geodesics
- Use Riemannian methods for geodesics to get existence proofs for new orbits

The Jacobi-Maupertuis Metric

Let

$$q \in \mathcal{U} \subset \mathbb{R}^n \qquad\qquad \mathcal{U} = \text{Configuration Space}$$

Consider a Lagrangian system on phase space $\mathcal{U} \times \mathbb{R}^n$ of the standard form:

$$L(q, v) = \frac{1}{2}\|v\|^2 + U(q)$$

$$\|v\|^2 = \text{Kinetic Energy Metric}$$

$$U(q) = \text{Minus Potential Energy}$$

Fix an energy level

$$E(q, v) = \frac{1}{2}\|v\|^2 - U(q) = h$$

and define the corresponding Jacobi-Mauptertuis metric

$$\mathbf{g}(v, v) = 2(U(q) + h)\|v\|^2.$$

It's a Riemannian metric on the *Hill's Region*

$$\mathcal{H}(h) = \{q \in \mathcal{U} : U(q) + h \geq 0\}.$$

Euler-Lagrange for L ⟷ Geodesics of **g**

Suppose the metric looks like $\|v\|^2 = v \cdot Mv$ for M a symmetric matrix. Let $p = L_v = Mv$. Then the Euler-Lagrange and energy equations for $L(q,v)$ are

$$\begin{aligned} M\dot{q} &= p \\ \dot{p} &= \nabla U(q) \\ \frac{1}{2}\|\dot{q}\|^2 - U(q) &= h \end{aligned}$$

On the other hand, unit speed geodesics for **g** satisfy the Euler-Lagrange equations for

$$\tilde{L}(q,v) = \frac{1}{2}\mathbf{g}(v,v) = (U(q)+h)\|v\|^2.$$

Let $p = (\tilde{L})_v = 2(U(q)+h)Mv$. Then

$$\begin{aligned} Mq' &= \frac{p}{2(U(q)+h)} \\ p' &= \frac{\nabla U(q)}{2(U(q)+h)} \\ 2(U(q)+h)\|q'\|^2 &= 1 \end{aligned}$$

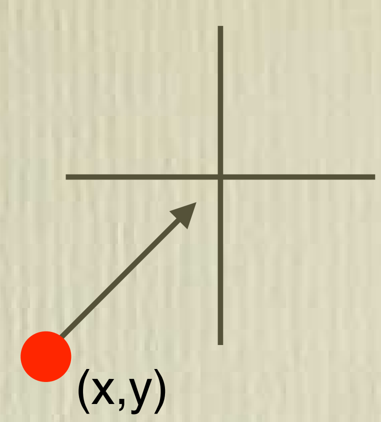
Just a change of timescale: $' = \dfrac{1}{2(U(q)+h)} \dot{}$

Two dimensional examples considered here

Kepler problem:

$$U(x,y) = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

$$\mathbf{g} = 2\left(\frac{1}{\sqrt{x^2+y^2}} + h\right)(dx^2+dy^2) = 2\left(\frac{1}{r}+h\right)(dr^2 + r^2\, d\theta^2)$$

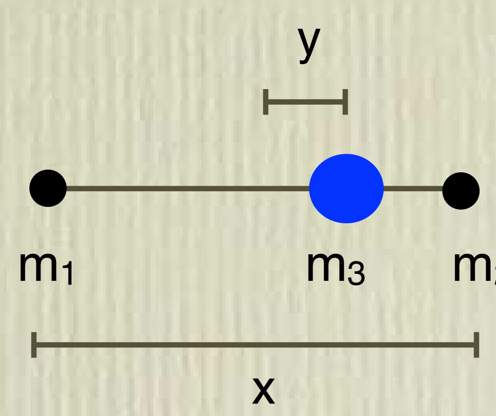


Collinear 3BP:

$$U(x,y) = \frac{1}{x} + \frac{m_3}{\frac{x}{2}-y} + \frac{m_3}{\frac{x}{2}+y} \qquad m_1 = m_2 = 1$$

$$\mathbf{g} = 2\,(U(x,y)+h)\,(\mu_1\, dx^2 + \mu_2 dy^2)$$

$$\mathcal{H}(h) = \{(x,y) : x \geq 0, -\frac{x}{2} \leq y \leq \frac{x}{2}, U(x,y) \geq 0\}$$



$$\mu_1 = \frac{m_1 m_2}{m_1+m_2} = \frac{1}{2}$$

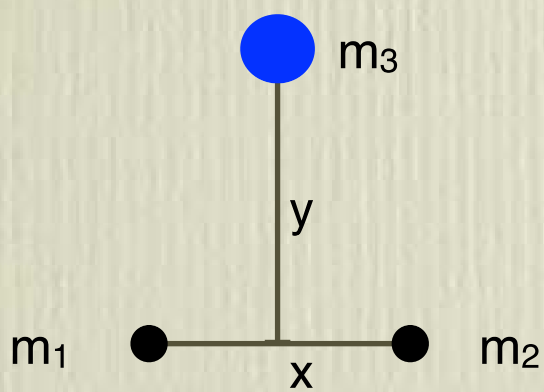
$$\mu_2 = \frac{(m_1+m_2)m_3}{m_1+m_2+m_3} = \frac{2m_3}{2+m_3}$$

Isosceles 3BP:

$$U(x,y) = \frac{1}{x} + \frac{2m_3}{\sqrt{\frac{x^2}{4}+y^2}} \qquad m_1 = m_2 = 1$$

$$\mathbf{g} = 2\,(U(x,y)+h)\,(\mu_1\, dx^2 + \mu_2 dy^2)$$

$$\mathcal{H}(h) = \{(x,y) : x > 0, U(x,y)+h \geq 0\}$$



The Kepler Surfaces

$$U(x,y) = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

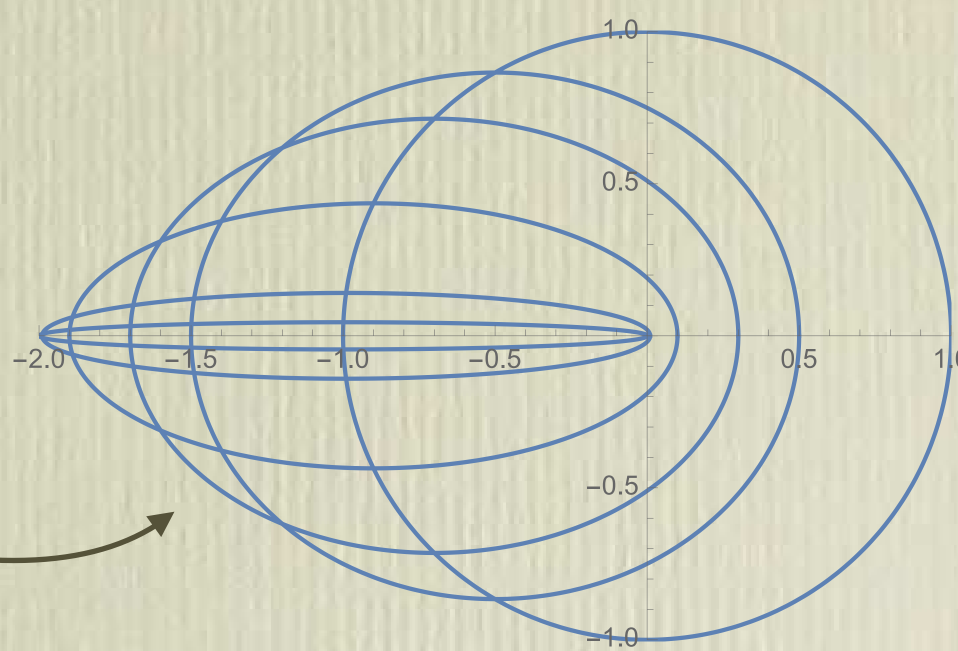
Central Force Problem — Rotation Invariant

$$\mathbf{g} = 2\left(\frac{1}{\sqrt{x^2+y^2}} + h\right)(dx^2+dy^2) = 2\left(\frac{1}{r}+h\right)(dr^2 + r^2\,d\theta^2)$$

The well-known solutions sweep out conic sections in the configuration space R^2

- Positive energy: Hyperbolas
- Zero energy: Parabolas
- Negative energy: Ellipses

- h=1
- h=0
- h=-1



Geodesics for the Jacobi-Mauptertuis metric

The Jacobi-Maupertuis metric describes an “abstract surface of revolution”.

Question: Can we find an embedding as a surface of revolution in R^3 ?

Embedding the Kepler Surfaces

$$Kepler: \qquad \mathbf{g} = \left(\frac{1}{r} + h\right)(dr^2 + r^2\,d\theta^2) = \left(\frac{1}{r} + h\right)dr^2 + r(1+hr)\,d\theta^2$$

$$\mathbb{R}^3: \qquad \mathbf{g}_{Euc} = d\rho^2 + dz^2 + \rho^2\,d\theta^2$$

Embedding as a surface of revolutions $\rho = f(r), z = g(r)$ which gives the metric

$$g = (f'(r)^2 + g'(r)^2)\,dr^2 + f(r)^2\,d\theta^2.$$

We must have

$$f(r) = \sqrt{r(1+hr)} \qquad f'(r)^2 + g'(r)^2 = \frac{1}{r} + h$$

which gives

$$g'(r)^2 = \frac{3+hr}{4r(1+hr)}.$$

Just solve this equation for $g(r)$ to get an embedding.

R.M.: *Embedding the Kepler problem as a surface of revolution,* to appear Reg.Ch.Dyn.
Volume 23, Issue 6, 2018

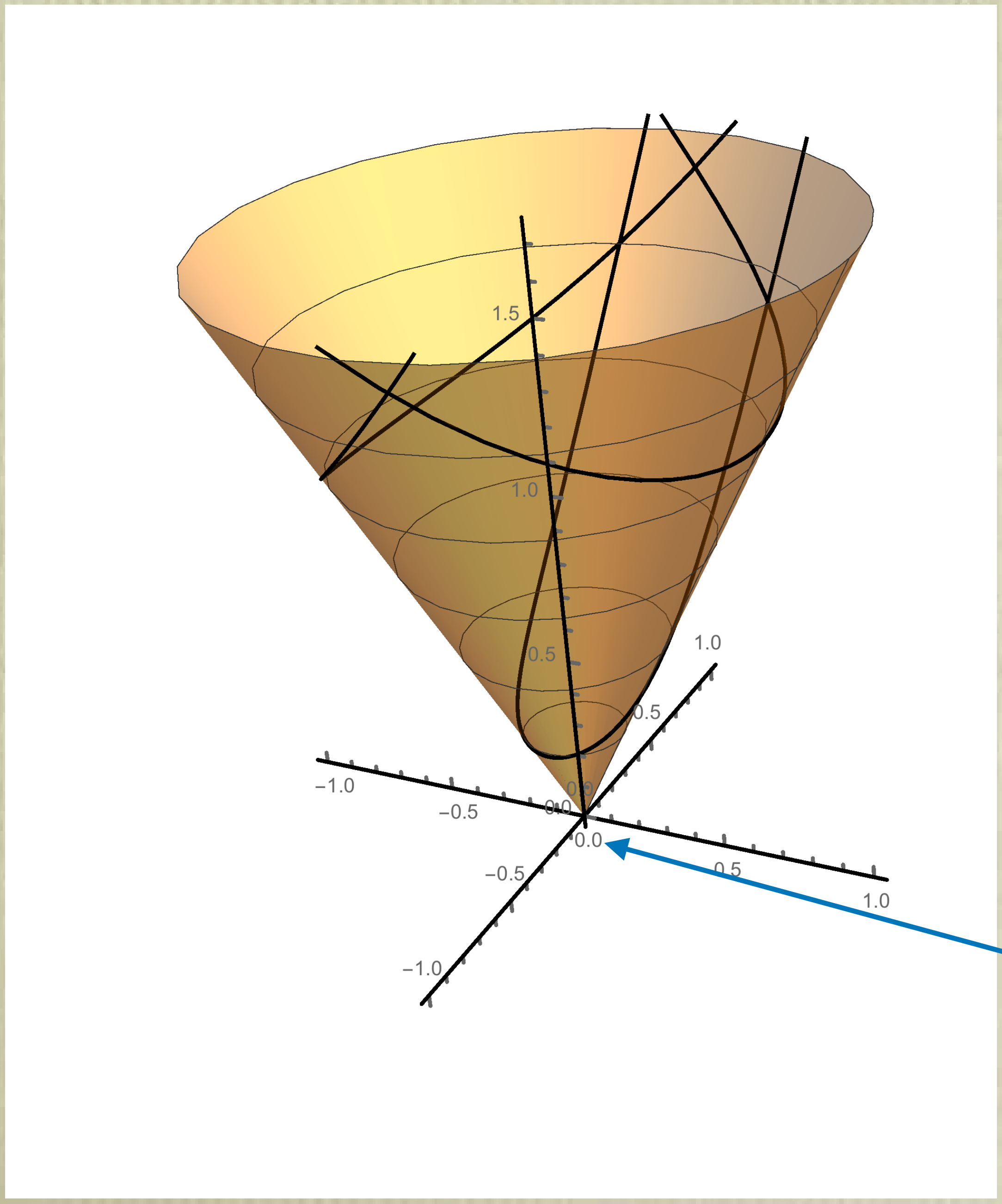
h=0 Kepler Surface is a Cone

For energy $h = 0$ we have an embedding in cylindrical coordinates (ρ, θ, z) where

$$\rho = f(r) = \sqrt{r} \qquad z = g(r) = \sqrt{3r}$$

so the generating curve is given by the line

$$z = \sqrt{3}\,\rho$$



The surface is a cone. The opening angle is π / 3.

You can make one from a sheet of paper by gluing the edges together.

Straight lines on the paper go to geodesics on the cone.

Note: Collision singularity becomes a cone point on the surface.

Positive Energy Kepler Surface

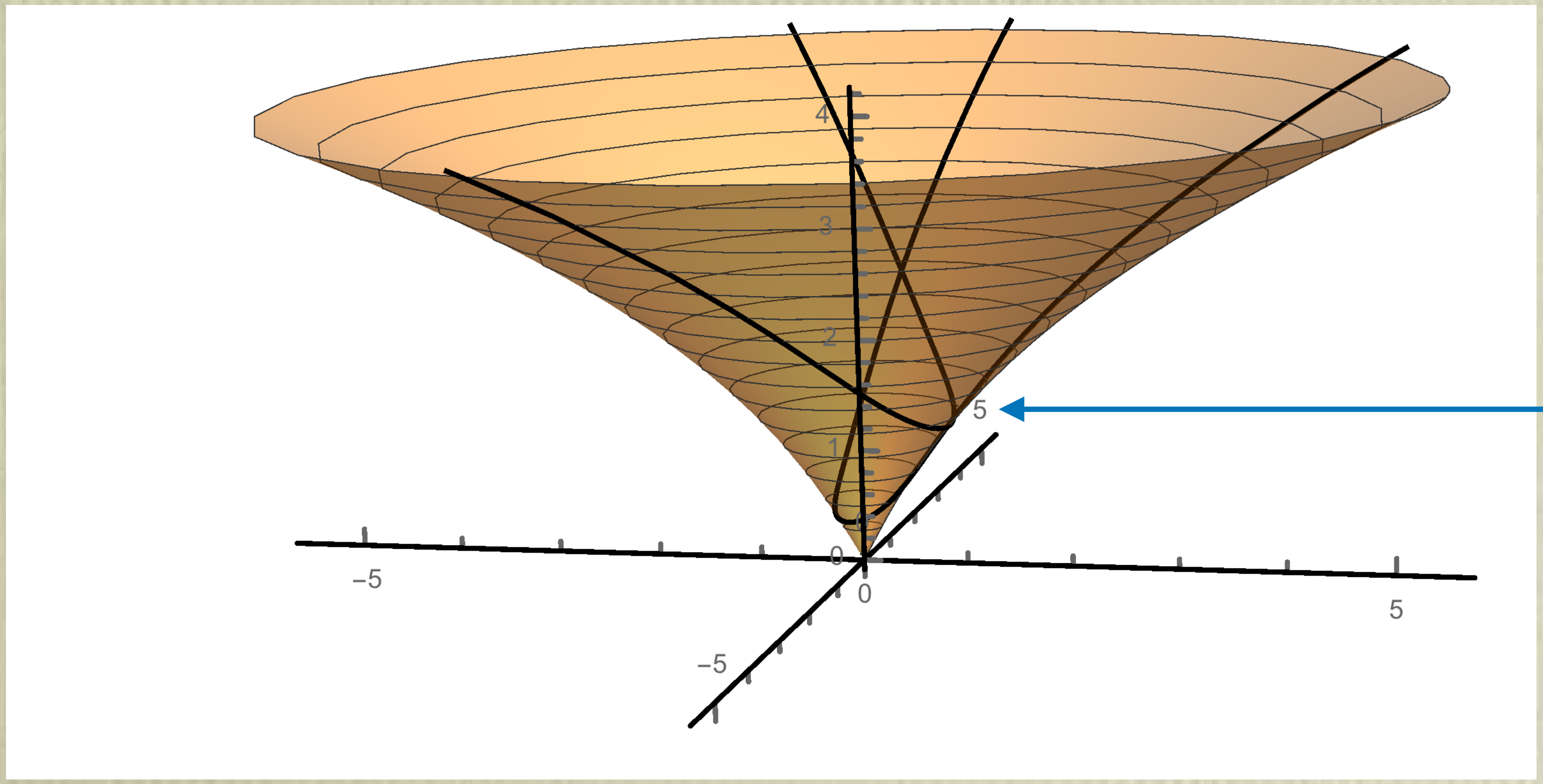
Next suppose $h = 1$. We find $\rho = f(r), z = g(r)$ where

$$f(r) = \sqrt{r(1+r)} \qquad g(r) = \int_0^r \sqrt{\frac{3+4r}{4r(1+r)}}\,dr = \int_{\sqrt{3}}^{t(r)} \frac{t^2\,dt}{\sqrt{(t^2-3)(t^2+1)}}$$

This is an elliptic integral which can be evaluated with some effort to give

$$g(r) = \frac{3}{2}u - 2E(u,k) + 2\mathbf{dn}(u,k)\mathbf{sc1}(u,k) \quad \text{where} \quad \mathbf{nc}(u,k) = \sqrt{1+\frac{4r}{3}}$$

where $\mathbf{dn}(u,k), \mathbf{sc1}(u,k), \mathbf{nc}(u,k)$ are Jacobi elliptic function and $E(u,k)$ is the Jacobi elliptic integral of the second kind with modulus $k = \frac{1}{2}$.

