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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Eva Miranda

Talk Title: From Celestial Mechanics to Fluid Dynamics: Contact Structures with Singularities

Date: 11 / 28 / 2018 **Time:** 11 : 00 **am** / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Singular symplectic structures are particular examples of Poisson structures and naturally appear in problems in celestial mechanics when doing the McGehee change of coordinates or other regularizations. The induced structure on level-sets of contact type is studied and it is proved that the singularities in the contact form allow us to do a plug-like construction for the Reeb flow and thereby produce compact examples without periodic orbits, which is in sharp contrast to the Weinstein conjecture in the smooth setting.

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From Celestial Mechanics to Fluid Dynamics: Contact structures with singularities

Hamiltonian systems, from topology to applications through analysis II

Eva Miranda (UPC & Observatoire de Paris & ICMAT)

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Outline

joint work with

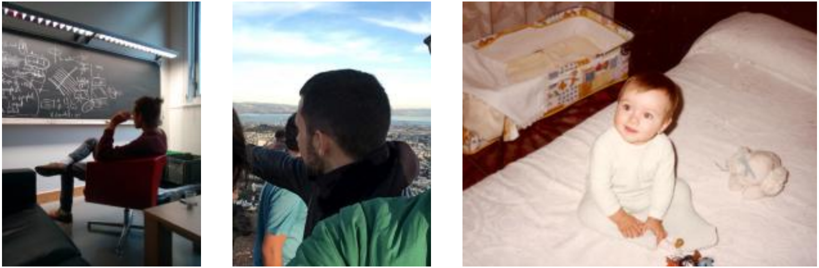


Figure: Cédric Oms, Robert Cardona and ...

joint work with

Figure: Cédric Oms, Robert Cardona and Daniel Peralta Salas

The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.

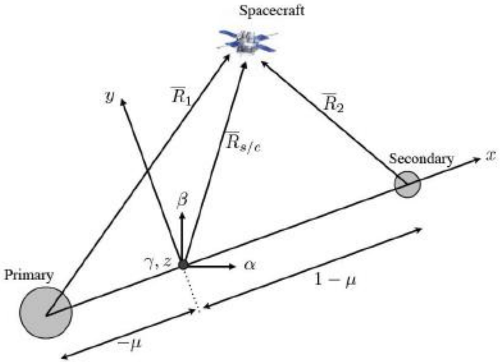


Figure: Circular 3-body problem

An example from Celestial Mechanics: Planar restricted 3-body problem

- The time-dependent self-potential of the small body is $U(q,t) = \frac{1-\mu}{|q-q_1|} + \frac{\mu}{|q-q_2|}$, with $q_1 = q_1(t)$ the position of the planet with mass $1-\mu$ at time t and $q_2 = q_2(t)$ the position of the one with mass μ.
- The Hamiltonian of the system is $H(q,p,t) = p^2/2 - U(q,t), \quad (q,p) \in \mathbf{R}^2 \times \mathbf{R}^2$, where $p = \dot{q}$ is the momentum of the planet.
- Consider the canonical change $(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$.
- Introduce **McGehee coordinates** (x, α, y, G), where $r = \frac{2}{x^2}, \quad x \in \mathbf{R}^+$, can be then extended to infinity ($x = 0$).
- The symplectic structure becomes a singular object

$$-\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$$

which extends to a b^3-symplectic structure on $\mathbf{R} \times \mathbb{T} \times \mathbb{R}^2$.

Symplectic and contact geometry of these systems

(b^m-symplectic)

$$\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}}\mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

or (m-folded)

$$\omega = \mathbf{x_1^m dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

Contact Geometry

The restriction to $H = ct$ induces a contact structure whenever there exists a *Liouville vector field* is transverse to it. This contact structure may admit singularities.

How are these singularities?

Symplectic	**Contact**
$\dim M = 2n$	$\dim M = 2n+1$
2-form ω, non-degenerate $d\omega = 0$	1-form α, $\alpha \wedge d(\alpha)^n \neq 0$
Hamiltonian $\iota_{X_H}\omega = -dH$	Reeb $\alpha(R) = 1$, $\iota_R \alpha = 0$ Ham. $\begin{cases} i_{X_H}\alpha = H \\ i_{X_H} d\alpha = -dH + R(H)\alpha. \end{cases}$

Topology of the circular restricted 3-body problem

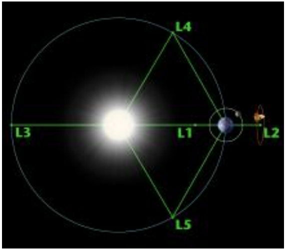


Figure: Lagrange points (Source: NASA/WMAP Science Team)

- For low energy levels $c \in \mathbb{R}$, $\Sigma_c = H^{-1}(c)$ has 3 connected components: Σ_c^E (the satellite stays close to the earth), Σ_c^M (to the moon), or it is far away.
- On the axis between earth and moon there is a critical point of the energy (L_1, the first Lagrange point). If $c > H(L_1)$, (the satellite can cross from the region around the earth to the region around the moon)$\rightsquigarrow$ there are two connected components, one bounded $\Sigma_c^{E,M}$ and an unbounded one.