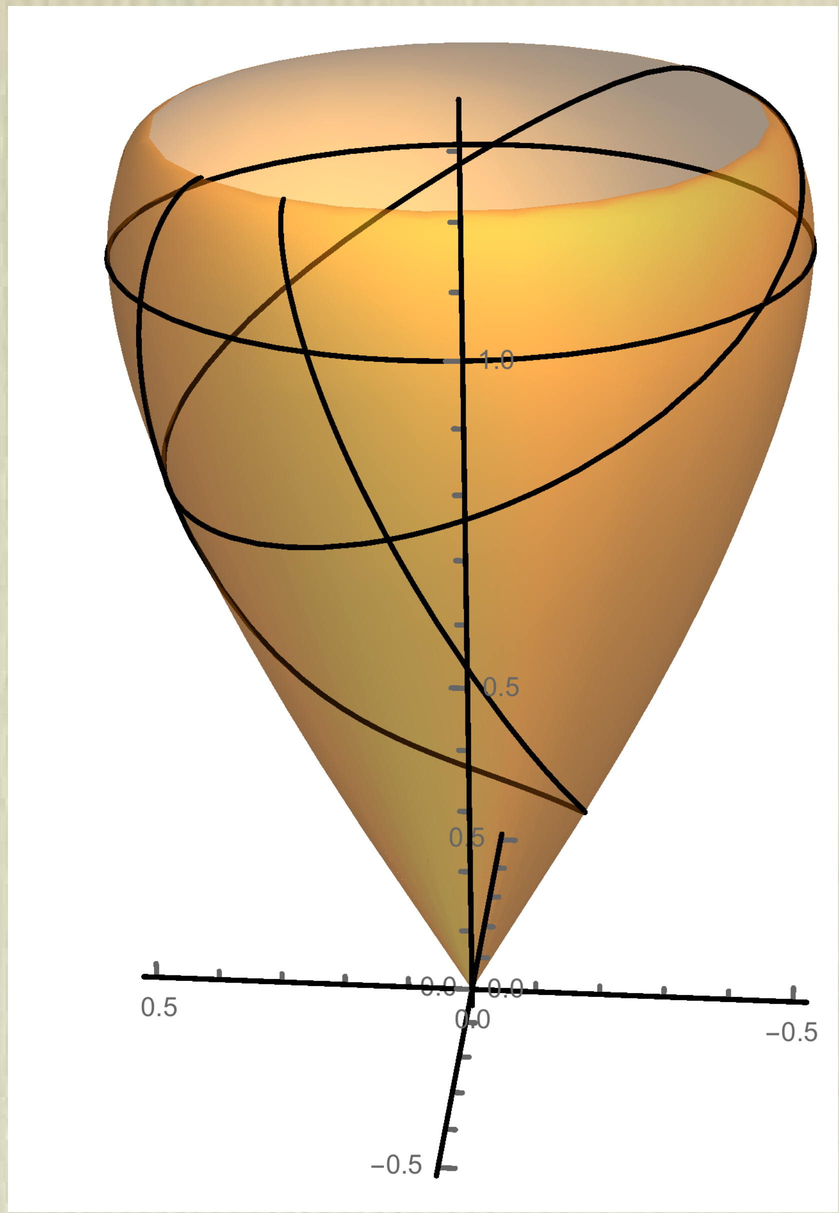


Geodesics correspond to hyperbolic Kepler orbits

Negative Energy Kepler Surface

Finally, suppose $h = -1$. The Jacobi-Mauptertuis metric is

$$\mathbf{g} = 2\left(\frac{1}{r} - 1\right)(dr^2 + r^2\,d\theta^2)$$



with Hill's region

$$\mathcal{H}(-1) = \{r \leq 1\} = \text{unit disk.}$$

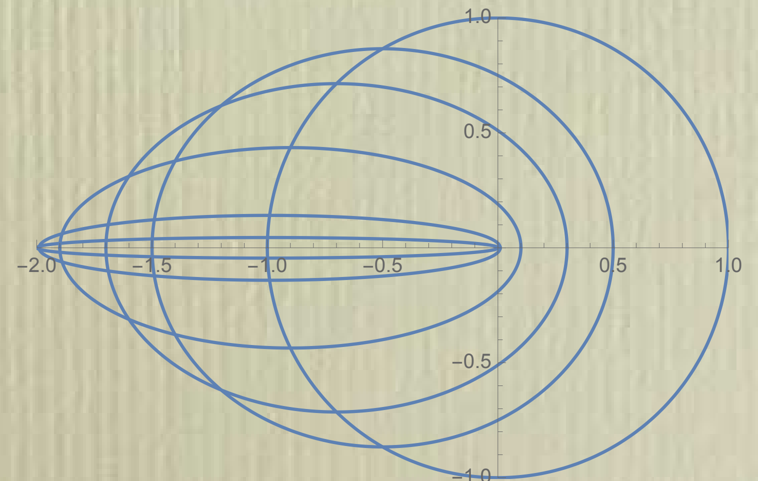
We have $\rho = f(r), z = g(r)$ where

$$f(r) = \sqrt{r(1-r)} \qquad g'(r)^2 = \frac{3-4r}{4r(1-r)}$$

Clearly the solution for $g(r)$ can only be valid for $0 \leq r \leq \frac{3}{4}$.

It is not possible to embed the entire surface !

There are problems near the Hill boundary $\{r = 1\}$.



Elliptical Kepler orbits with eccentricities e ≤ 1/2 appear as geodesics on the surface of revolution. The others pass too close to the Hill boundary and hit the edge of the embedded part of the surface.

More Non-Embedding Results

Further investigation reveals that the negative energy Kepler surface cannot be embedded ***as a surface of revolution*** in

- Euclidean space R^n for any n
- A round sphere S^n for any n
- Hyperbolic three-space

In each case, there is a problem near the Hill boundary r = 1

For example, in the hyperbolic case $g(r)$ would satisfy

$$4r(1-r)(r^2-r-\alpha^{-2})g'(r)^2+4r(1-r)(2r-1)g(r)g'(r)+(3-4r)g(r)^2=0.$$

For $r \approx 1$ it turns out that $g'(r)/g(r)$ would have to be nonreal.

How about an embedding in three-dimensional Minkowski space ?

$$Minkowski: \qquad \mathbf{g}_{Mink} = d\rho^2 + \rho^2\, d\theta^2 - dz^2$$

Setting $\rho = f(r), z = g(r)$ gives

$$f(r) = \sqrt{r(1-r)} \qquad g'(r)^2 = -\frac{3-4r}{4r(1-r)}.$$

The sign change on $g'(r)^2$ means we can embed the part with $\frac{3}{4} \leq r < 1$ but not the previous part !

There is an embedding in four-dimensional Minkowski space, but this seems like cheating.

Even More Non-Embedding Results

The lack of a symmetrical embedding in Euclidean space turns out to be a general property of ***central force*** problems when there is a Hill boundary. For example, one cannot even embed the Jacobi-Maupertuis metric of the harmonic oscillator with h = 1/2 as a surface of revolution

$$\mathbf{g}_{harm} = \left(1 - r^2\right)\left(dr^2 + r^2\,\delta\theta^2\right)$$

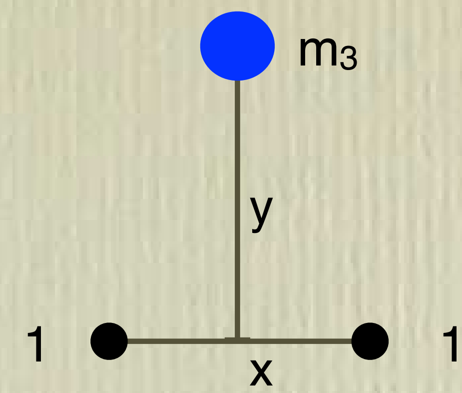
Theorem: Suppose $U(r)$ is an analytic function such that $r = r_0 > 0$ is a boundary circle for the Hill's region $\mathcal{H}(h) = \{U(r) + h \geq 0\}$. Then there is $\delta > 0$ such that the Jacobi-Mauptertuis metric on the part of the Hill region with $0 < |r - r_0| < \delta$ does not admit an embedding as a surface of revolution in $\mathbb{R}^3$.

Proof idea: A local analysis of a hypothetical embedding $\rho = f(r), z = g(r)$ shows that the $g'(r)^2 < 0$ near $r = r_0$.

Minimal Geodesics for the Isosceles 3BP

Isosceles 3BP:

$$U(x,y) = \frac{1}{x} + \frac{2m_3}{\sqrt{\frac{x^2}{4} + y^2}} \qquad m_1 = m_2 = 1$$



$$\mathbf{g} = 2\left(U(x,y) + h\right)\left(\mu_1\, dx^2 + \mu_2 dy^2\right)$$

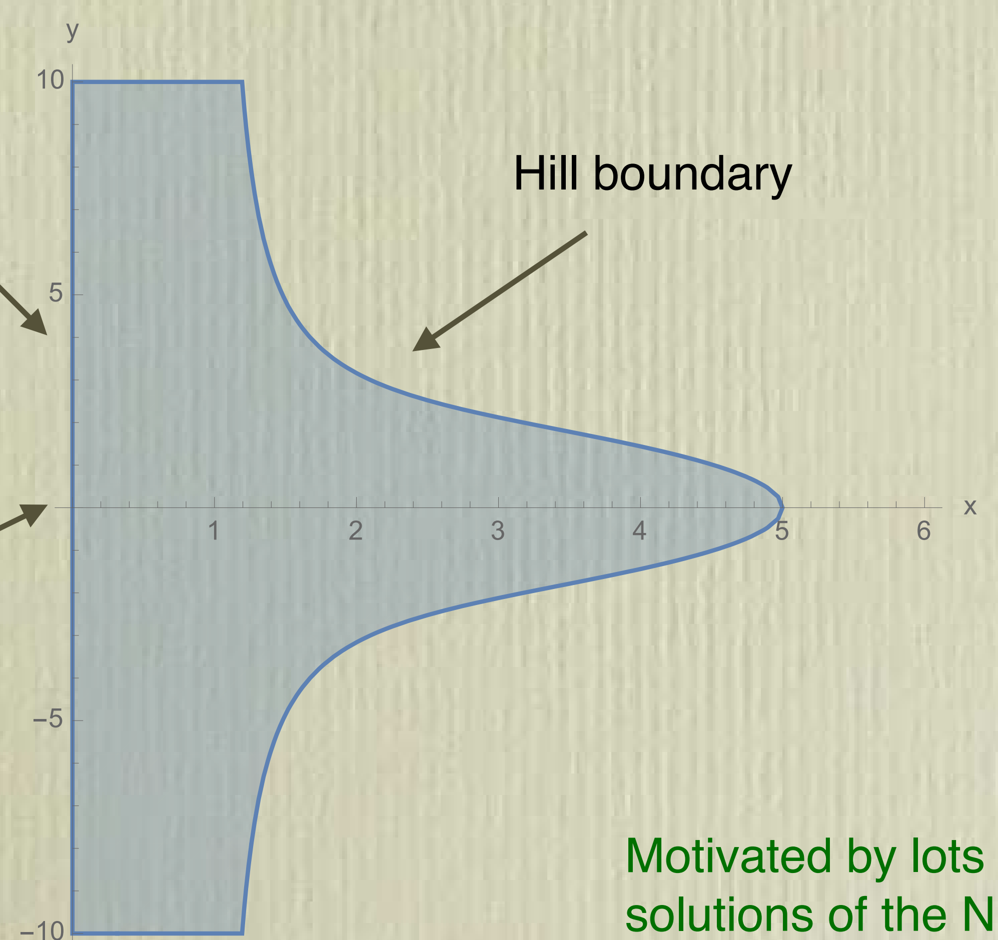
$$\mathcal{H}(h) = \{(x,y) : x > 0, U(x,y) + h \geq 0\}$$

Note: The conformal factor 2(U(x,y) + h) is infinite at double and triple collisions and it's 0 on the Hill boundary. Nevertheless we want to consider solutions which hit these sets.

double collision, x=0

Hill boundary

triple collision
x=y=0



This is a negative energy case.

For h ≥ 0 we can avoid the Hill boundary !

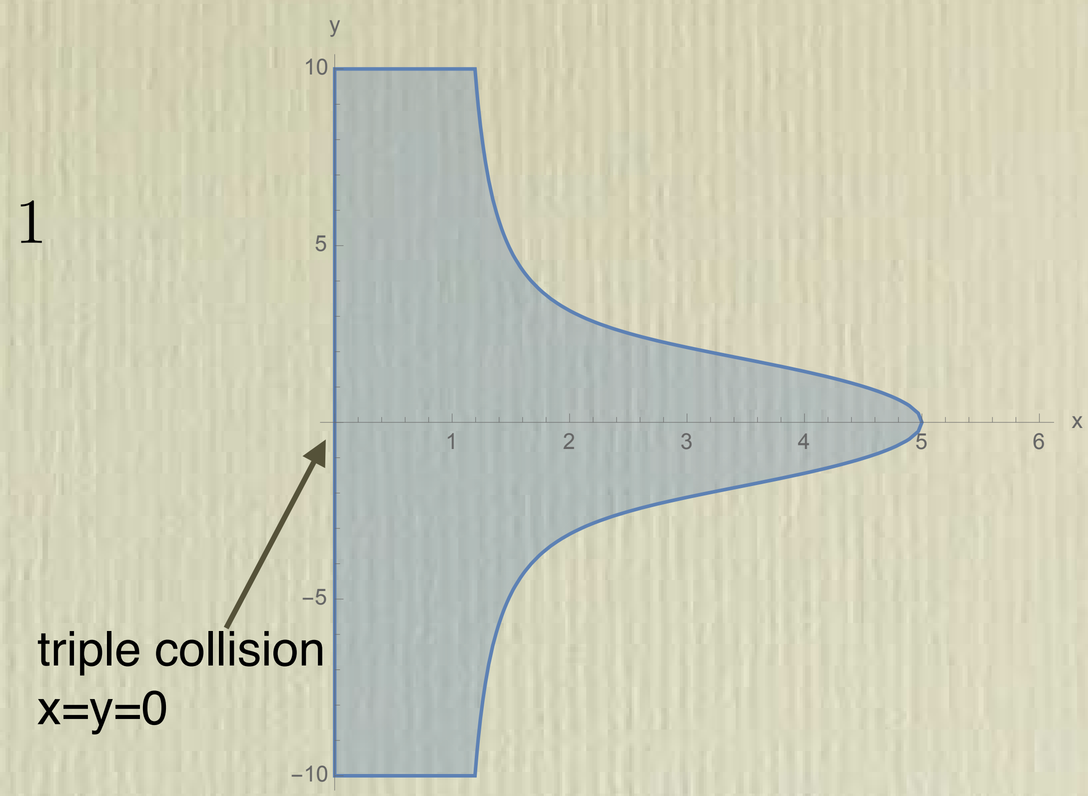
Motivated by lots of recent work on action minimizing solutions of the N-body problem by Maderna, Venturelli, Terracini, Barutello, Sanchez-Morgado, Montgomery, …

Blow-up of Triple Collision

$$U(x,y) = \frac{1}{x} + \frac{2m_3}{\sqrt{\frac{x^2}{4} + y^2}} \qquad m_1 = m_2 = 1$$

$$\mathbf{g} = 2\left(U(x,y) + h\right)\left(\mu_1\, dx^2 + \mu_2 dy^2\right)$$

$$\mathcal{H}(h) = \{(x,y) : x > 0, U(x,y) + h \geq 0\}$$



Blow-up Triple Collision

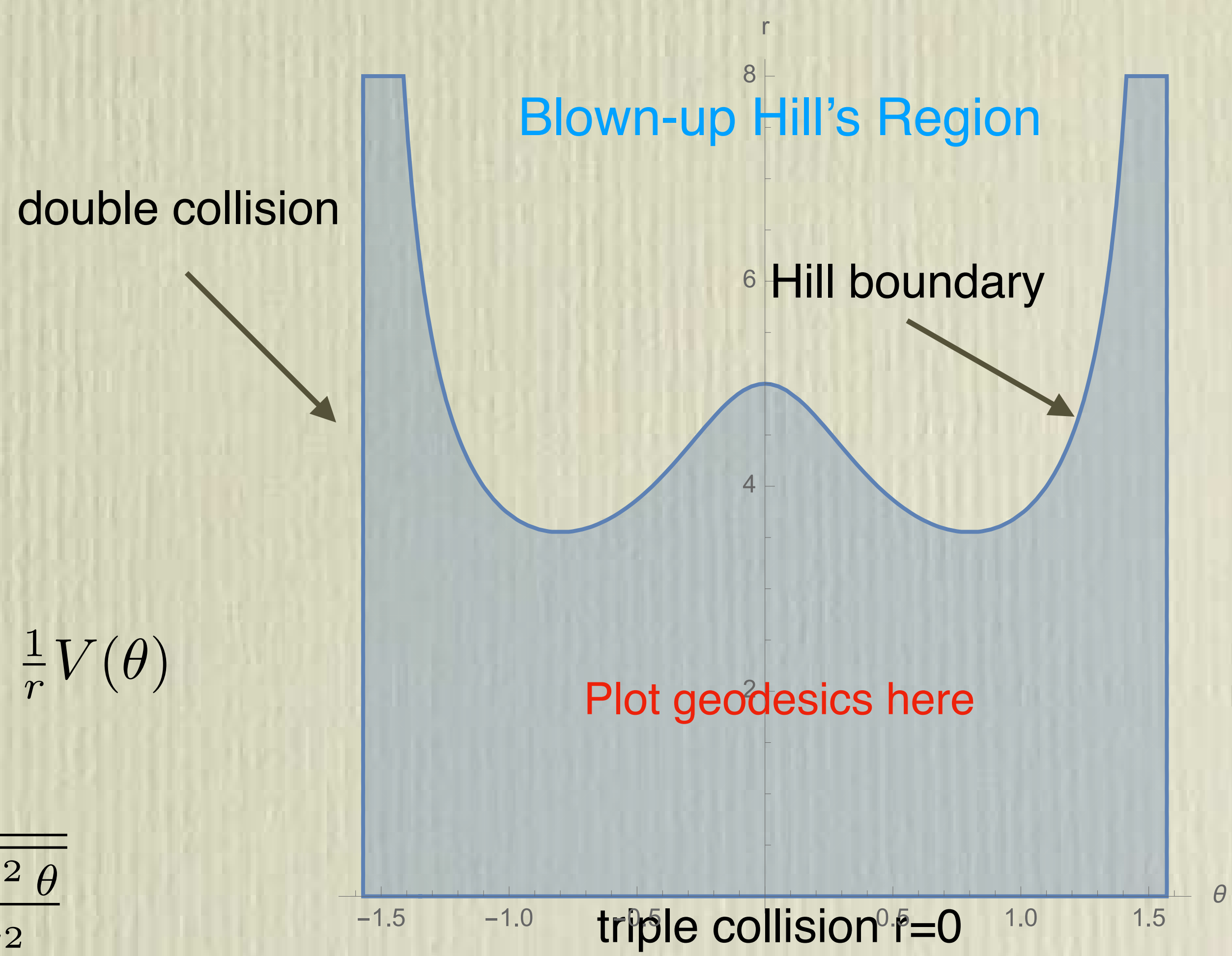
$$x = \frac{r\cos\theta}{\sqrt{\mu_1}} \qquad y = \frac{r\sin\theta}{\sqrt{\mu_2}}$$