Moser regularization of the restricted 3-body problem

- To deal with the singularities of the Kepler problem, Moser (1970) introduced a regularization procedure. This can be applied to the planar circular restricted 3-body problem.
- Via Moser's regularization Σ_c^E and Σ_c^M can be compactified to $\overline{\Sigma}_c^E$ and $\overline{\Sigma}_c^M$ diffeomorphic to $\mathbb{R}P^3$.
- Moser's regularization $\overline{\Sigma}_c^{E,M}$ is diffeomorphic to $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Contact Geometry of the restricted 3-body problem

Theorem (Albers-Frauenfelder-Van Koert-Paternain)

For $c < H(L_1)$ both connected components $\overline{\Sigma}_c^E$ and $\overline{\Sigma}_c^M$ admit a compatible contact form λ. Moreover, there exists $\epsilon > 0$ such that if $c \in (H(L_1), H(L_1) + \epsilon)$ the same assertion holds true for $\overline{\Sigma}_c^{E,M}$.

Corollary (Albers-Frauenfelder-Van Koert-Paternain)

For $c < H(L_1)$ the contact structures $(\overline{\Sigma}_c^E, \ker \lambda)$ and $(\overline{\Sigma}_c^M, \ker \lambda)$ coincide with the tight $\mathbb{R}P^3$ and for $c \in (H(L_1), H(L_1) + \epsilon)$ the contact structure $(\overline{\Sigma}_c^{E,M}, \ker \lambda)$ coincides with the tight $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Contact geometry of the restricted 3-body problem

Theorem (Albers-Frauenfelder-Van Koert-Paternain)

For any value $c < H(L_1)$, the regularized planar circular restricted three body problem has a closed orbit with energy c.

- **What if we consider the b^3-symplectic model?**
- **Does this contact structure have singularities?**
- **Can we still prove the existence of periodic orbits?**
- **Can we localize these periodic orbits with respect to the line at infinity?**

Other physical examples and (singular) symplectic and contact structures

Classical Hamiltonian systems	• Symplectic structures
Elliptic restricted 3-body problem in Mc-Gehee coordinates	• b^3-symplectic structure
McGehee regularization 3-body problem	• Folded-type symplectic structures
Kustaanheimo-Stiefel regularization for n-body problem	• Folded symplectic structure
McGehee type change for double collision	• b^m-symplectic structures • Folded-m symplectic structures

Contact structures show up by restriction to a *good* hypersurface.

Motivating examples from Fluid Dynamics

Figure: Arnold, Khesin, Ghrist, Etnyre, Peralta, Enciso

Euler equations for fluids in 3-manifolds

Euler equations model the dynamics of an inviscid and incompressible fluid flow for a Riemannian 3-manifold (M, g)

$$\frac{\partial u}{\partial t} + \nabla_u u = -\nabla P$$
$$\operatorname{div} u = 0.$$

(u the velocity, P the pressure) Using the Riemannian volume form μ, the second equation reads

$$\mathcal{L}_u \mu = 0.$$

The vorticity is the only vector field satisfying

$$\iota_\omega \mu = d\alpha,$$

where $\alpha = \iota_u g$. Using the vorticity the equations are the same as in the Euclidean case

$$\frac{\partial u}{\partial t} - u \times \omega = -\nabla B$$
$$\operatorname{div} u = 0,$$

with $B = P + \frac{1}{2} g(u, u)$ (Bernoulli function)

Stationary solutions

(Velocity field does not depend on time) $\nabla_u u = -\nabla P, \ \operatorname{div} u = 0.$
Using the Bernoulli function

$$u \times \omega = \nabla B, \ \operatorname{div} u = 0.$$

- if $B = ct$ the vorticity is proportional to u $(\operatorname{curl} u = fu)$ and u is a Beltrami flow. Then α is contact.
- If B not constant and analytic, its critical set $Cr(B) := \{p \in M | \ \nabla B(p) = 0\}$ has codimension at least 1 and

Theorem (Arnold structure theorem)

If the flow is assumed tangent to the boundary, then $M \backslash Cr(B)$ consists of finitely many domains M_i

1. *M_i is trivially fibered by invariant tori of u and on each torus the flow is conjugated to the linear flow.*
2. *or M_i is trivially fibered by invariant cylinders of u, and its flow is periodic.*

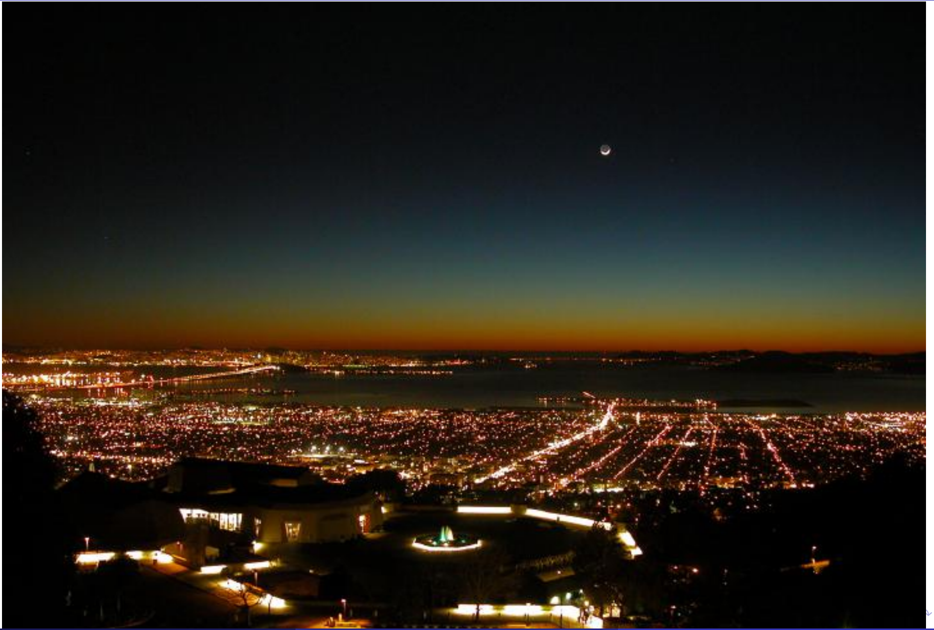
Singularities and Euler equations

- (joint with R. Cardona and D. Peralta-Salas) Morse-Bott (singular) symplectic description of the fibers B with the structure $i^*\mu_2$ where

$$\mu_2 = \iota_{\frac{\nabla B}{|\nabla B|^2}} \nu$$

- Applications of Contact Topology tools to the study of periodic orbits of the Beltrami Flows on manifolds with boundary (Etnyre-Ghrist).
- **What happens when there is a singularity on the boundary? ("vortons")**

Zooming in and out on Symplectic/Contact Geometry



b-Symplectic/b-Poisson structures

Definition

Let (M^{2n}, Π) be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a b-**Poisson structure** on (M, Z).

b-symplectic, log-symplectic

Batakidis, Braddell, Cardona, Cavalcanti, Delshams, Frejlich, Gualtieri, Guillemin, Kiesenhofer, Klaasse, Lanius, Songhao Li, Marcut, Martínez-Torres, Melrose Miranda, Nest, Oms, Osorno, Pelayo, Pires, Planas, Radko, Ratiu, Scott, Tsygan, Vera, Villatoro, Weitsman

Darboux normal forms

Theorem

For all $p \in Z$, *there exists a Darboux coordinate system* $x_1, y_1, \ldots, x_n, y_n$ *centered at* p *such that* Z *is defined by* $x_1 = 0$ *and*

$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

Darboux for b^m-symplectic structures

$$\Pi = x_1^m \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

or dually

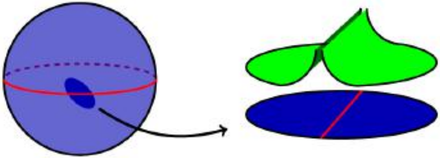
$$\omega = \frac{1}{x_1^m} dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$$

Radko's classification of b-Poisson surfaces

Radko classified these structures on compact oriented surfaces:

- Geometrical invariants: The topology of S and the curves γ_i where Π vanishes.
- Dynamical invariants: The periods of the "modular vector field" along γ_i.
- Measure: The regularized Liouville volume of S, $V_h^\varepsilon(\Pi) = \int_{|h|>\varepsilon} \omega_\Pi$ for h a function vanishing linearly on the curves $\gamma_1, \ldots, \gamma_n$ and ω_Π the "dual "form to the Poisson structure.

Other classification schemes: For b^m-symplectic structures (not necessarily oriented) $\rightsquigarrow$ Scott, M.-Planas.



Other compact examples.

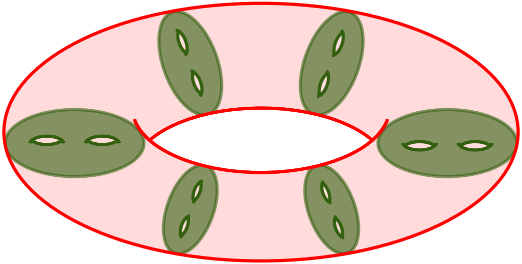
- The product of (R, π_R) a Radko compact surface with a compact symplectic manifold (S, ω) is a b-Poisson manifold.
- corank 1 Poisson manifold (N, π) and X Poisson vector field $\Rightarrow$ $(S^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi)$ is a b-Poisson manifold if,
  1. f vanishes linearly.
  2. X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

The singular hypersurface

Theorem (Guillemin-M.-Pires)

If $\mathcal{L}$ contains a compact leaf L, then Z is the mapping torus of the symplectomorphism $\phi : L \to L$ determined by the flow of a Poisson vector field v transverse to the symplectic foliation.

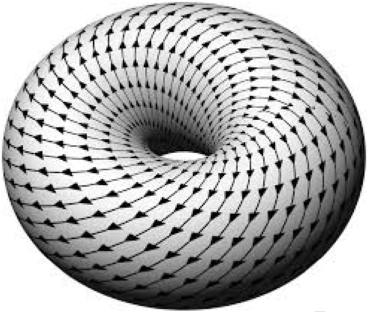


This description also works for b^m-symplectic structures.