$$\mathbf{g} = \frac{2}{r} V(\theta)(dr^2 + r^2 d\theta^2)$$

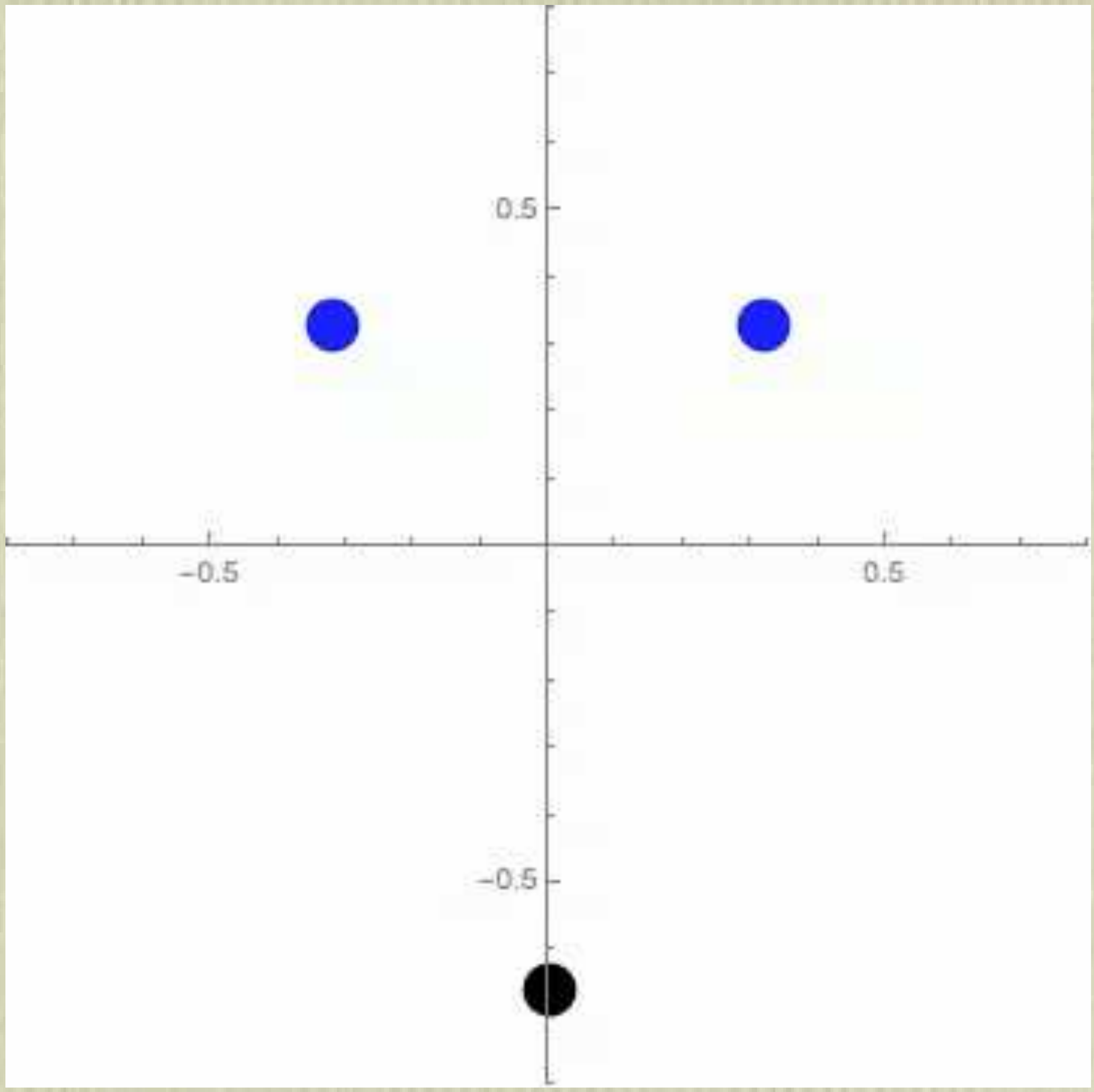
$$\tilde{\mathcal{H}}(h) = \{(r,\theta) : r \geq 0, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$$

Shape potential $V(\theta)$ such that $U(x,y) = \frac{1}{r}V(\theta)$

$$V(\theta) = \frac{\sqrt{\mu_1}}{\cos\theta} + \frac{2m_3}{\sqrt{\frac{\cos^2\theta}{4\mu_1} + \frac{\sin^2\theta}{\mu_2}}}$$

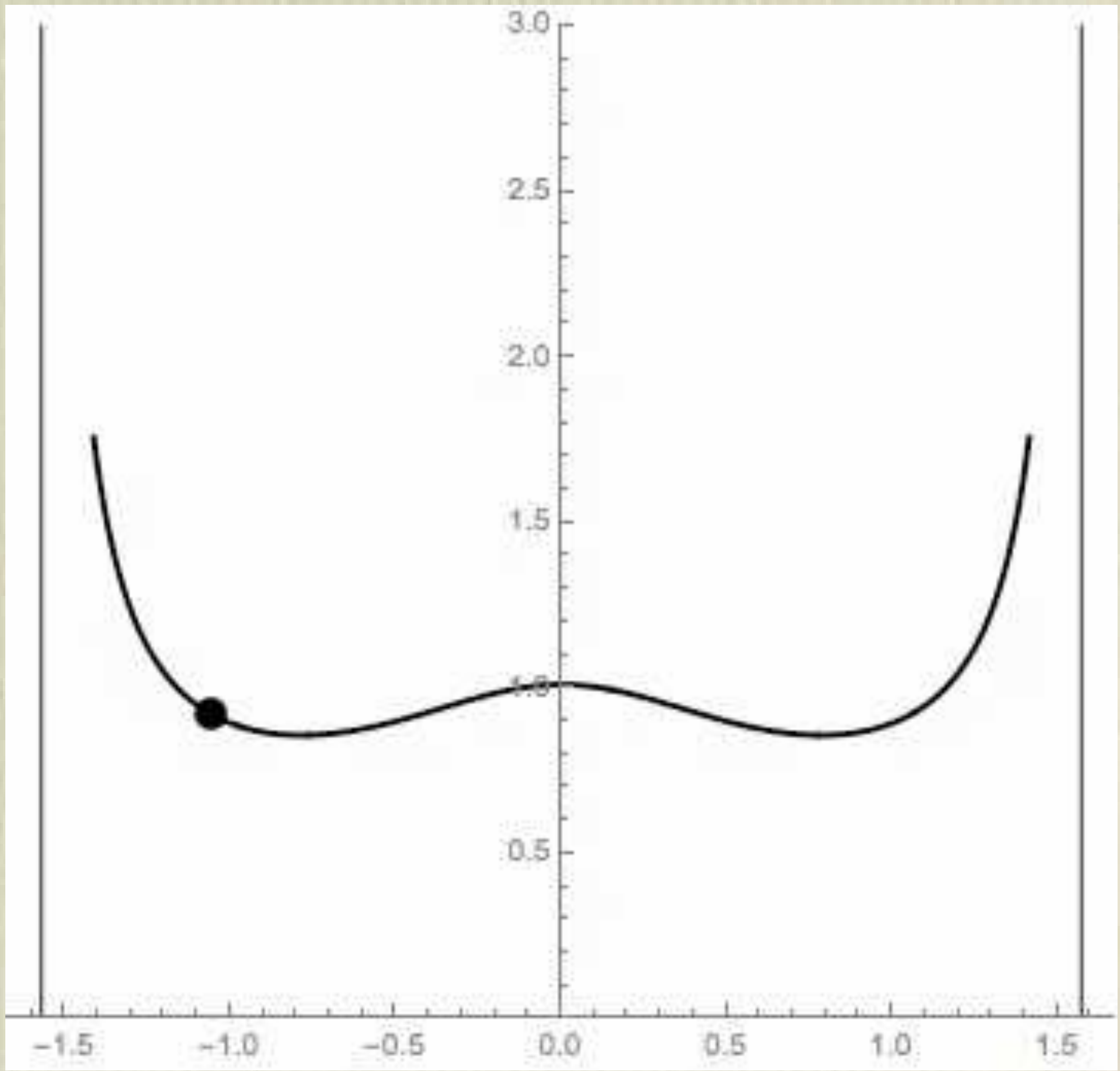


Examples of isosceles solutions/geodesics



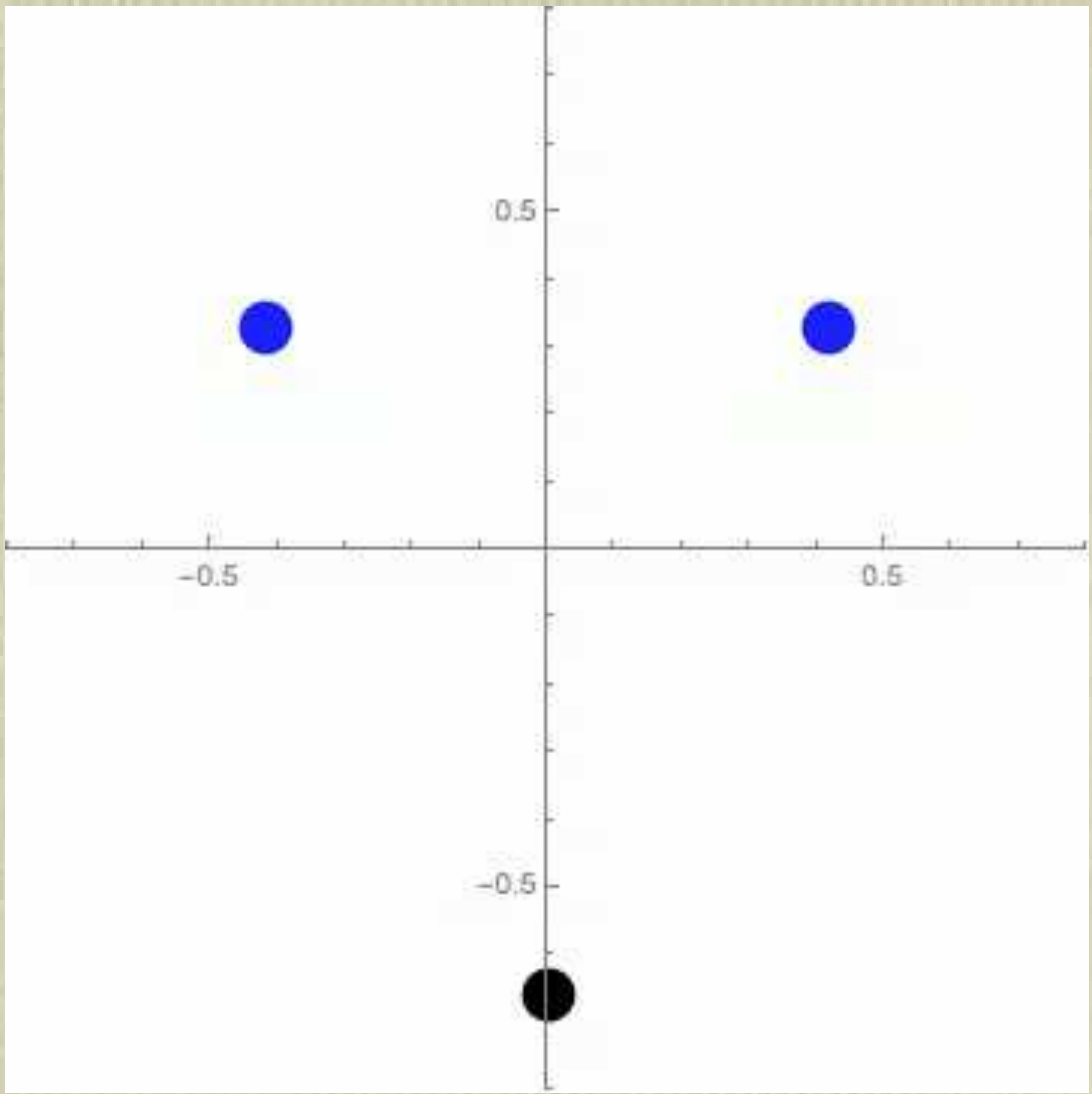
Sample solution with negative energy

Corresponding geodesic in the blown-up Hill's region



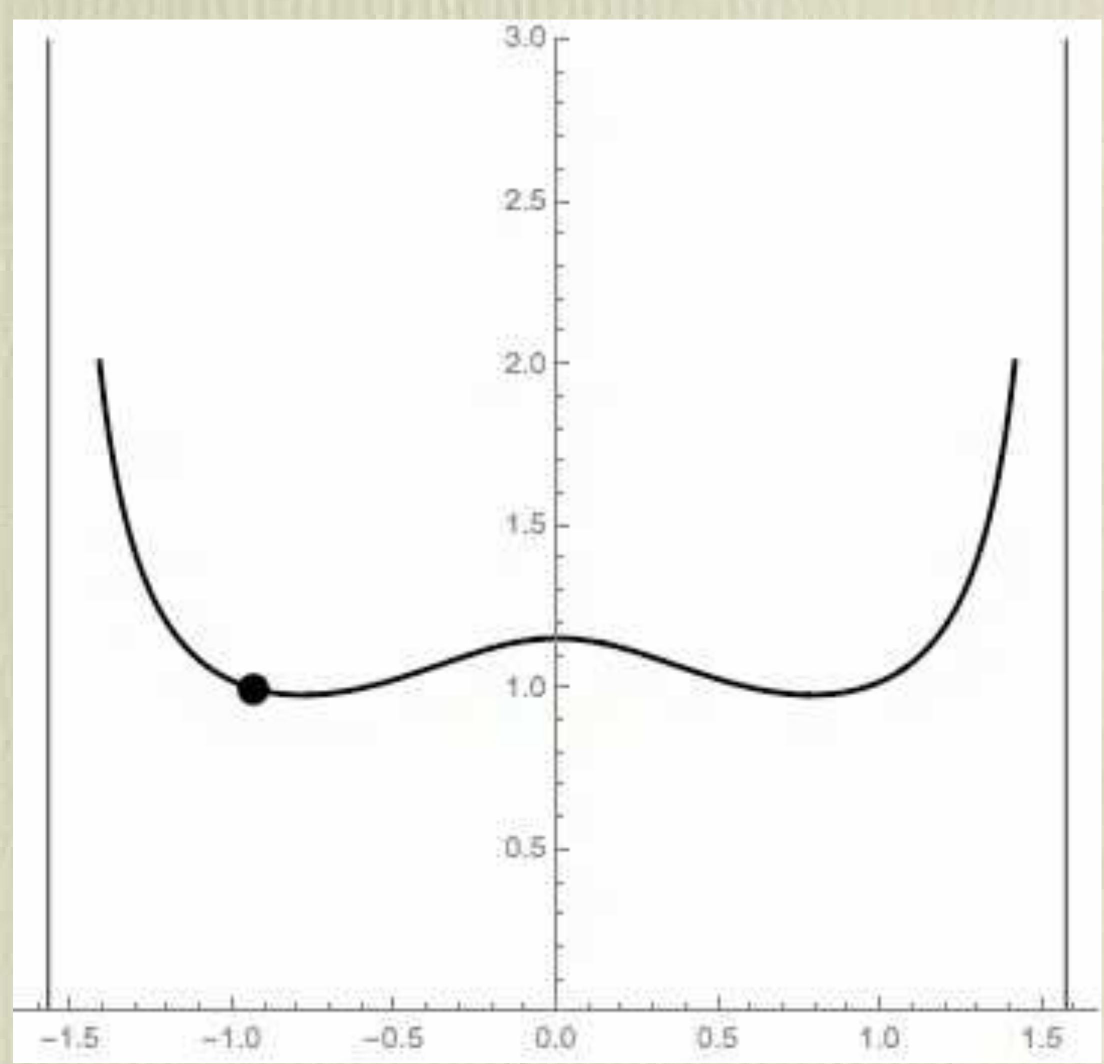
Note: We have to "regularilize" double collisions. Altered timescale is slower near collisions.

Examples of isosceles solutions/geodesics



Periodic Break Orbit

Corresponding geodesic in the blown-up Hill's region



Orbits in Phase Space vs. Geodesics in Configuration Space

Two approaches to the problem:

- Qualitative study of the flow in blown-up phase space
- Variational study of geodesics

Blown-up ODE in McGehee coordinates

$$r' = vr\cos\theta$$

$$v' = W(\theta) - \frac{1}{2}v^2\cos\theta + 2r\,h\cos\theta$$

$$\theta' = w$$

$$w' = W'(\theta)\cos\theta - W(\theta)\sin\theta - \frac{1}{2}vw\cos\theta - (2rh - v^2)\sin\theta\cos\theta$$

New timescale: $' = r^{\frac{3}{2}}\cos\theta$.