Zooming in again...

- b-Poisson structures can be seen as symplectic structures modeled over a Lie algebroid (the b-cotangent bundle).
- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$



b-forms

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

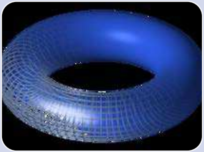
$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

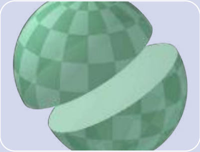
- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2.
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.
- We can introduce **b-contact structures on a manifold** M^{2n+1} as b-forms of degree 1 for which $\alpha \wedge (d\alpha)^n \neq 0$.
- The b-cotangent bundle can be replaced by other algebroids (E-symplectic) known to Nest and Tsygan.

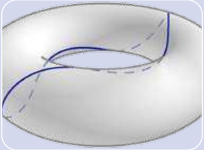
(Singular) symplectic manifolds

- b^m-Symplectic
- Symplectic
- Folded symplectic

Déjà-vu...







Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle coordinates

b-Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle theorem

Folded symplectic manifolds

- Darboux theorem (Martinet)
- Delzant-type theorems (Cannas da Silva-Guillemin-Pires)
- Action-agle theorem (M-Cardona)

Examples

Orientable Surface	$\mathbb{C}P^2$	S^4
• Is symplectic • Is folded symplectic • (orientable or not) is b-symplectic	• Is symplectic • Is folded symplectic • Is not b-symplectic	• Is not symplectic • Is not b-symplectic • Is folded-symplectic

Desingularizing b^m-symplectic structures

Theorem (Guillemin-M.-Weitsman)

Given a b^m-symplectic structure ω on a compact manifold (M^{2n}, Z):

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If $m = 2k + 1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

Consequences

Corollary

A manifold admitting a b^{2k}-symplectic structure also admits a symplectic structure.

Corollary

A manifold admitting a b^{2k+1}-symplectic structure also admits a folded symplectic structure.

Theorem (Cannas da Silva)

Any orientable compact 4-manifold admits a folded structure.

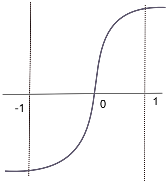
The converse is not true.

S^4 admits a folded structure but no b-symplectic structure.

Deblogging b^{2k}-symplectic structures

$$\omega = \frac{dx}{x^{2k}} \wedge \left(\sum_{i=0}^{2k-1} \alpha_i x^i\right) + \beta \qquad (1)$$

- Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be an odd smooth function satisfying $f'(x) > 0$ for all $x \in [-1, 1]$,



and such that outside $[-1, 1]$,

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1 \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1 \end{cases}$$

Deblogging b^{2k}-symplectic structures (Proof)

- Scaling:

$$f_\epsilon(x) := \frac{1}{\epsilon^{2k-1}} f\left(\frac{x}{\epsilon}\right). \tag{2}$$

Outside the interval $[-\epsilon, \epsilon]$,

$$f_\epsilon(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x < -\epsilon \\ \frac{-1}{(2k-1)x^{2k-1}} + \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x > \epsilon \end{cases}$$

- Replace $\frac{dx}{x^{2k}}$ by df_ϵ to obtain

$$\omega_\epsilon = df_\epsilon \wedge \left(\sum_{i=0}^{2k-1} \alpha_i x^i\right) + \beta$$

which is symplectic.

Symplectic character

- $d\alpha_i = 0 \rightsquigarrow \omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} \alpha_i x^i) + \beta$ closed.
- Outside U, ω_ϵ coincides with ω.
- In U but away from Z,

$$\omega_\epsilon^n = \frac{df_\epsilon}{dx} x^{2k} \omega^n$$

which is nowhere vanishing.
- To check that ω_ϵ is symplectic at Z, observe that

$$\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} x^i \alpha_i) + \beta = \epsilon^{-2k} f'\left(\frac{x}{\epsilon}\right) dx \wedge (\sum_{i=0}^{2k-1} x^i \alpha_i) + \beta$$

which on the interval $|x| < \epsilon$ is equal to $\epsilon^{-2k}(f'(\frac{x}{\epsilon})\, dx \wedge \alpha_0 + \mathcal{O}(\epsilon)) + \beta$ and hence

$$\omega_\epsilon{}^n = \epsilon^{-2k}(f'\left(\frac{x}{\epsilon}\right) dx \wedge \alpha_0 \wedge \beta^{n-1} + \mathcal{O}(\epsilon))$$

which is non-vanishing for ϵ sufficiently small $dx \wedge \alpha_0 \wedge \beta^{n-1} \neq 0$.

Convergence

- To check

$$\omega_\epsilon^{-1} = \epsilon^{2k} g\left(\frac{x}{\epsilon}\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} + \cdots + \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n} \qquad (3)$$

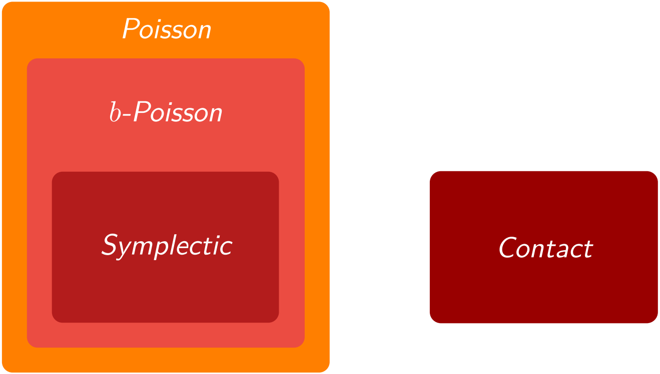
where $g(x) = \frac{1}{f'(x)}$, converges to

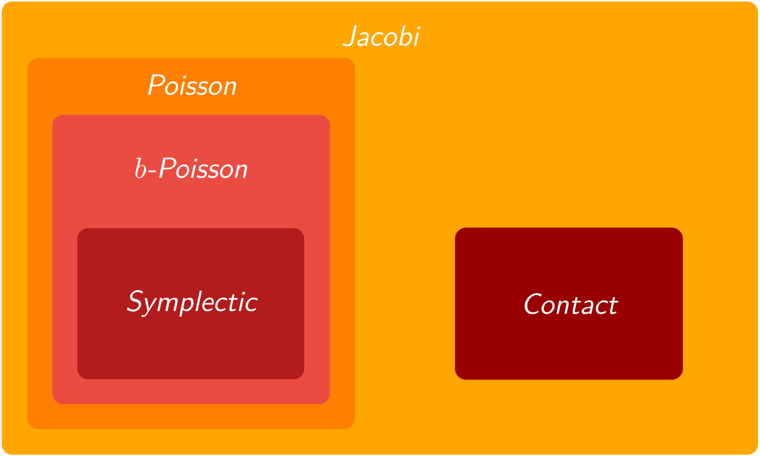
$$\omega^{-1} = x^{2k} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} + \cdots + \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n} \qquad (4)$$

as ϵ tends to zero.
Consider $h(x) = (\frac{d}{dx})^{2k-1} g(x)$.

- Then ω_ϵ^{-1} converges to ω^{-1} in the C^{2k-1} topology if $\epsilon h\left(\frac{x}{\epsilon}\right)$ converges in the uniform norm to $2kx$. But $x^{2k} = \epsilon^{2k} g\left(\frac{x}{\epsilon}\right)$ for $|x| > \epsilon$, so for $\epsilon < |x|$, $\epsilon h\left(\frac{x}{\epsilon}\right)$ is equal to $2kx$, and for $\epsilon > |x|$ both functions are bounded by a constant multiple of ϵ.
- Hence $\epsilon h\left(\frac{x}{\epsilon}\right)$ converges in the uniform norm to $2kx$ when $\epsilon \to 0$. and this gives the C^{2k-1}-convergence of (3) to (4).

Another general picture...


Poisson
b-Poisson
Symplectic
Contact


Jacobi
Poisson
b-Poisson
Symplectic
Contact

Jacobi manifolds

Theorem (Lichnerowicz, Kirillov)

A Jacobi bracket is of the form

$$\{f,g\} = \Lambda(df,dg) + f(Rg) - g(Rf),$$

where $\Lambda \in \mathfrak{X}^2(M)$ *and* $R \in \mathfrak{X}(M)$ *satisfy*

- $[\Lambda,\Lambda] = 2R\wedge\Lambda$,
- $[\Lambda,R] = \mathcal{L}_R\Lambda = 0$.

Examples

- Poisson manifolds: $R = 0$.
- Contact manifolds $(M, \ker \alpha)$: R Reeb vector field, $\Lambda(df, dg) := d\alpha(X_f, X_g)$.
- Locally conformally symplectic (l.c.s.) manifolds (M, ω, α): $\Lambda(df, dg) := dg(\omega^\sharp df)$ and $R := \omega^\sharp \alpha$.

Remark

If (M, Λ, R) Jacobi then $(M \times \mathbb{R}, e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge R))$ is Poisson.

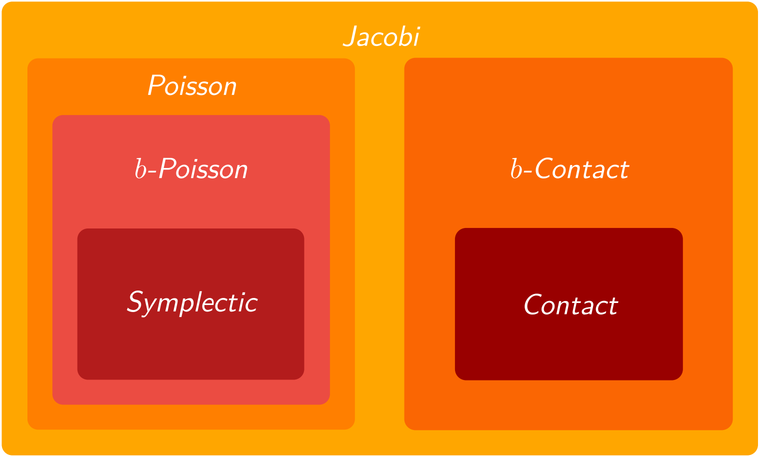
Characteristic leaves

Definition

The Hamiltonian vector fields are defined by $X_f := \Lambda^\sharp(df) + fR$.