Slow down near collisions

and the energy equation is

$$\frac{1}{2}(v^2\cos^2\theta + w^2) - W(\theta)\cos\theta = rh\cos^2\theta.$$

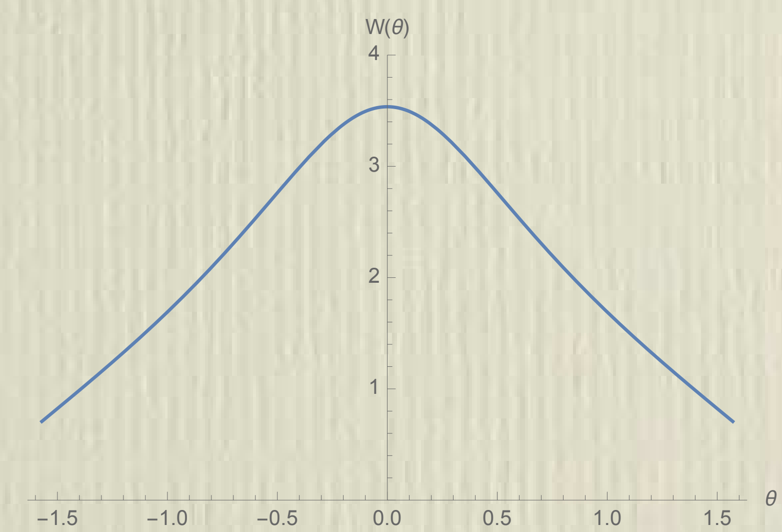
Variables

$r \in [0,\infty) =$ Size of the Triangle $\qquad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] =$ Shape of triangle

$v =$ Radial Velocity $\qquad w =$ Angular Velocity

Regularized Potential

$$W(\theta) = V(\theta)\cos\theta = \frac{1}{\sqrt{2}} + \frac{2m_3\cos\theta}{\sqrt{\frac{\cos^2\theta}{2} + \frac{\sin^2\theta}{\mu}}}.$$



Zero Energy Isosceles 3BP

Focus on the easiest case — Energy h = 0

Blown-up ODE in McGehee coordinates

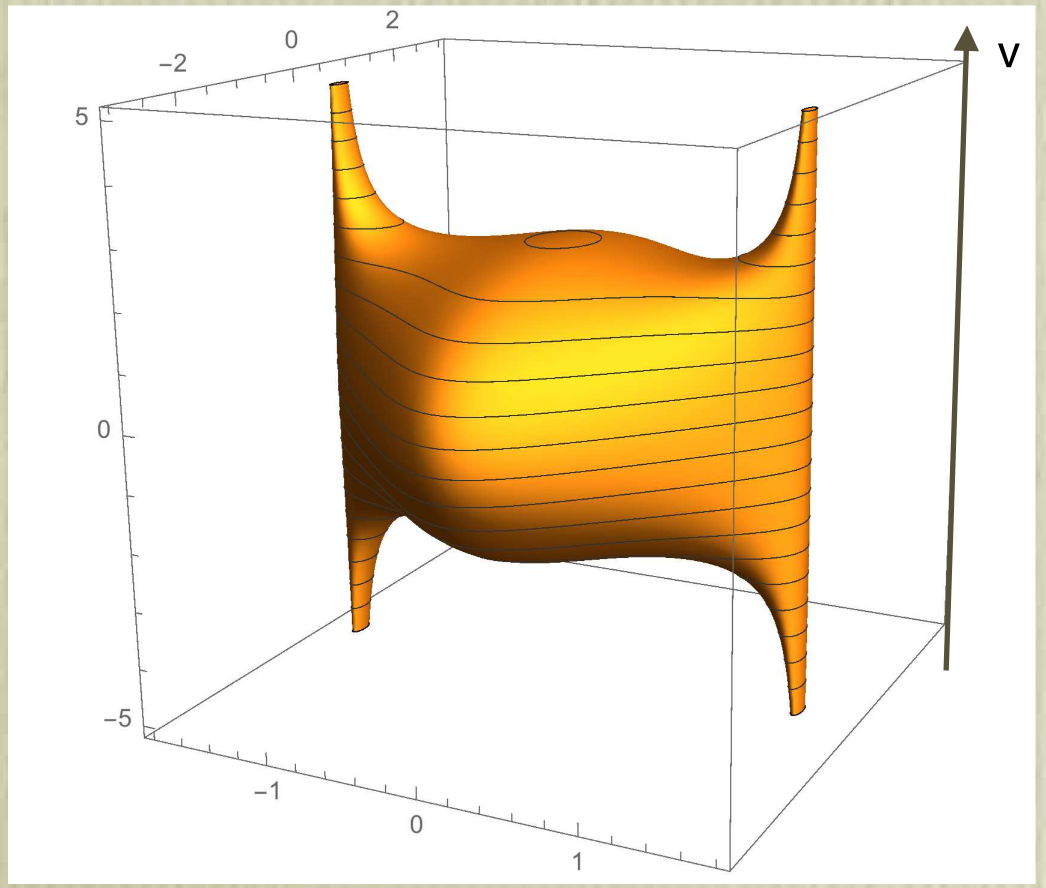
$$
\begin{aligned}
r' &= vr\cos\theta \\
v' &= W(\theta) - \frac{1}{2}v^2\cos\theta \\
\theta' &= w \\
w' &= W'(\theta)\cos\theta - W(\theta)\sin\theta - \frac{1}{2}vw\cos\theta + v^2\sin\theta\cos\theta
\end{aligned}
$$

Do not depend on r

and the energy equation is

$$\frac{1}{2}(v^2\cos^2\theta + w^2) - W(\theta)\cos\theta = 0.$$

“Collision Manifold”



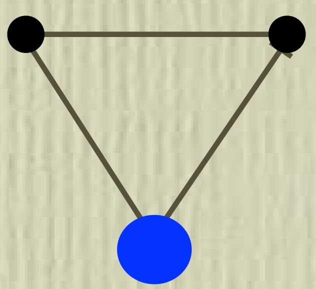
- Flow is a skew-product over the flow on the collision manifold in (θ, w, v) space
- r variable found from linear ODE

$$r'(\tau) = v(\tau)\cos\theta(\tau)\,r(\tau)$$

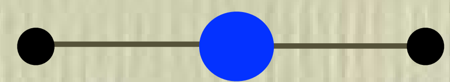
- v > 0, size r increasing
- v < 0, size r decreasing
- Flow on collision manifold is gradient-like with respect to v
- Six equilibrium points

Equilibrium Points, Homothetic Orbits

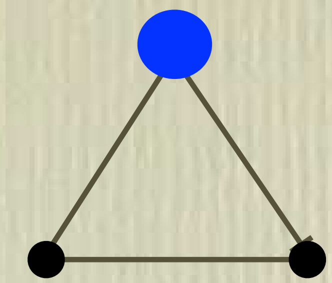
Three Central Configurations (CCs)



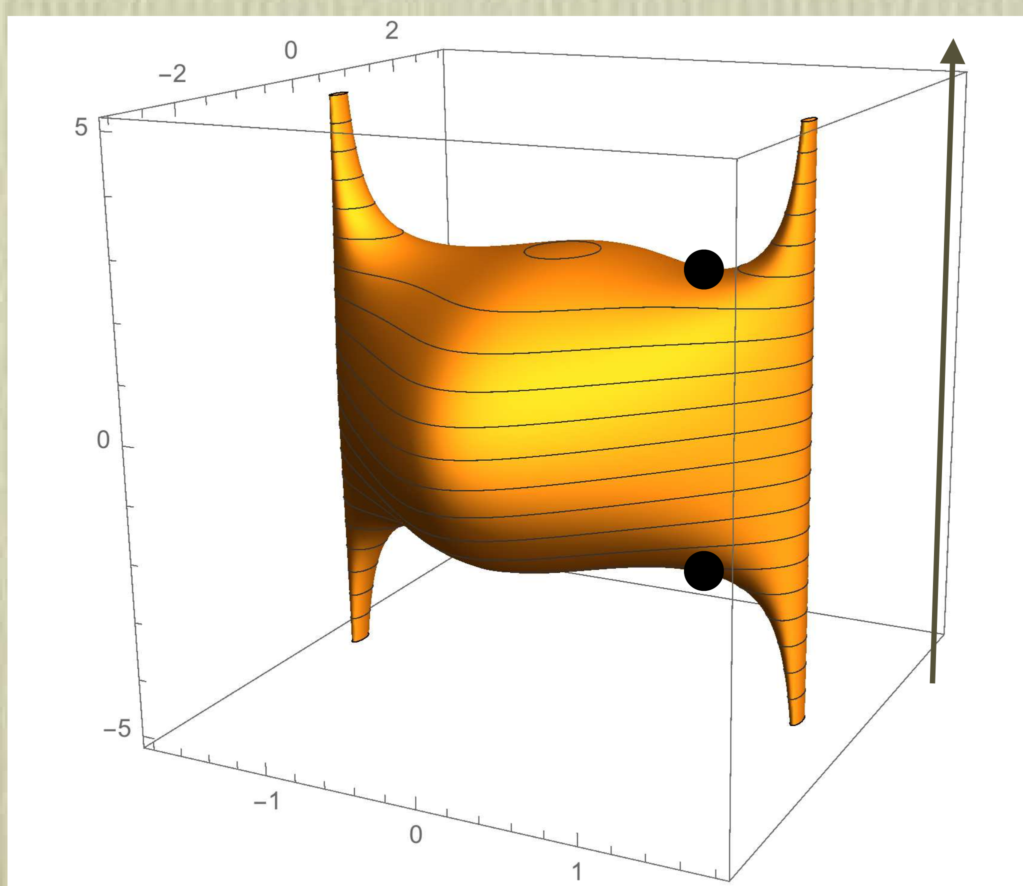
$$\theta = -\theta_\Delta$$



$$\theta = 0$$



$$\theta = \theta_\Delta = \arctan\sqrt{\frac{3m_3}{2+m_3}}$$



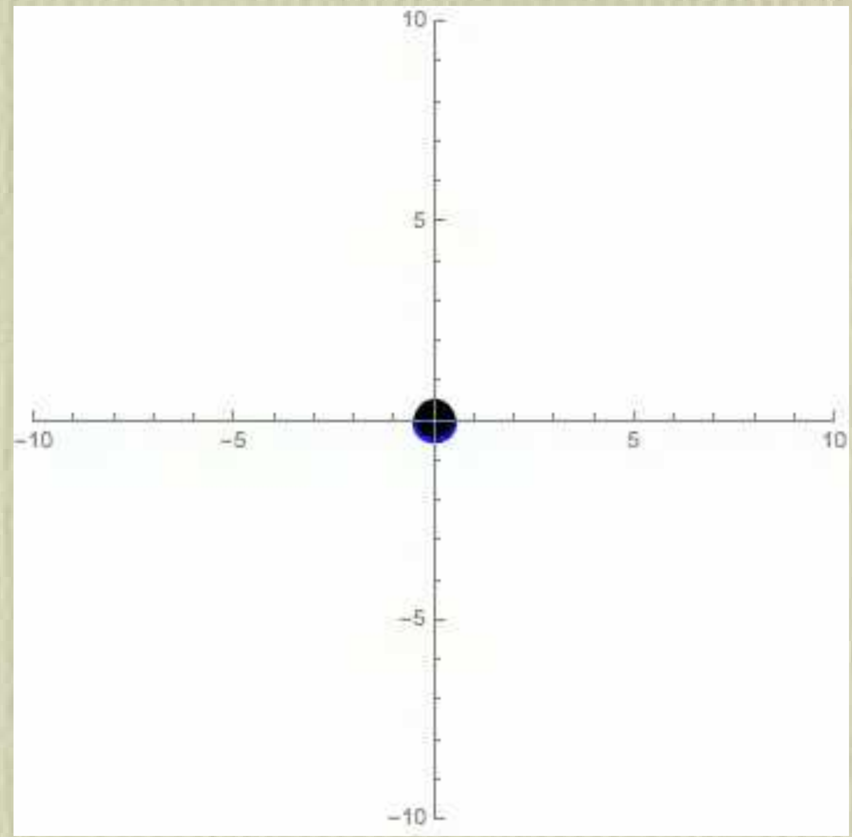
Each CC determines two equilibrium points

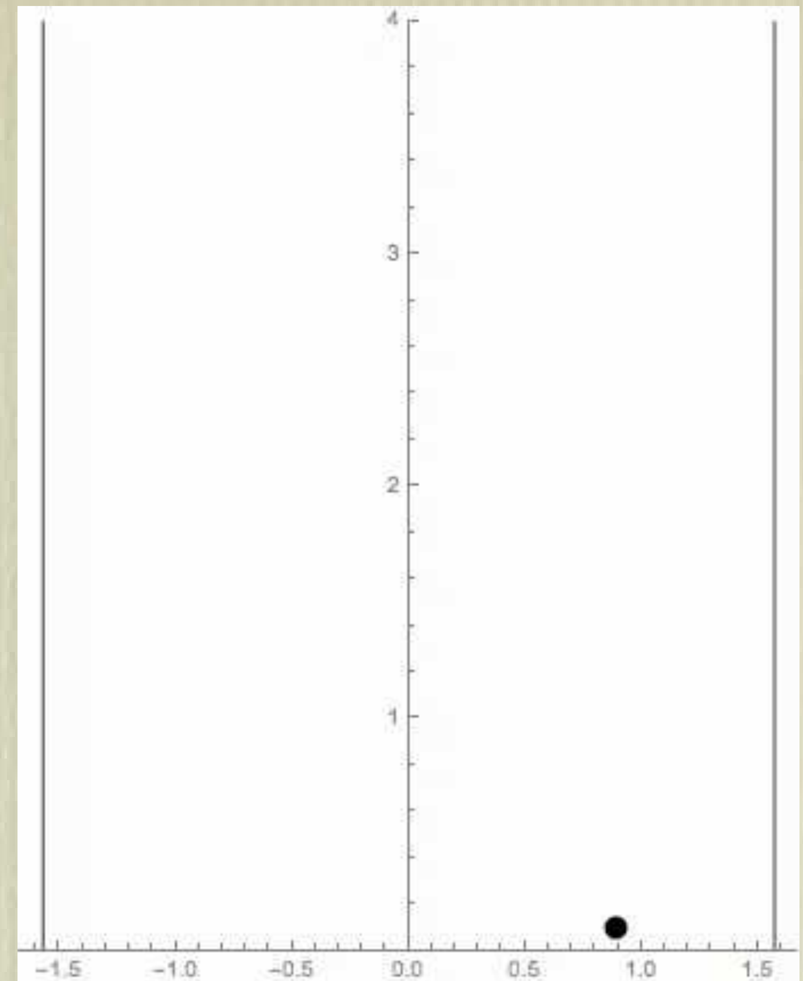
v > 0 : size increases from 0 to ∞ with constant shape

v < 0 : size decreases from ∞ to 0 with constant shape

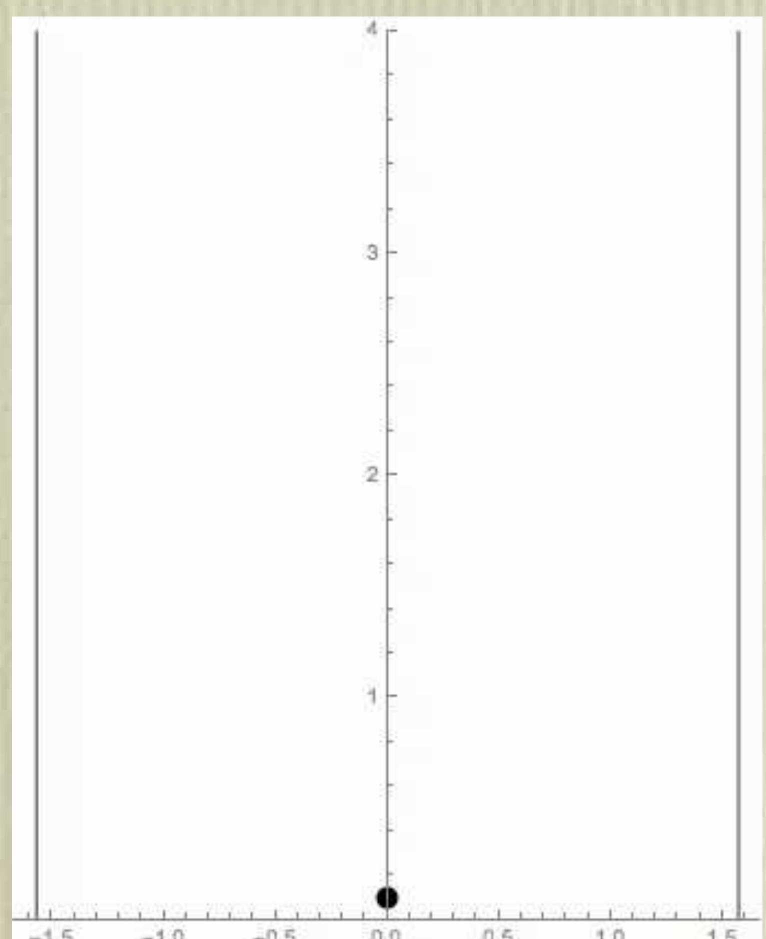
Homothetic Orbits and Corresponding Geodesics

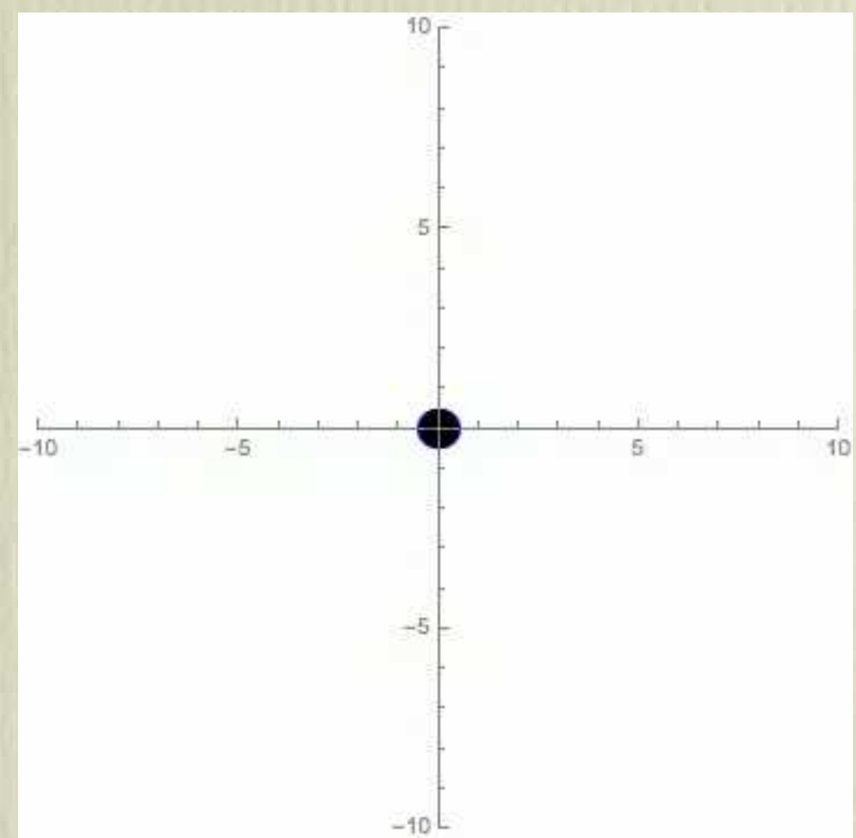
Lagrange
v > 0



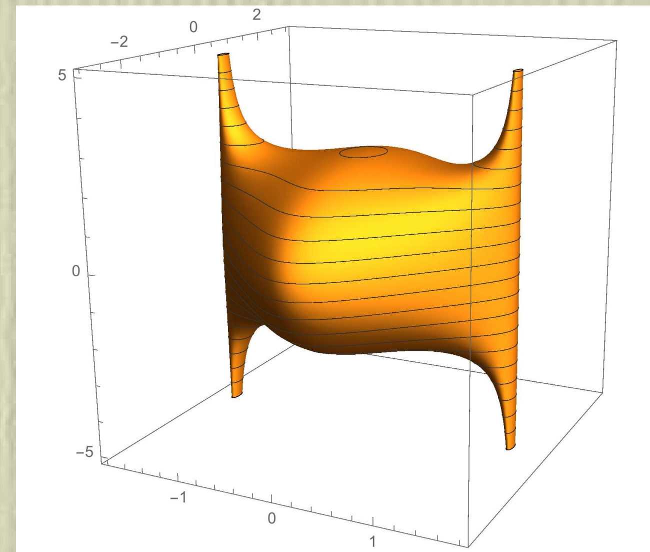


Euler
v > 0

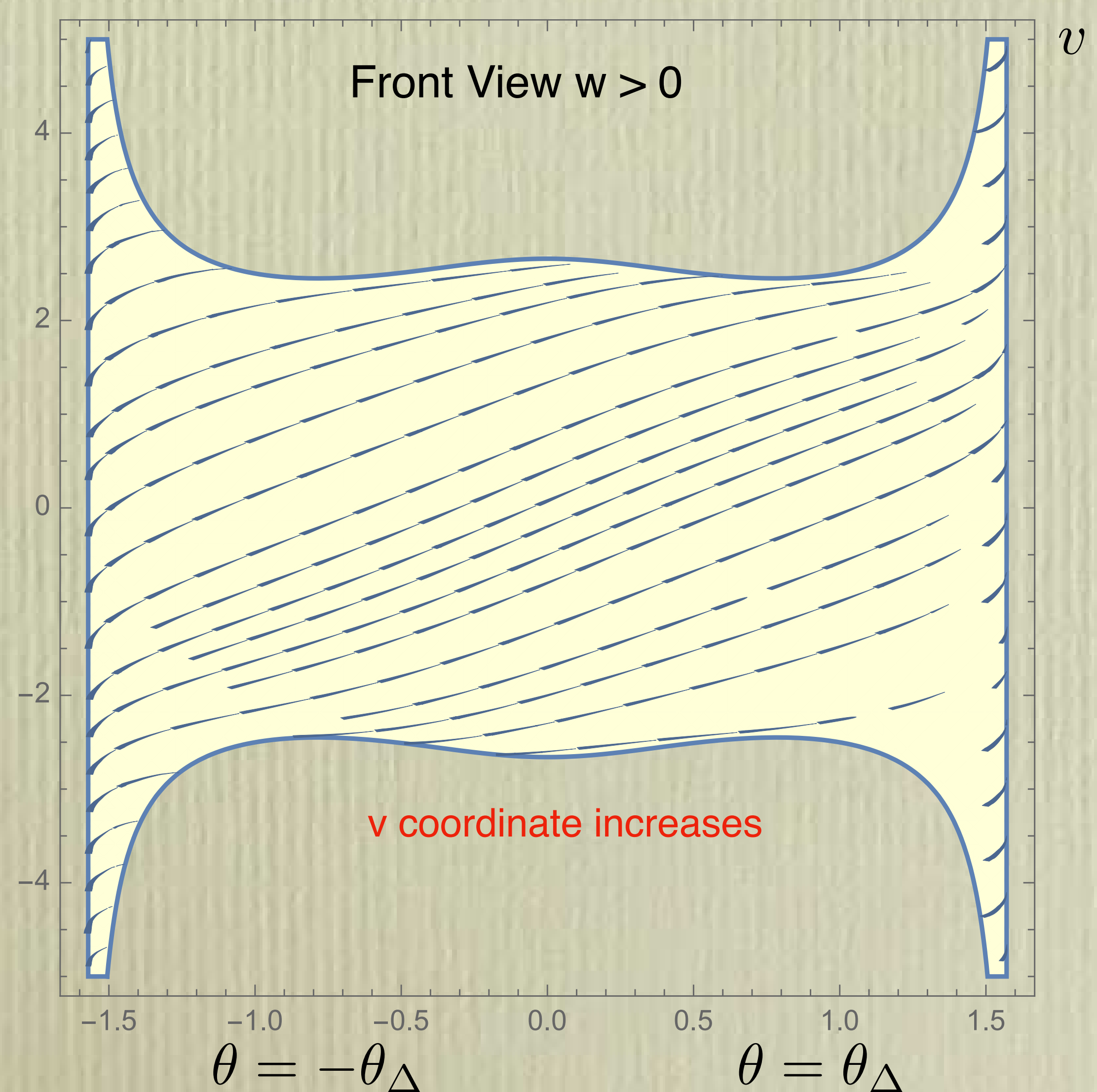


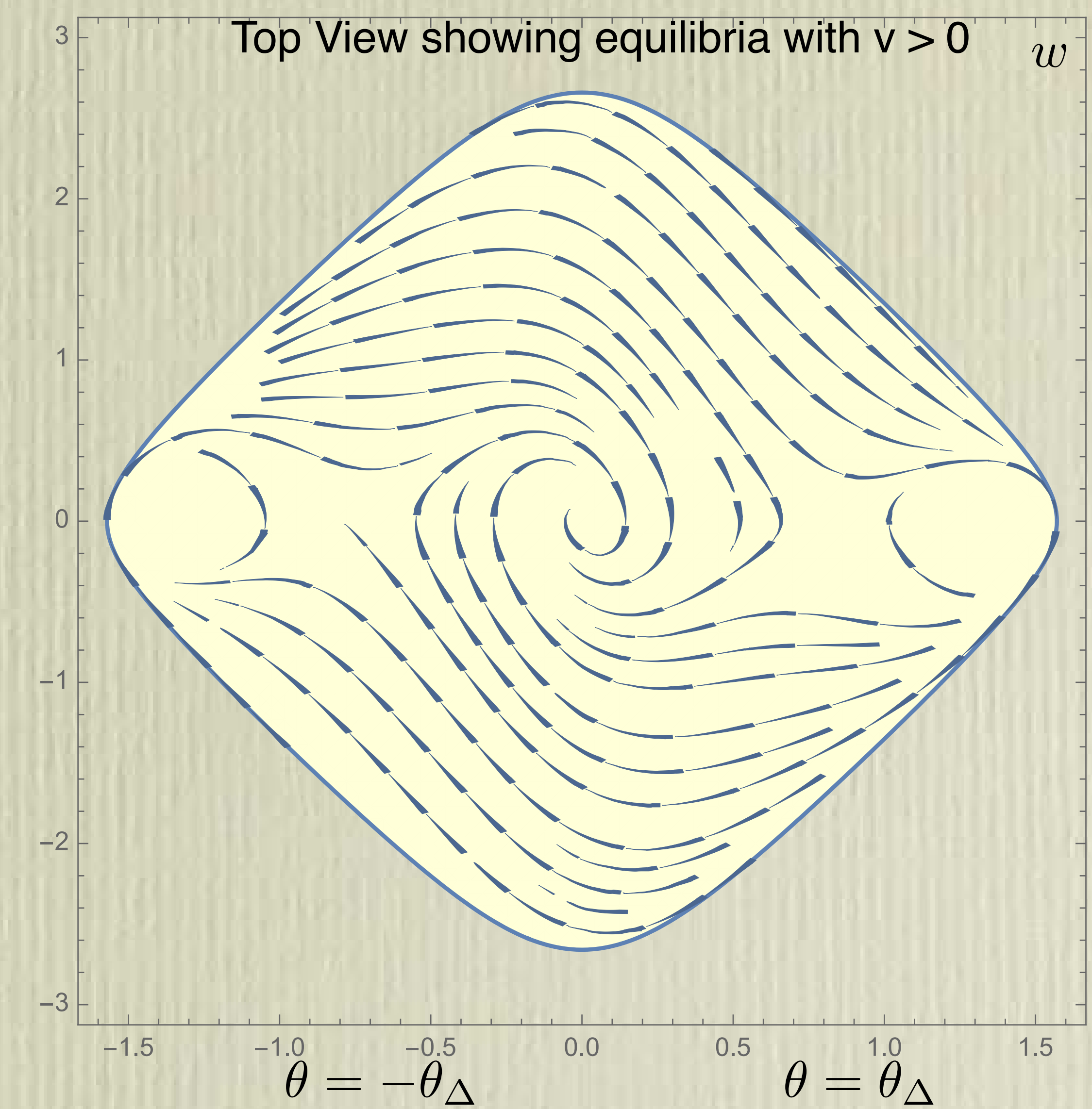


More about the Collision Manifold Flow



Lagrange equilbria are saddles
Euler equilibrium with v > 0 is a source
Euler equilibrium with v > 0 is a sink
Euler spiralling for $m_3 < 55/4$





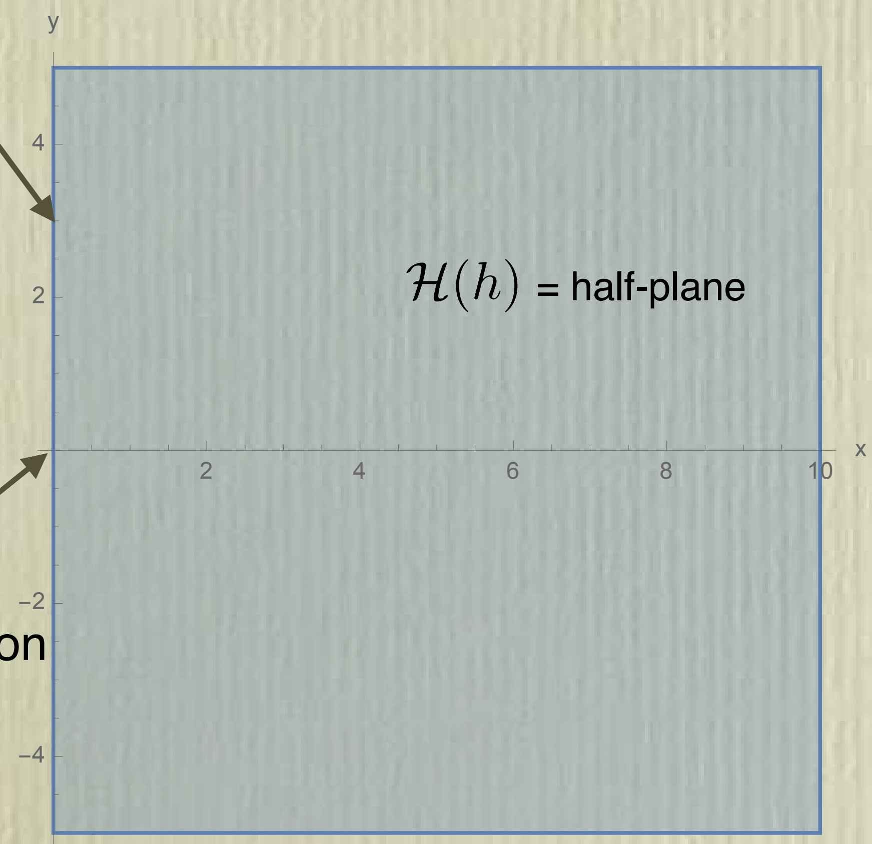
Minimal Geodesics when h = 0

Think of the Jacobi Maupertuis metric as a singular Riemannian metric on the closed Hill's region $\mathcal{H}(h)$

There is no Hill boundary but there are collision singularities where the metric blows up. But the singularities are integrable and one can still define arclength for curves, Riemannian distance, geodesics, etc.

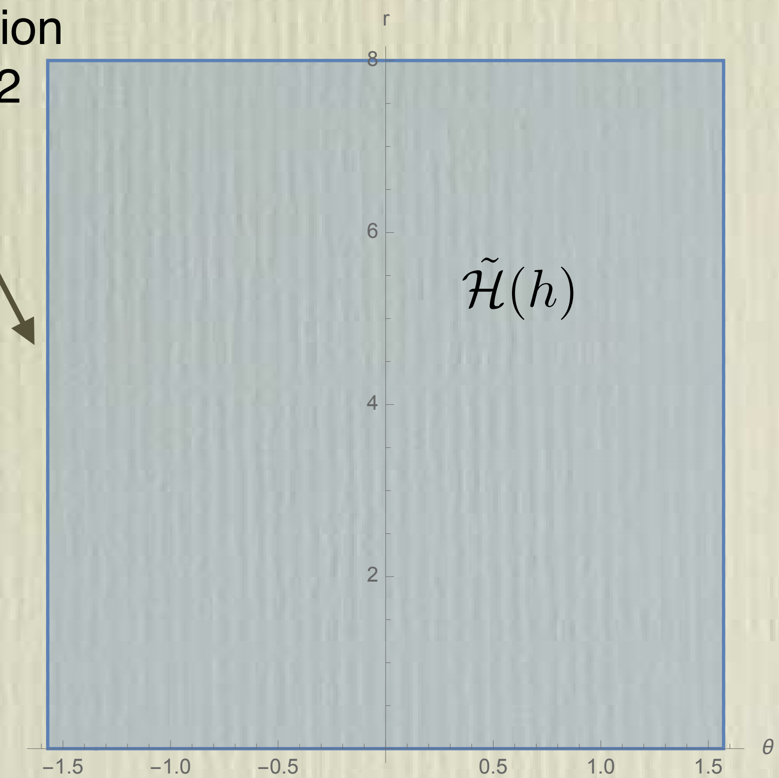
We are going to look for minimal geodesics, that is, curves which are the shortest curves connecting any two points on them. We will plot these in the blown-up Hill's region $\tilde{\mathcal{H}}(h)$

double collision x=0



triple collision
x=y=0

double collision
θ = -π/2 , π/2



triple collision r=0

Existence of Minimal Geodesics

Arclength: $\gamma(t), t \in J \subset R$ piecewise smooth curve and $[a,b] \subset J$:

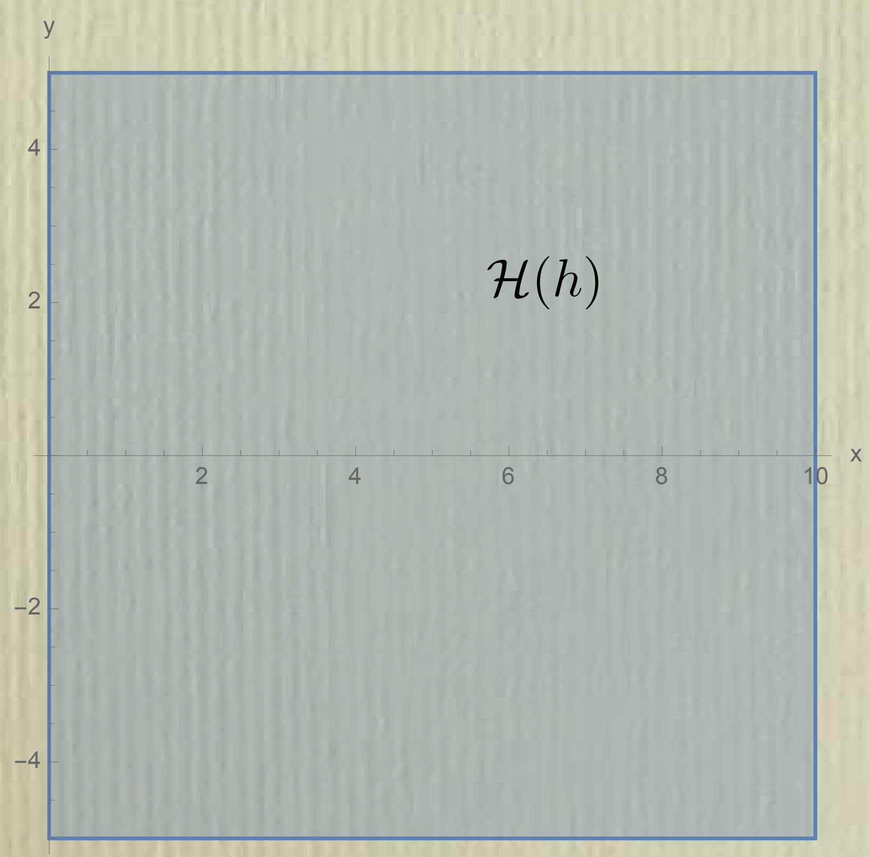
$$l(\gamma,[a,b]) = \int_a^b \sqrt{\mathbf{g}(\dot\gamma,\dot\gamma)}\,dt \in [0,\infty]$$

Riemannian Distance: $$d(p,q) = \inf_{\gamma} \; l(\gamma,[a,b])$$

where γ is a piecewise smooth curve with $\gamma(a) = p, \gamma(b) = q$.