$\mathfrak{F}(M) = \{X_f | f \in C^\infty(M)\} = \mathrm{Im}\Lambda^\sharp + \langle R \rangle$ is integrable.

- $R \in \mathrm{Im}\Lambda^\sharp$: even-dimensional leaves: l.c.s.
- $R \notin \mathrm{Im}\Lambda^\sharp$: odd-dimensional leaves: contact.



Definition

A Jacobi manifold (M^{2n+1}, Λ, R) is b-Jacobi if $\Lambda^n \wedge R \pitchfork 0$.

There is a one to one correspondence between b-Jacobi structures on an odd dimensional manifold and b-contact forms.

Local normal forms and geometry of the critical set

Darboux theorem

There are 3 local models depending on the singularities of the Reeb vector field and the contact structure.

Theorem

Let $(M^{2n+1}, \xi = \ker \alpha)$ *be a* b*-contact manifold and* $p \in Z$. *We denote* $\mathcal{F}_p$ *the leaf of the singular foliation* $\mathcal{F}$ *going through* p. *Then*

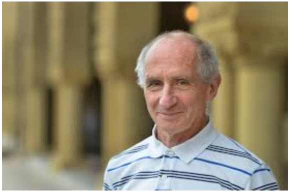
1. *if* ξ_p *is regular, that is* $\mathcal{F}_p$ *of dimension* $2n$, *then the induced structure on* $\mathcal{F}_p$ *is locally conformally symplectic;*
2. *if* ξ_p *is singular, that is* $\mathcal{F}_p$ *of dimension* $2n-1$, *then the induced structure on* $\mathcal{F}_p$ *is contact.*

Existence of contact structures

All 3-dimensional manifolds are contact (Martinet-Lutz) in higher dimensions.

Theorem (Borman-Eliashberg-Murphy)

Any almost contact closed manifold is contact.


Existence of b^{2k}-contact structures

Theorem (Even singularization, M-Oms)

For any pair (M, Z) of contact manifold and convex hypersurface there exists a b^{2k}-contact structure for all k having Z as critical set.

Corollary (of Giroux theorem)

For any 3-dimensional manifold and any generic surface Z, there exists a b^{2k}-contact structure on M realizing Z as the critical set.

Key point

- *Singularization (even case)*
  - Using the transverse contact vector field, $\alpha = udt + \beta$, where t is the coordinate on $\mathbf{R}$, $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$.
  - Let us take a smooth function f_ϵ such that
    1. $f_\epsilon(x) = x$ for $x \in \mathbf{R} \setminus [-2\epsilon, 2\epsilon]$
    2. $f_\epsilon(x) = -\frac{1}{x^{2k-1}}$ for $x \in [-\epsilon, 0[\cup]0, \epsilon]$
    3. $f'_\epsilon(x) > 0$ for all $x \in \mathbf{R}$.

    Consider $\alpha_\epsilon = udf_\epsilon + \beta$. It is a b^{2k}-form that coincides with α outside $Z \times (\mathbf{R} \setminus [-2\epsilon, 2\epsilon])$.
- Realization problem for odd m.

Theorem (Odd singularization, M-Oms)

Let (M, α) be a contact manifold and let Z be a convex hypersurface, then M admits a b^{2k+1}-contact structure for all k that has two diffeomorphic connected components Z_1 and Z_2 as critical set. One of the hypersurfaces can be chosen to be Z.

What about periodic orbits?

What about periodic orbits?

Weinstein's conjecture 

The Reeb vector field of a contact compact manifold admits at least one periodic orbit.

- Taubes proved it in dimension 3 for regular contact structures.
- What about singular contact structures?

Key point: Desingularization

Theorem (M-Oms)

Given a b^{2k}-contact manifold with convex critical set Z, there exists a family of contact forms agreeing with a b^{2k}-contact form α outside of an ϵ-neighbourhood of Z. The family of bi-vector fields Λ_ϵ and the family of vector fields $\mathbf{R}_\epsilon$ associated to the Jacobi structure of the contact form α_ϵ converges to the bivector field Λ and to the vector field R in the C^{2k-1}-topology as $\epsilon \to 0$.

Theorem (M-Oms)

Let (M, α) be a closed b^{2k}-contact manifold of dimension 3, then there exists a family of periodic orbits $\mathcal{O}_\epsilon$ associated to the Reeb vector fields R_ϵ.

A variational principle to detect periodic orbits.

Periodic orbits on M $\leftrightsquigarrow$ smooth maps $x : \mathbb{R}/\mathbb{Z} \longrightarrow M$ This set is called the loop space, $\mathcal{L}M$.