Length of C^0 curves: $l(\gamma,[a,b]) = \sup_P \sum d(\gamma(t_i),\gamma(t_{i+1}))$, P a partition of $[a,b]$

Theorem: *The zero energy Hill's region is a complete metric space with respect to the Jacobi-Maupertuis distance function.*



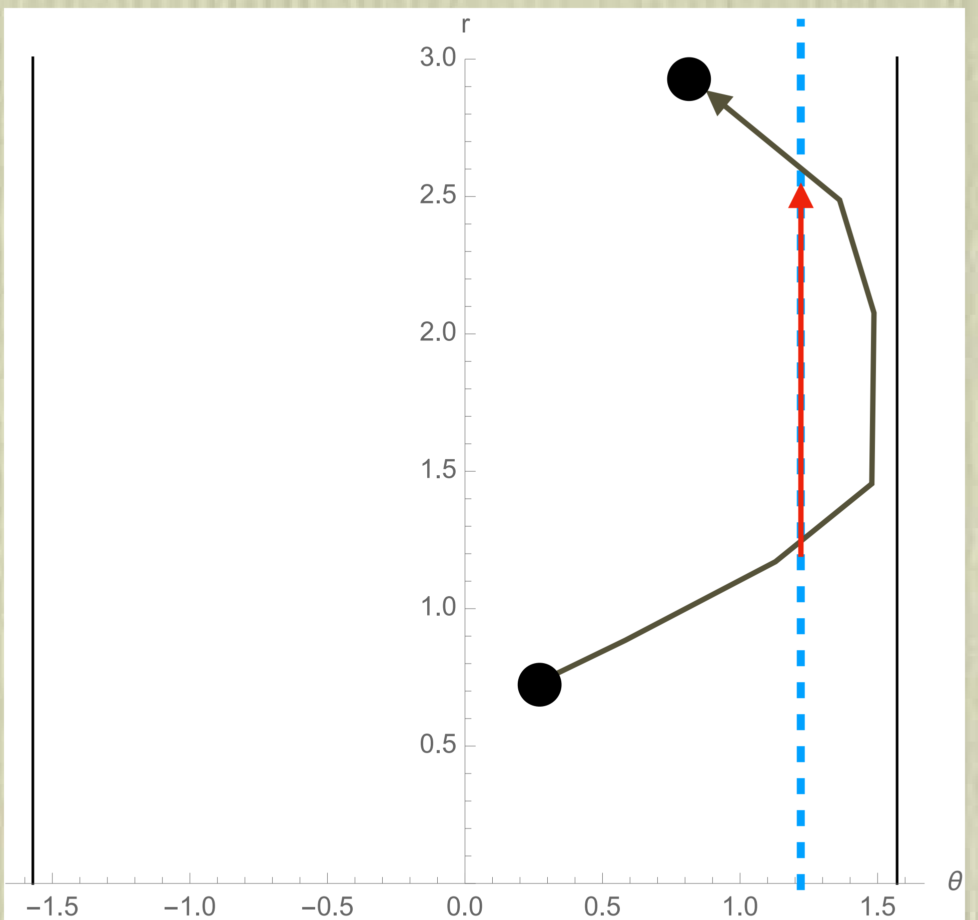
- $d_{JM}(p.q)$ is finite and nonzero if $p \neq q$
- Topology of $(\mathcal{H}, d_{JM})$ agrees with the subspace topology from R^2
- A subset is d_{JM} bounded iff it's bounded in R^2
- $(\mathcal{H}, d_{JM})$ is boundedly compact (bounded closed sets are compact)
- $(\mathcal{H}, d_{JM})$ is a complete metric space

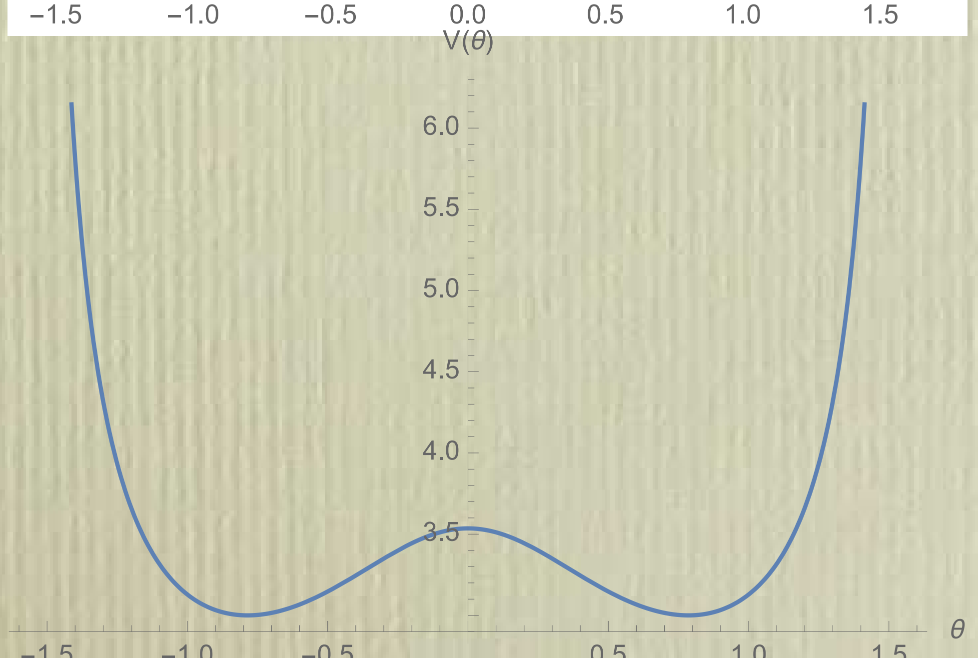
Corollary: *For any p, q in $\mathcal{H}$ there exists a minimal geodesic from p to q, i.e., a continuous curve with length d(p,q)*

A version of the Hopf - Rinow theorem

Minimal Geodesics Avoid Collision

Marchal’s lemma does not work for one-dimensional shape spaces.



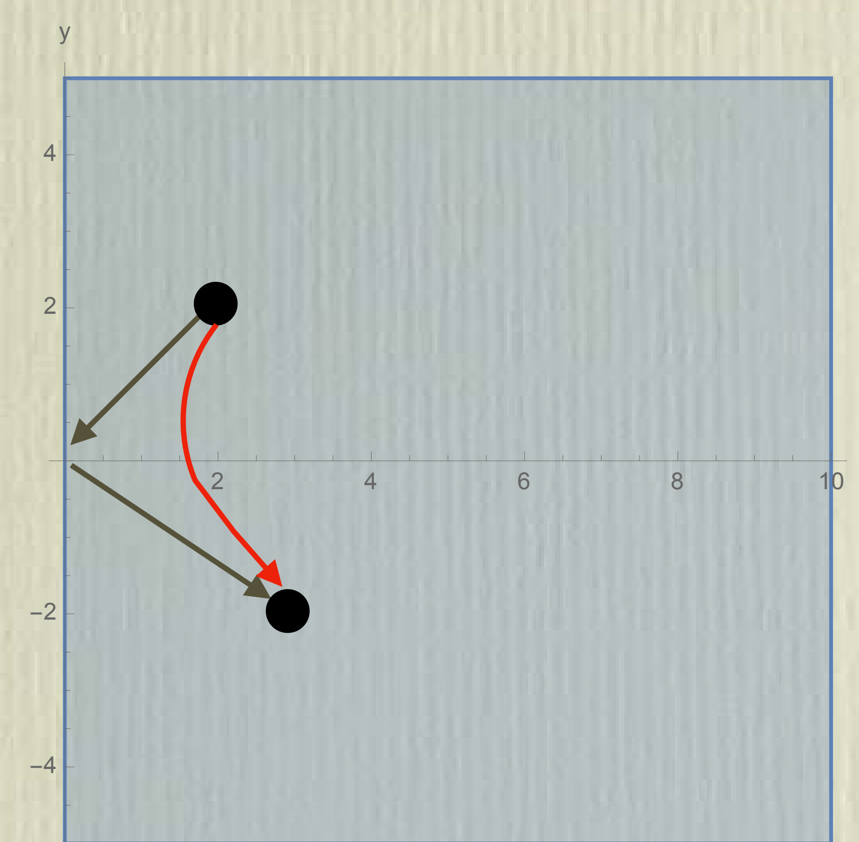


Curve segment approaching double collision and returning cannot be a minimizer. Modified curve with red segment is shorter.

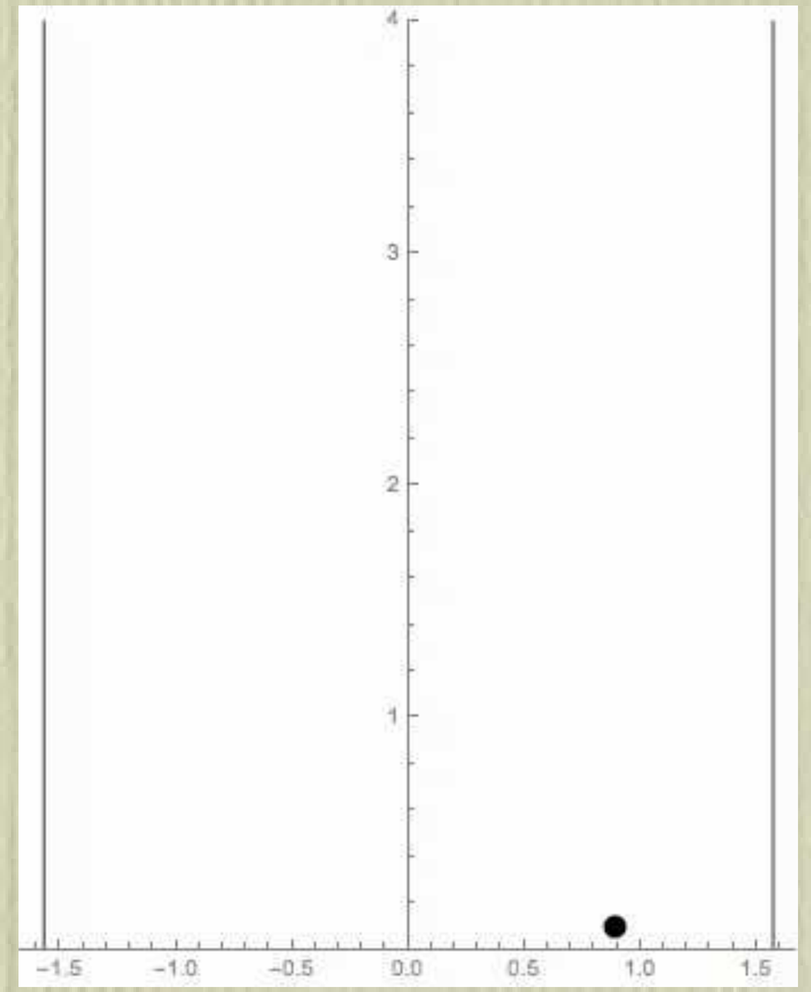
$$l(\gamma) = \int_a^b \sqrt{\frac{2}{r} V(\theta)(dr^2 + r^2 d\theta^2)}$$

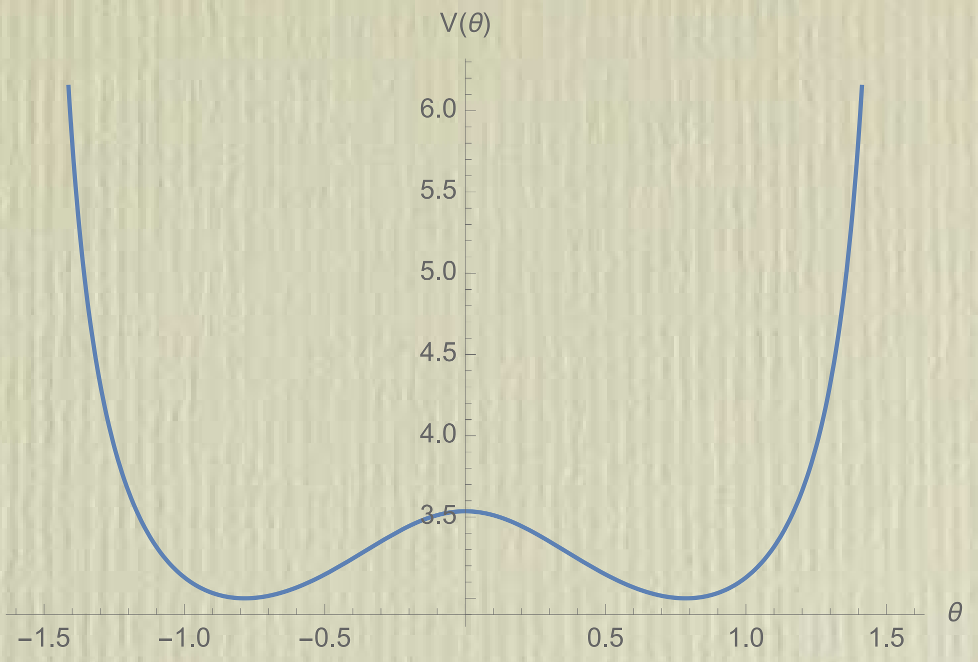
Triple Collision involves a more complicated argument to show that the path through collision is not the shortest one.

triple
collision
x=y=0



Lagrange Homothetic Orbit as a Minimal Geodesic





Lagrange homothetic orbits. Lagrange shape gives the minimum of the shape potential V(θ)

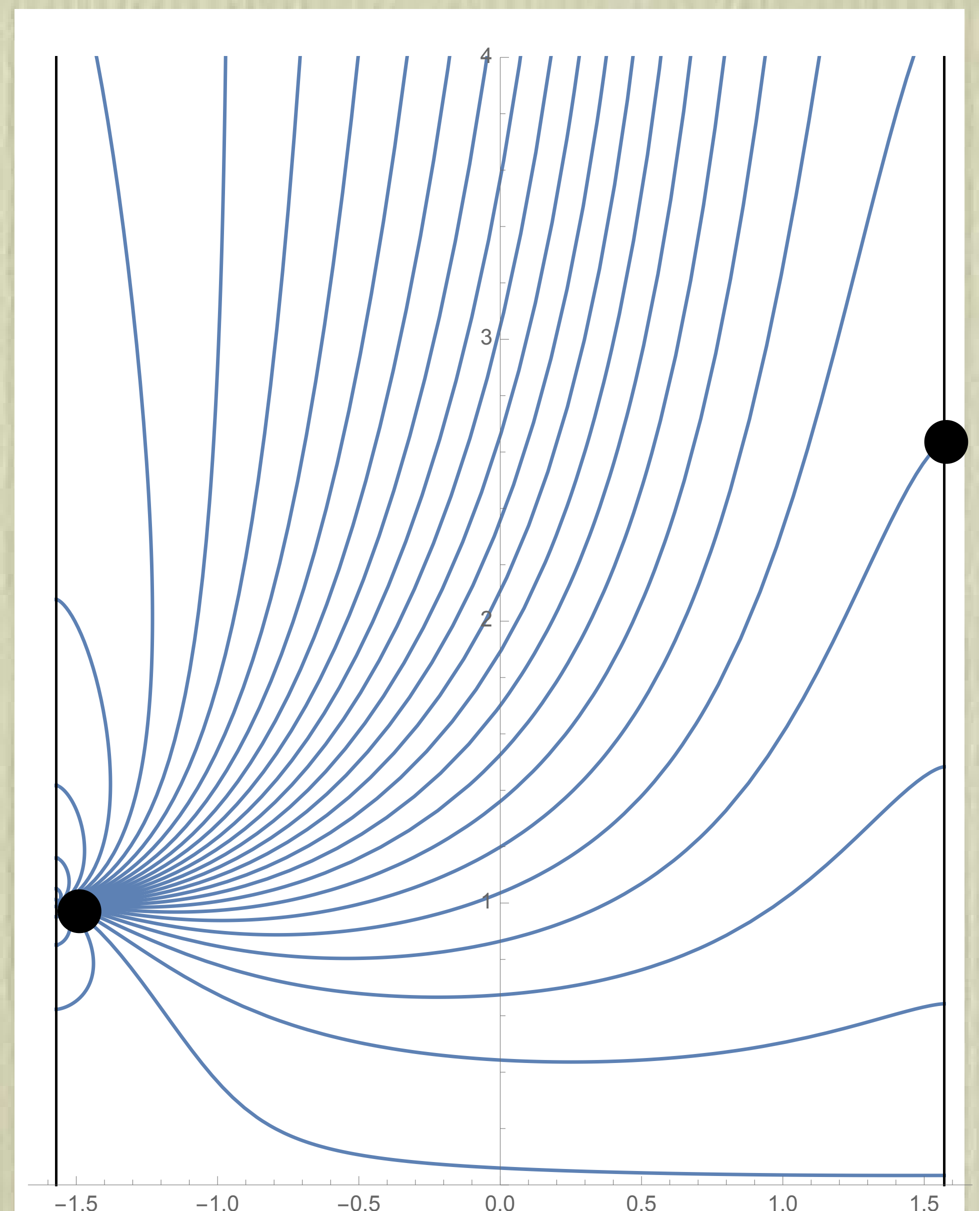
$$l(\gamma) = \int_a^b \sqrt{\frac{2}{r} V(\theta)(dr^2 + r^2 d\theta^2)}$$

This geodesic is globally minimal, that is, it's the shortest curve between any two of its points.

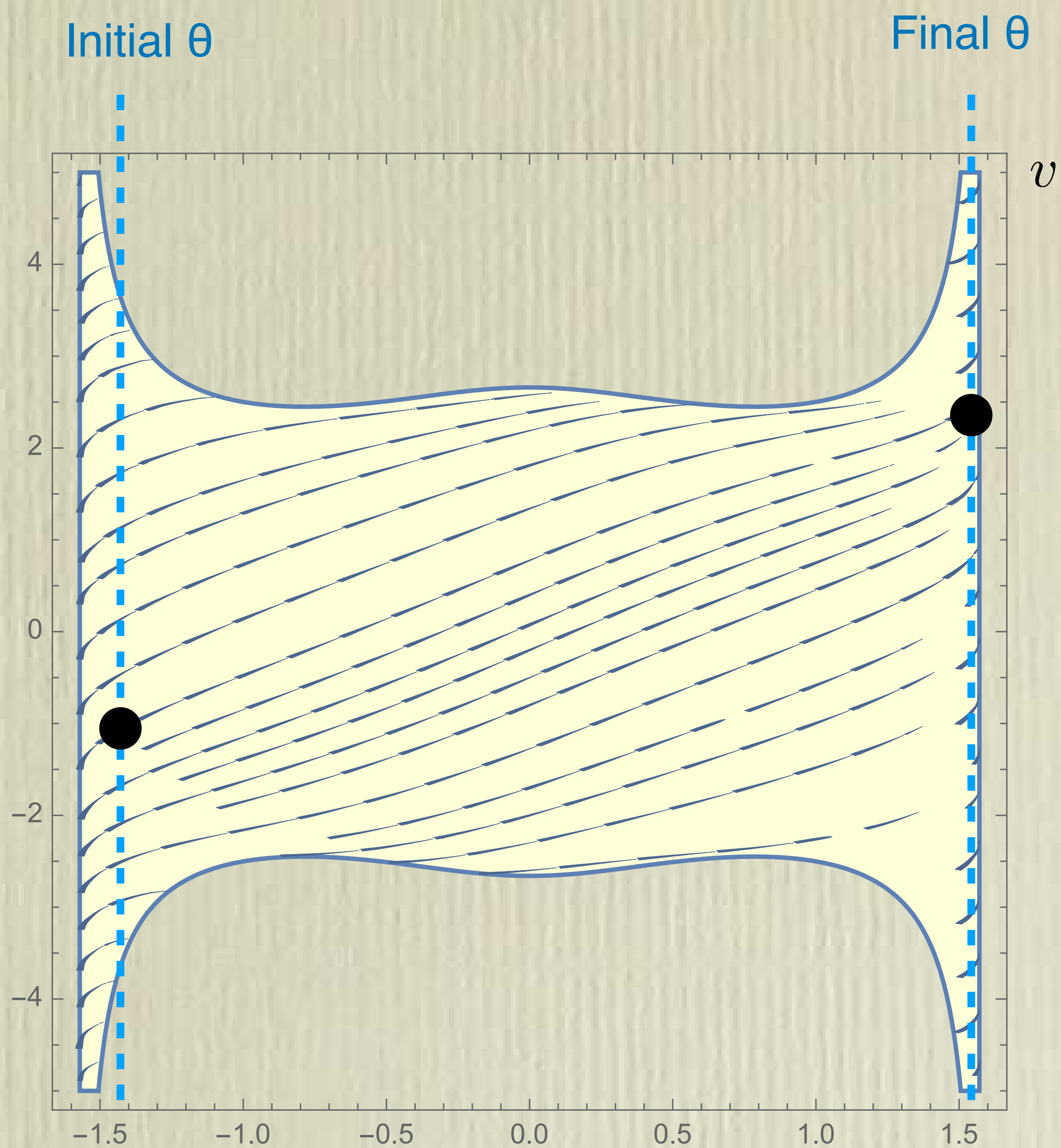
This generalizes to the NBP for minimal CCs, i.e., CCs which are minima of the shape potential.

Minimal geodesics starting at a point

Starting at a given point p in $\mathcal{H}$, there must exist lots of minimal geodesics. We can reach any other point q in $\mathcal{H}$ with a minimal geodesic.



Geodesics in $\mathcal{H}$

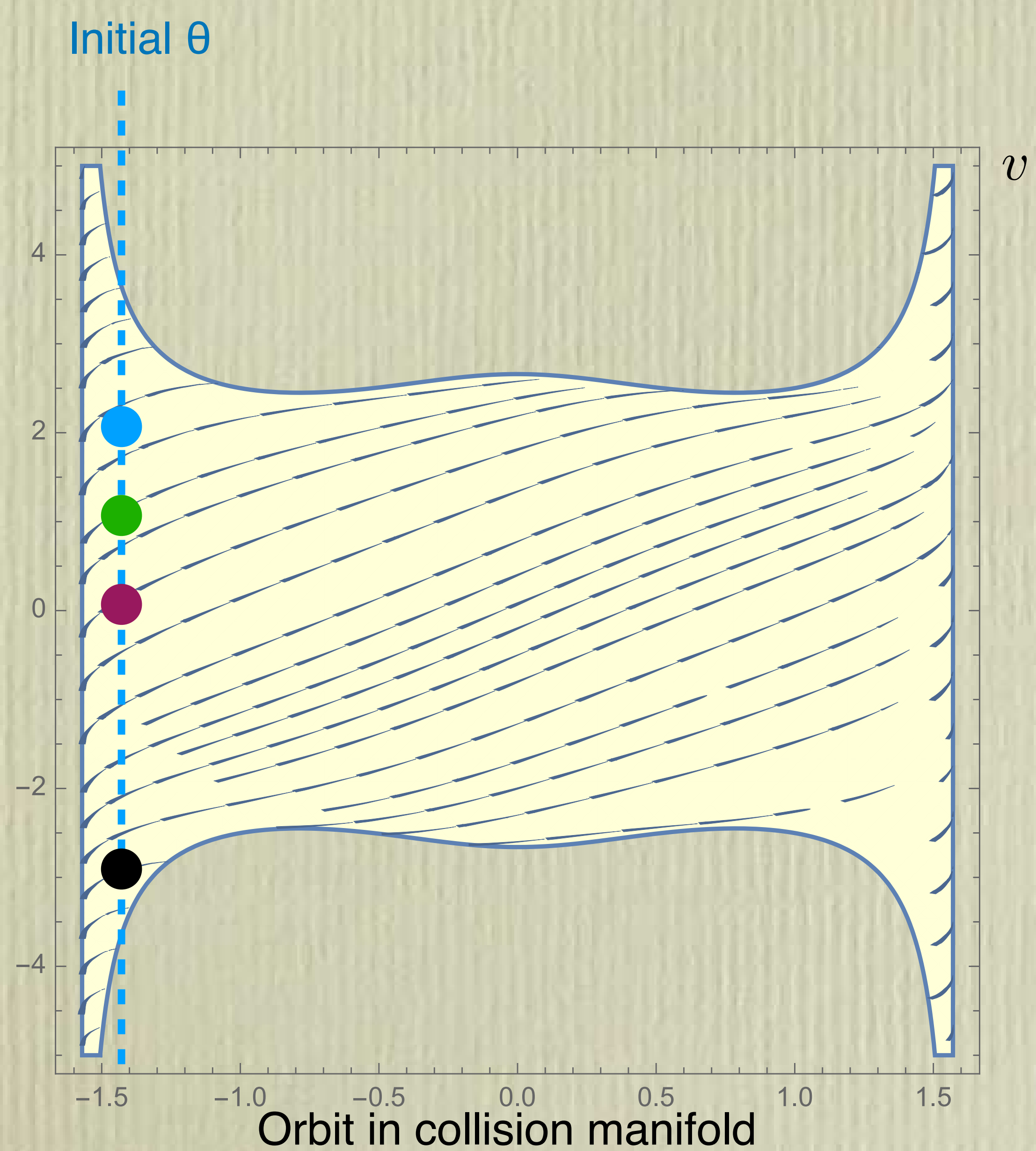


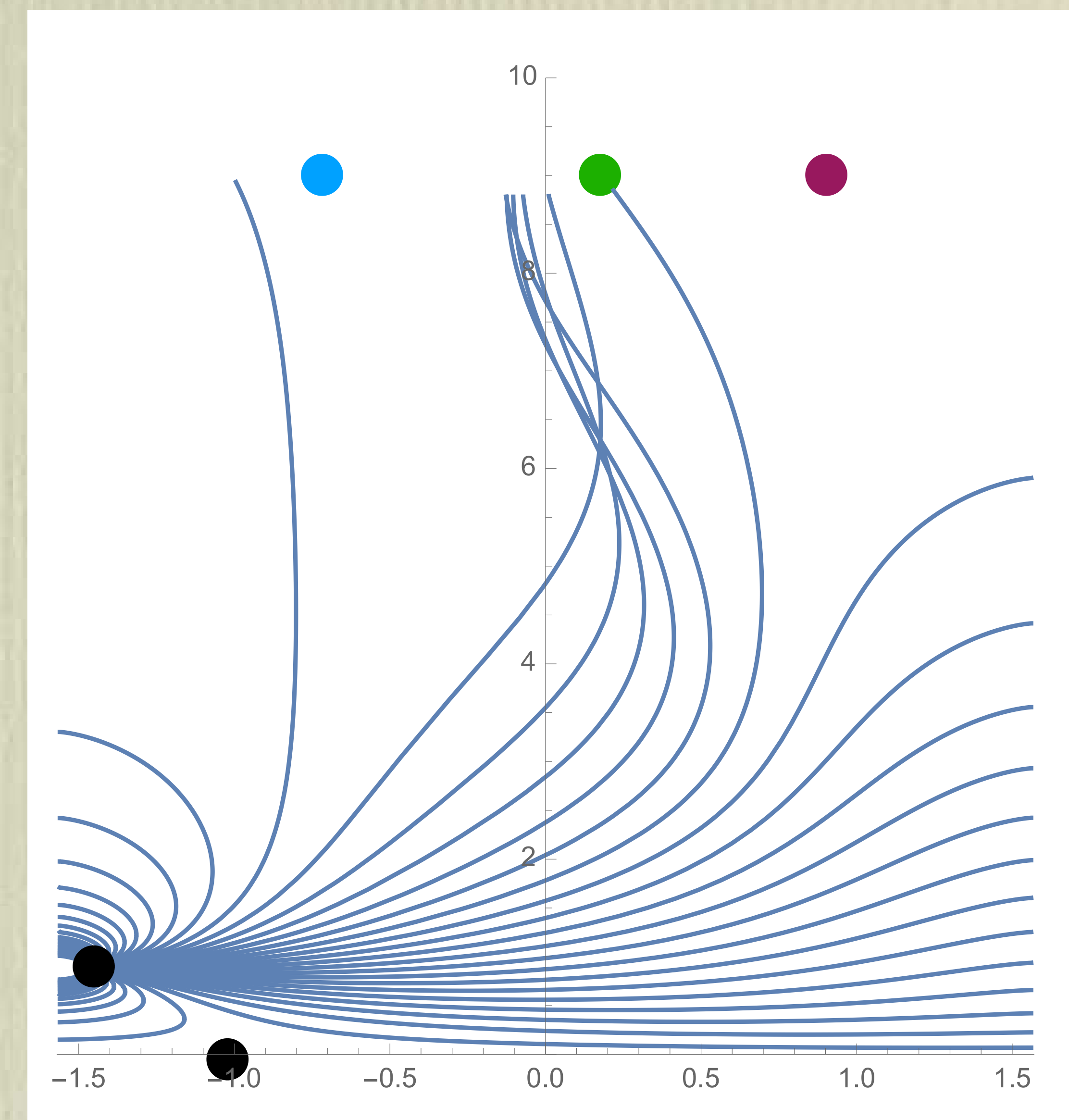
Orbit in collision manifold

Geodesics from p correspond to orbits on the collision manifold starting at a given initial θ slice. By choosing the right orbit, we can hit any given final r and θ.

Geodesics and stable manifolds

Starting at a given point p in $\mathcal{H}$, there are geodesics for which the corresponding orbits on the collision manifold converge to a restpoint. These geodesics are asymptotic to homothetic orbits or to triple collision.

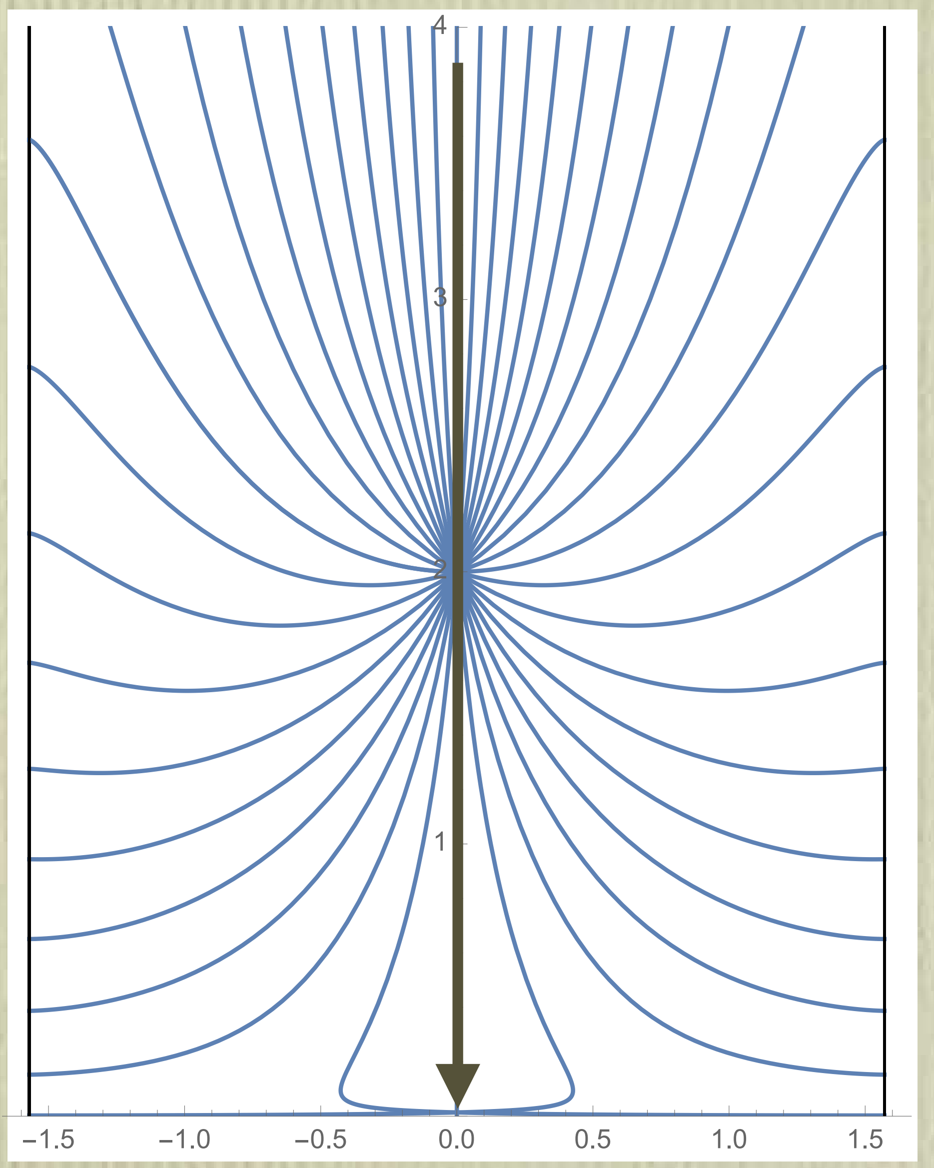


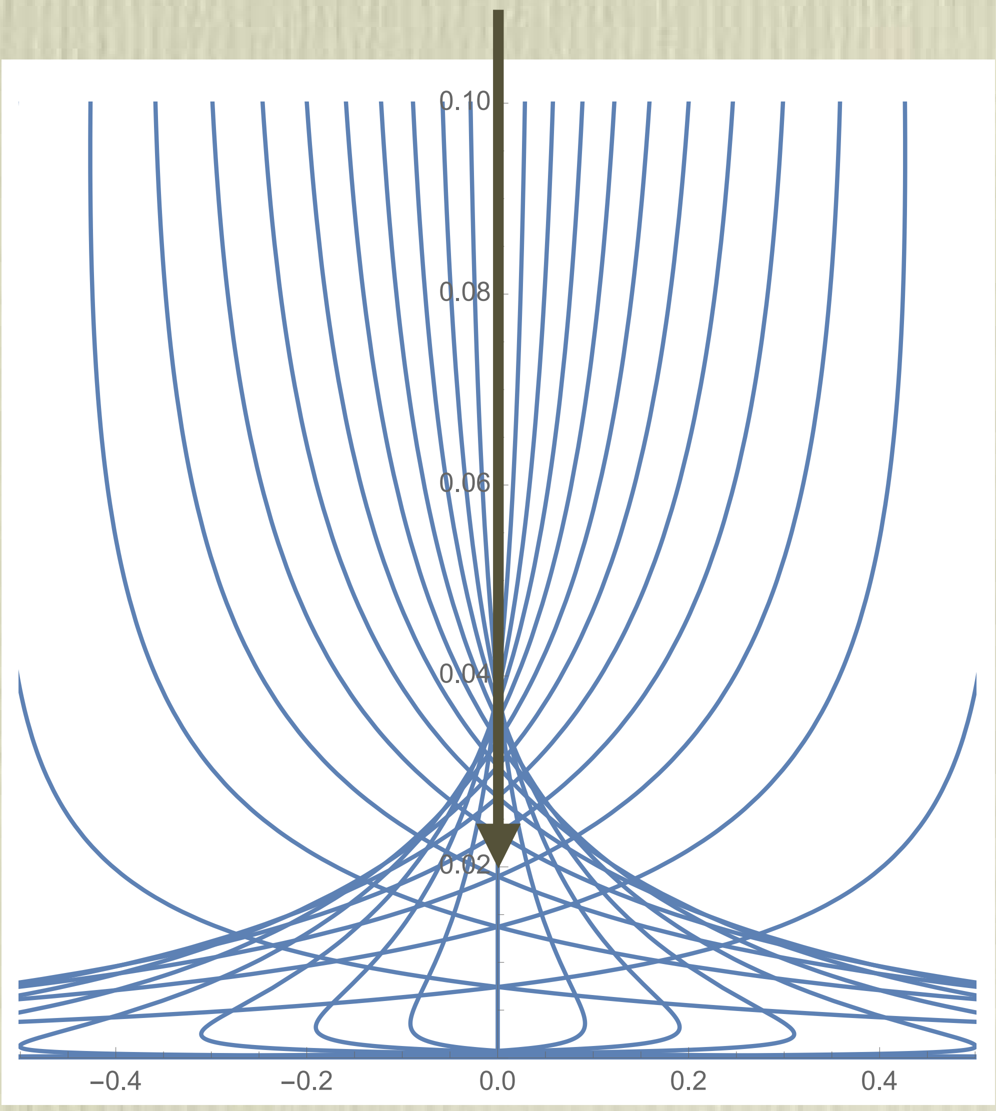


- ● Triple collision Δ-
- ● Asymptotic to Lagrange homothetic Δ-
- ● Asymptotic to Lagrange homothetic Δ+
- ● Asymptotic to Euler homothetic

Minimality of the Euler homothetic orbit ?