If $\Pi_2(M) = e$ the action functional is well-defined:

$$\mathcal{A}_H(x) := -\int_D u^*\omega + \int_0^1 H_t(x(t))\,dt,$$

(where u is an extension of x to the disk and we assume $H_t = H_{t+1}$)

Theorem

A loop x is a critical point of the action functional $\mathcal{A}_H(x)$ if and only if $t \mapsto x(t)$ is a periodic solution of the Hamiltonian system

$$\dot{x} = X_t(x(t)).$$

Key point:

$$d\mathcal{A}_H(x)(Y) = \int_0^1 \omega(\dot{x} - X_t(x(t)), Y)dt.$$

$\forall Y$ and ω is non-degenerate and this works in the b-symplectic case too.

Periodic orbits via desingularization

What one would like to do: Take the limit $\epsilon \to 0$.

Problem: γ_ϵ need not be continuous.

The periodic orbit of γ_ϵ can be outside the critical set (type (1)), or be contained in it (type (2)) or cut it (type (3)).

Lemma

If there exist $\epsilon > 0$, such that there exists a periodic orbit of R_{α_ϵ} of type (1), then this is a periodic Reeb orbit of (M, α).

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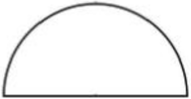
A counterexample to the singular Weinstein conjecture

Inspirational: Find periodic "solutions" that go to infinity in the restricted three body problem.



Theorem (M.-Oms)

There are compact b^m-contact manifolds in any dimension for all $m \in \mathbb{N}$ without periodic Reeb orbits.

Corollary (Counterexample to the Hamiltonian Seifert conjecture, M.-Oms)

There are b^m-symplectic manifolds with proper smooth Hamiltonian whose level-set does not contain periodic orbits of the Hamiltonian flow.

A counterexample to Weinstein's conjecture for singular contact structures

- Take a compact contact manifold with a finite number of isolated periodic Reeb orbits (for instance an ellipsoid) and change the contact form to a b^m-contact form by inserting a copy of Oms' plug for every periodic Reeb orbit.
- How to construct Oms' plug: Consider the standard contact structure α_{st} on the cylinder $D(2) \times [-2, 2]$ given by $\alpha_{st} = dz + xdy$, where $D(2)$ is the disk of radius 2 and z is the coordinate on the interval. The Reeb vector field is given by $\frac{\partial}{\partial t}$.
- Take $S^2 \subset D(2) \times [-2, 2]$.We identify the unit sphere as the critical set of a b^m-contact form that agrees with the standard one on the boundary of the cylinder. (The unit sphere is a convex surface (the vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$ is a contact vector field), $\rightsquigarrow$ semilocally $\alpha_{st} = udt + \beta$, where $X = \frac{\partial}{\partial t}$, $u \in C^\infty(S^2)$ and $\beta \in \Omega^1(S^2)$.)
- Check that it works!

A picture is worth a thousand words!

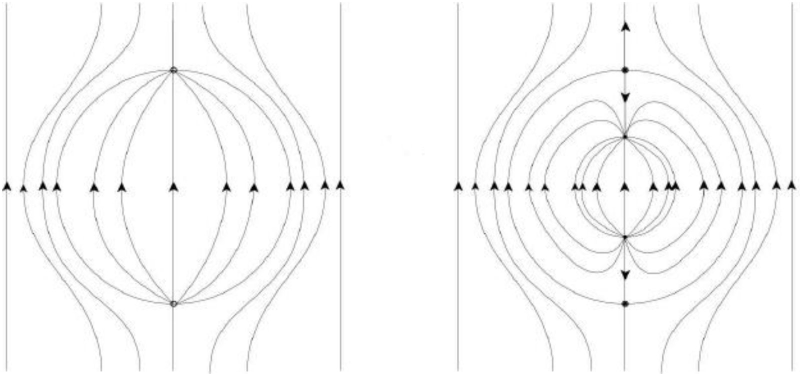


Figure: Oms' plug for even and odd m

FROM CELESTIAL MECHANICS TO FLUID DYNAMICS
CONTACT STRUCTURES WITH SINGULARITIES

EVA MIRANDA Notes by Jeffrey Heninger

<u>Example from Celestial Mechanics</u> - circular, restricted, planar 3 body problem

regularization transformation - McGehee coordinates

almost a symplectic structure - symplectic away from $x=0$.

what happens to this singularity when you restrict to a level set of the Hamiltonian?

b^3-symplectic structure singularity $\sim \frac{1}{x^3}$

"b" from boundary (not Barcelona)

sometimes also called log-symplectic

(M^{2n}, ω) $\omega^n \nearrow \omega^n \to 0$ folded-type [Martinet] these are often dual

$\searrow \omega^n \to \infty$ b^m-symplectic [...]

contact geometry

good contact structure - Liouville vector field $L_X\omega = \omega$ transverse to it

Symplectic vs. contact geometry

Reeb vector field - $\alpha(R)=1$, $i_R\, d\alpha = 0$

CR3BP without McGehee coordinates

look at topology - connected to Lagrange points

Moser regularization - makes the level sets compact

completely characterized contact geometry of energy level sets

What if ask similar questions using McGehee coordinates?

should be able to deal with the line at infinity

<u>Example from Fluid Dynamics</u> - Euler equations for fluids in 3-manifolds (inviscid, incompressible)

look for stationary solutions

if $u \propto \omega$, Beltrami flow, $B=$const. ← contact integrable ↓

if B not constant & analytic → Arnold structure thm (like Arnold-Liouville Thm)

what if B smooth, but not analytic → Morse-Bott (singular) symplectic description of the fibers

can use contact topology to study periodic orbits of Beltrami flows on manifold with boundary

what happens if there is a singularity on the boundary? "vortons"

K2

Def Poisson structure

described using a vector field instead of a Poisson bracket

$\pi(df, dg) = \{f, g\}$

$[\pi, \pi] = 0$ Schouten bracket

look at a surface where Casimir = const.

↳ Symplectic foliation (M, π)

$D = \{X_f, f \in C^\infty(M)\}$

where X_f is the Hamiltonian vector field $\pi(df, \cdot) = X_f$

(M^{2n}, π) Poisson manifold

wedge π n times: $\pi \wedge \cdots \wedge \pi$

you have a Darboux Thm

$\omega = \frac{1}{x_1^m} dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n$ — we have examples for any m.

only difference from usual Darboux thm

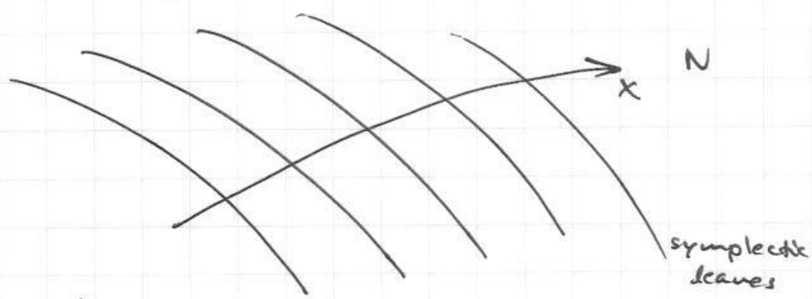
Radko's Classification of b-Poisson structure (like Moser's classification)

making higher dim. ex. can be difficult — when making products, check transversality

take a codim I foliation

assume this is a symplectic vector field transverse to all leaves

take $N \times S^1$



critical set is always symplectic mapping torus

Change Language to Symplectic Geometry

enlarge tangent bundle

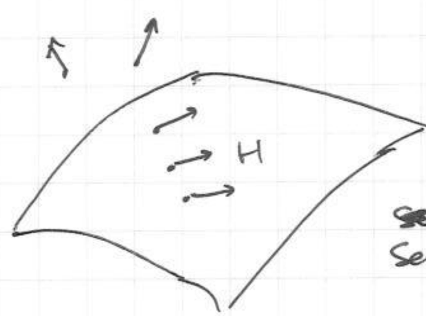
take a hypersurface on your manifold

take vector fields that are tangent to the hypersurface along the hypersurface

take {these vector fields}

is there a bundle such that these fields are sections of the bundle?

Yes - Melrose - b tangent bundle



Serre-Swan

then dualize bundle → b cotangent bundle

take exterior powers of b cotangent bundle — take sections of this = b-forms

K3

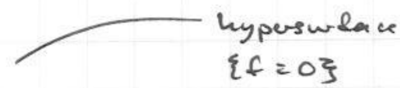
like having two pairs of glasses - one can see singularities in one, not the other
hide the singularities in the bundle

do symplectic geometry on this new bundle

how to find the differential:

ω: b-form Z with (M, Z)

$\omega \in {}^b\Omega^k(M)$



$$\omega = \frac{df}{f} \wedge \alpha + \beta \qquad \alpha \in \Omega^{k-1}(M), \quad \beta \in \Omega^k(M)$$

[de Rham] $d\omega = \frac{df}{f} \wedge d\alpha + d\beta$

$${}^k H^b(M) = H^k_{DR}(M) \oplus H^{k-1}_{DR}(Z)$$

↑ Thm [Mazzeo - Melrose]

Now look at b-contact structures
impose contact condition $\qquad \alpha^n (d\alpha)^n \neq 0$

Jacobi Manifolds
generalizes a lot of things - including contact geometry
2 integrability equations $\qquad$ R - "Reeb"

$d\omega = \alpha \wedge \omega$, where α is a closed form

Hamiltonian vector field for Jacobi
different structures for even/odd dimensional leaves

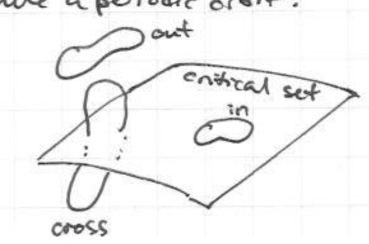
Existence of Contact Surfaces

Thm - any almost contact closed manifold is compact
do you have this result for singular contact structure?
we can desingularize / singularize these
odd case - regularization thm - topological constraint needed

Periodic Orbits

Weinstein Conjecture
Reeb vector field on contact compact manifold - is there a periodic orbit?
counterexample by Ons
limiting procedure - does the family of periodic orbits converge?

b-contact → b-symplectic, back



limit is multiple heteroclinic orbits -

limit → multiple heteroclinic orbits - not a periodic orbit