For $m_3 < 55/4$, Euler's geodesic is not globally minimizing, that is, long segments are not minimal. Nearby geodesics oscillate around it producing cut points (recall the spiralling near the Euler restpoint on the collision manifold).





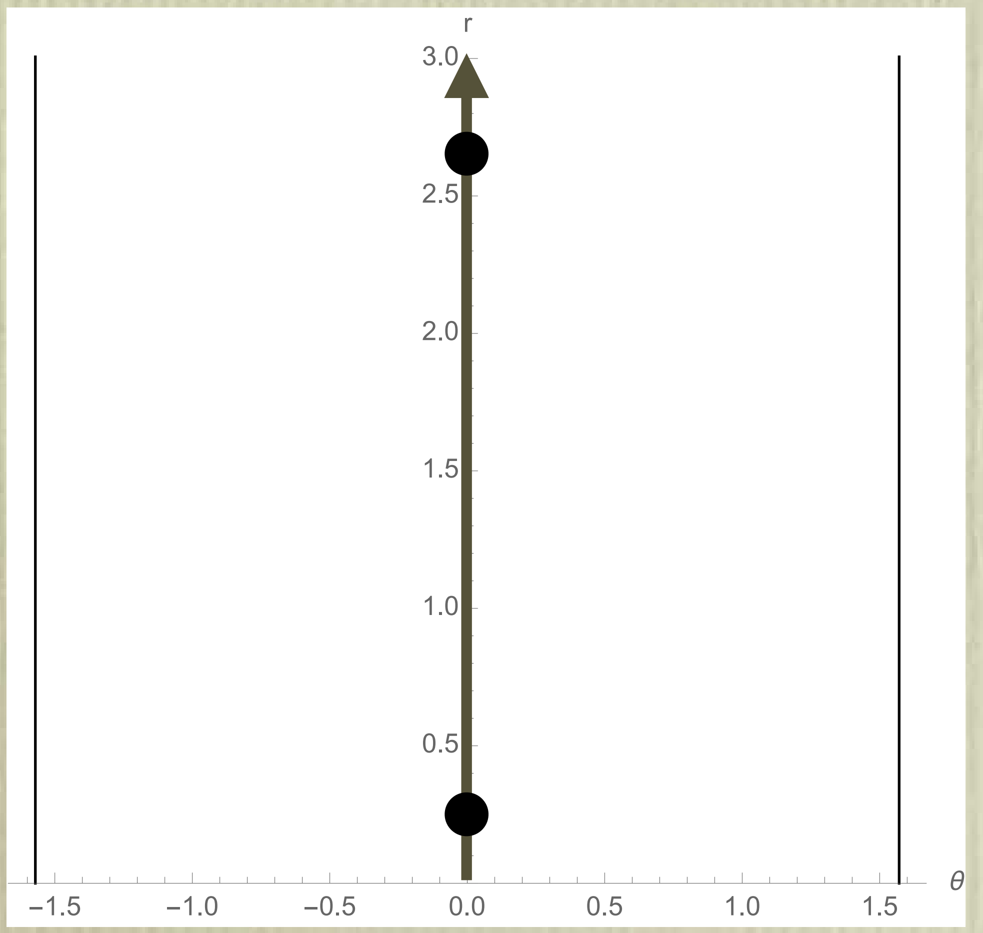
Barutello - Secchi
RM, Montgomery, Sanchez-Morgado

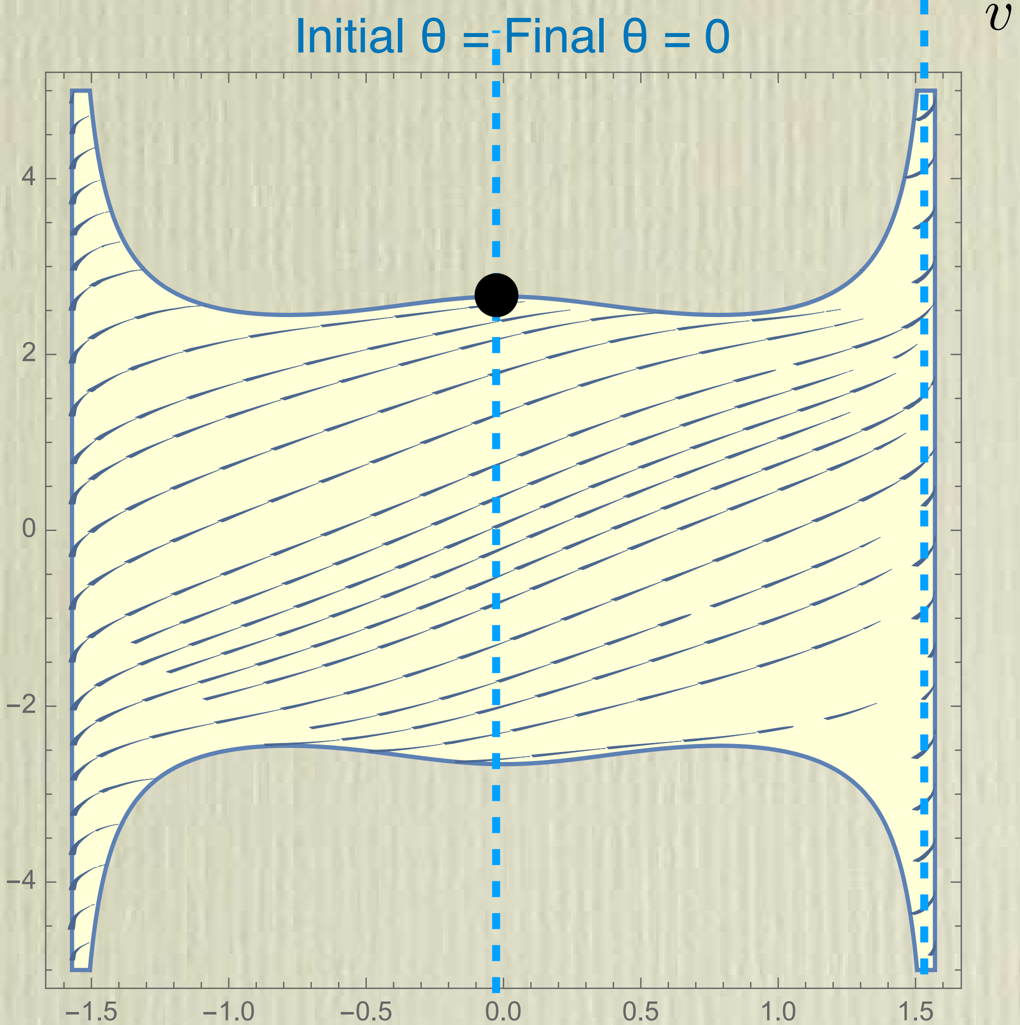
Minimality of the Euler homothetic orbit ?

For $m_3 > 55/4$, there is no spiralling near the Euler restpoint on the collision manifold. We can use the flow on the collision manifold to prove:

Theorem: *For* $m_3 > 55/4$ Euler's geodesic is globally minimizing in the h = 0 isosceles problem (even though it is a local maximum for the shape potential).

Proof: Given two points p, q on Euler's geodesic, we know there exists a minimal geodesics between them. In the collision manifold, the corresponding orbit must connect the slice θ = 0 to itself. One way to do this is to choose the Euler restpoint. This gives Euler's geodesic.

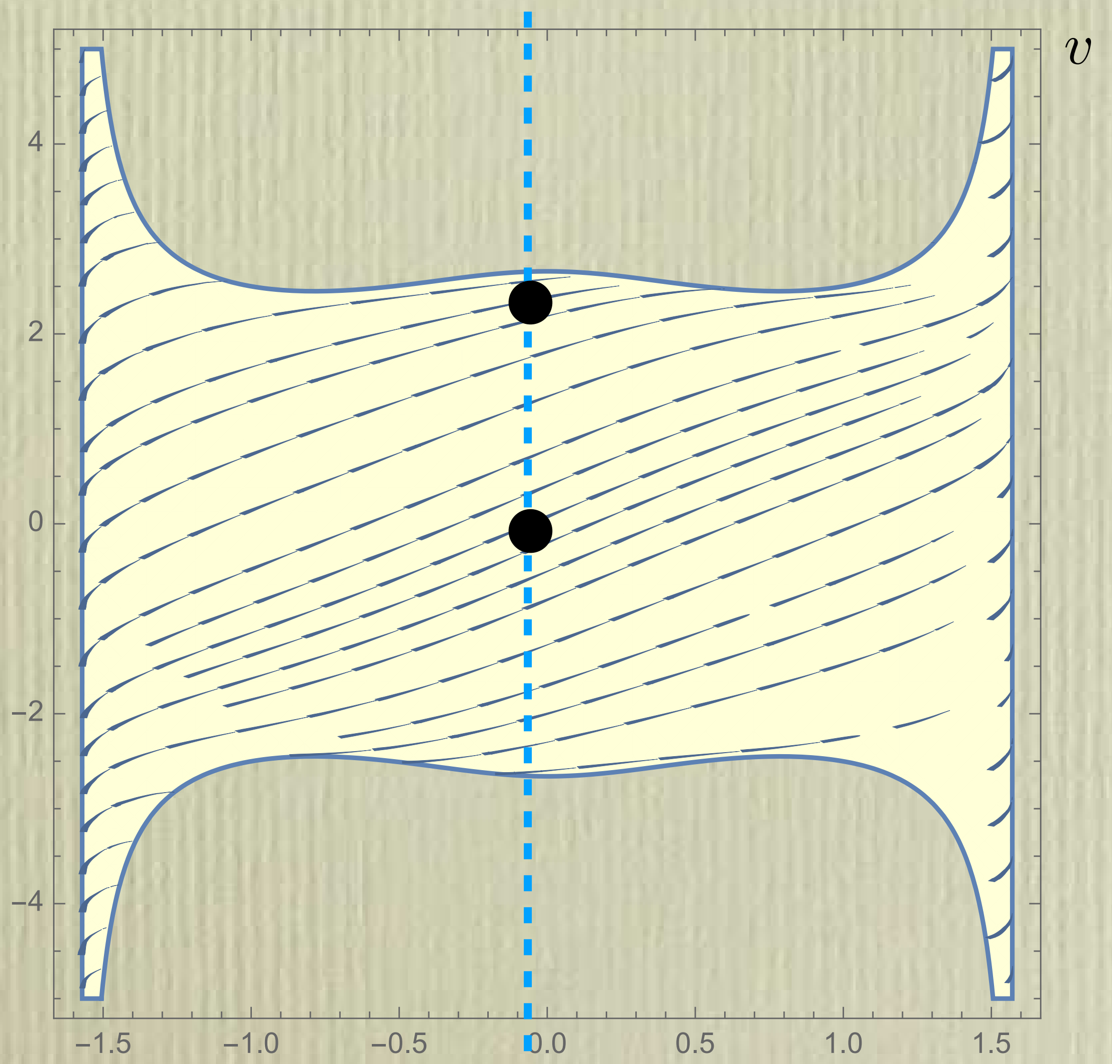




Proof of Minimality of the Euler homothetic orbit (continued)

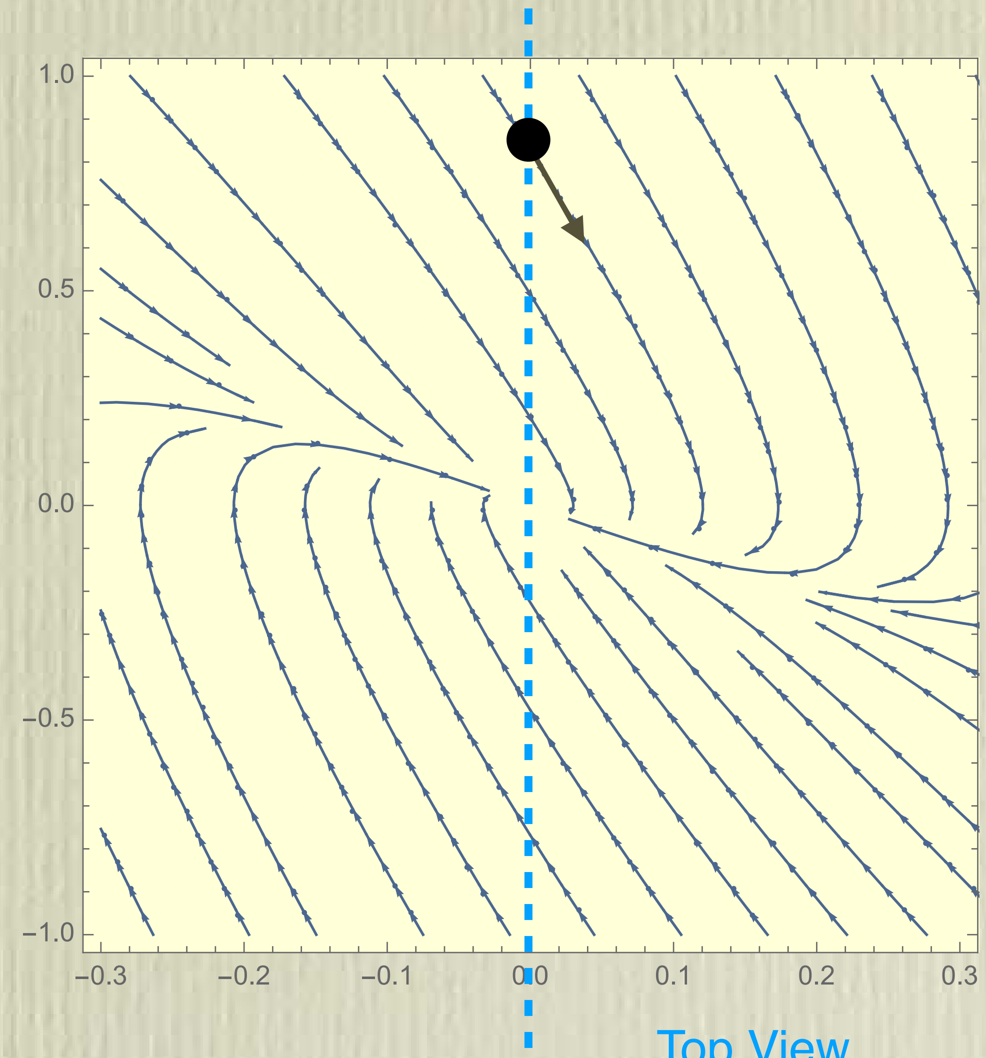
We know that any minimal geodesic avoids collisions, so the corresponding orbit in the collision manifold would have to return to θ = 0 without hitting θ = π/2, -π/2. With no spiralling near the equilibrium point, such orbits just do not exist. So Euler's geodesic must be the minimal one.

Initial θ = Final θ = 0



Front View

Solutions crossing θ = 0 never return, although they do approach asymptotically.



Top View

The End

Thanks

TWO DIMESIONAL EXAMPLES OF THE JACOBI–MAUPERTUIS METRIC

RICHARD MOECKEL   Notes by Jeffrey Heninger

understand J-M metric – start with low dim. examples so we can draw pictures

Lagrangian system with 2 d.o.f. with fixed energy h

→ Riemannian metric in configuration space

solutions of Euler–Lagrange ⟷ geodesics of g

$$g(v,v) = (U(q) + h)\,\|v\|^2$$

Hill's region = {this > 0}
needed for metric to be Riemannian

Simple Examples:

- Kepler
- Isosceles 3 Body Problem
- (Colinear 3 Body Problem) – not in this talk

Maupertuis Metric for Kepler Problem

geodesics are Kepler's solutions

what does the surface look like?

"abstract surface of revolution" – a bunch of circles – can we imbed it in $\mathbb{R}^3$?

try to use cylindrical coordinates for the embedding

$h = 0$ ⟹ Kepler surface is a cone with slope $\sqrt{3}$

collision singularity is the point of the cone

$h > 0$ ⟹ Kepler surface in terms of elliptic functions

cone singularity   gets flatter as you go out

$h < 0$

nontrivial Hill's region = $\{ r \le 1 \}$ = unit disk   problems near Hill boundary ($r = 1$)

can only solve for surface if $0 \le r \le 3/4$

can't embed the Kepler surface into $\mathbb{R}^3$ as a surface of revolution.

$r = 3/4$ ⟷ eccentricity of ellipse = $1/2$

tried embedding $h < 0$ in other spaces

in each case, there is a problem near the Hill boundary

can embed in 4D Minkowski space

for any central force problem with a Hill boundary, a region near the Hill boundary can't be embedded

Minimal Geodesics for the Isosceles 3BP

this has a Hill boundary if $h < 0$ & singularities when there are collisions

we will focus on $h \ge 0$

can blow up the singularity using mass weighted polar coordinates

this set up automatically has zero angular momentum.

Two approaches to problem:

- look at solutions in phase space
- look at geodesics

Technique to regularize collisions

- variables: r = size of triangle, θ = shape of triangle
- slow down time near collisions → smooth set of differential equations

Easiest case: $h = 0$.

- $\dot{\theta}$, $\ddot{\theta}$ do not depend on r.
- flow on collision manifold is gradient-like with respect to v. — v always increases
- equilibrium points connected to central configurations
  - $\theta = \pm \arctan\sqrt{\frac{3m_3+1}{2+m_3}}$, $\theta = 0$
  - each can also have $v > 0$ or $v < 0$

  } 6 equilibria — 4 Lagrange – saddles; 2 Euler – source, sink (spiral)
- geodesic point of view:
  - singular metric at the wall for collisions. – still finite to reach the wall
- zero energy Hill's region is a complete metric space under the Jacobi-Maupertuis metric
  - ⇒ there exists a minimal geodesic between any 2 points (not both on boundary)
  - the minimizer does not include a collision
- Lagrange homothetic orbit is a (global) minimal geodesic
- we can see the saddles & sinks from before in the geodesics
- Euler homothetic orbit is not globally minimizing
  - a sufficiently long segment is not a minizer
  - it's a local maximizer for shape potential, so it's shorter to leave & come back
  - if m_3 is sufficiently large, Euler homothetic orbit is globally minimizing
    - this corresponds to spiral → node
    - aren't other orbits that leave & come back if you don't have a spiral