MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Wilfrid Gangbo

Talk Title: A Weaker Notion of Convexity for Lagrangians not Depending Solely on Velocities and Positions

Date: 11 / 30 / 2018 **Time:** 9 : 30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: For a fluid system satisfying the continuity equation and initial and boundary conditions, the action has a minimizer if the minimizer of the action of the convex envelope of the Lagrangian is achievable by the action of the Lagrangian. Extend this to a similar magnetic field problem which satisfies the homogeneous Maxwell's equations instead of the continuity equation. Using quasiconvexity, defined using Jensen's Inequality, a vector potential, and a specific gauge condition, we convert this problem into the fluid problem.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

P1

A Weaker Notion of Convexity for Lagrangians Not Depending Solely on Velocities and Positions

WILFRID GANGBO   Notes by Jeffrey Heninger

[joint with B. Dacorogna]

Well Established Theory

Data $\Omega \subset \mathbb{R}^3$ convex, bounded, open   (later: $\Omega \subset \mathbb{R}^n$, bounded, contractable)

$L : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ smooth   ($L : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ l.s.c)

$c_1 (|v|^2 + |x|^2) - c_2 \le L(x,v) \quad \forall\, x, v \in \mathbb{R}^3 \qquad c_1 > 0$

Unknown

$\rho : [0,1] \times \Omega \to [0,\infty)$ $\rho(t,\cdot)$ density function

$v : [0,1] \times \Omega \to \mathbb{R}^3$ $v(t,\cdot)$ velocity vector field

$m : [0,1] \times \Omega \to \mathbb{R}^3$ $m = \rho v$ momentum

Continuity equation: $\partial_t \rho + \nabla \cdot m = 0$ on $(0,1) \times \Omega$

Boundary: $m \cdot \nu = 0$ on $(0,1) \times \partial\Omega$

Initial & Final Conditions: $\rho(0,\cdot) = \rho_0$, $\rho(1,\cdot) = \rho_1$ prescribed

Use these to find (ρ, m). $\mathcal{C}(\rho_0, \rho_1)$

Action $A_L(\rho, m) = \int_0^1 dt \int_\Omega L(\rho(t,x), m(t,x))\, dx$

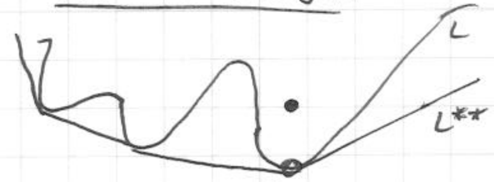
QUESTION: What is the ~~correct~~ correct condition on L to ensure $\inf_{(\rho,m) \in \mathcal{C}(\rho_0,\rho_1)} A_L(\rho, m)$ admits a minimizer

Notation Denote as L^{**} the convex envelope of L

$$L^{**} = \sup\{\varphi : \varphi \text{ convex}, \varphi \le L\}$$

Thm $\inf_{\mathcal{C}(\rho_0,\rho_1)} A_L(\rho,m) = \min_{\mathcal{C}(\rho_0,\rho_1)} A_{L^{**}}(\rho, m)$

Finite Dim analogue



min of L^{**}. L is discontinuous here. (doesn't have to be discont.)

A necessary & sufficient condition for A_L to have a minimizer if the minimizer of $A_{L^{**}}$ is in A_L.

New Exploration (A particular case)

Unknowns $E, B : [0,1] \times \Omega \to \mathbb{R}^3$
electric magnetic

Assume:

$\mathcal{C}(B_0, B_1)$
(E, B)

$\nabla_x \cdot B = 0$ on $(0,1) \times \Omega$ Gauss's law for Magnetism / No Monopoles

$\partial_t B + \nabla \times E = 0$ on $(0,1) \times \Omega$ Faraday's law

$\nu_\Omega \cdot B = 0$ $\nu \times E = 0$ on $(0,1) \times \partial\Omega$

$B(0,\cdot) = B_0$ $B(1,\cdot) = B_1$

Lagrangian $\bar{L} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$A_{\bar{L}}(E, B) = \int_0^1 dt \int_\Omega \bar{L}(E, B)\, dx$

QUESTION What is the condition on $\bar{I}$ for $\inf_{\mathcal{C}(B_0, B_1)} A_{\bar{I}} = \min!$

Review on C.B. Morrey Condition

Assume $F : \mathbb{R}^{N\times m} \to \mathbb{R}$ function on $N\times m$ matrices

For $u : \mathcal{O} \subset \mathbb{R}^m \to \mathbb{R}^n$ define

$$I[u] = \int_{\mathcal{O}} F(\nabla u(z))\, dz$$

Note: $\mathcal{O} \neq 0$

Given $u_* \in W^{1,2}(\mathcal{O}, \mathbb{R}^N)$

Consider $\inf_{u \in u_* + \underbrace{C_c^1(\mathcal{O}, \mathbb{R}^N)}_{W_0^{1,2}(\mathcal{O}, \mathbb{R}^N)}} I[u]$ $\quad \|u\|_{W^{1,2}} = \|u\|_{L^2} + \|\nabla u\|_{L^2}$

Remark Let φ be a function of compact support. $\varphi \in C_c^1(\mathcal{O}, \mathbb{R}^N)$. Then

(i) $\int_{\mathcal{O}} \nabla \varphi(x)\, dx = 0$

Take $u_\xi(x) = \xi x + \varphi(x)$, $\xi \in \mathbb{R}^{N\times m}$

$$\int_{\mathcal{O}} \nabla u_\xi(x)\, dx = \xi \operatorname{meas}(\mathcal{O})$$

(ii) Assume $m = N$

$$\int_{\mathcal{O}} \det(\nabla u_\xi(x))\, dx = \det(\xi) \operatorname{meas}(\mathcal{O})$$

$$\Rightarrow \int_{\mathcal{O}} |\det \nabla u_\xi(x)|\, dx \geq \left| \int_{\mathcal{O}} \det \nabla u_\xi(x)\, dx \right| = |\det \xi| \operatorname{meas}(\mathcal{O})$$

Def F is called quasiconvex if it satisfies Jensen's inequality

$$\int_{(0,1)^m} F(\nabla u_\xi(x))\, dx \geq F\left(\int_{(0,1)^m} \nabla u_\xi(x)\, dx \right) = F(\xi)$$

All convex functions is quasiconvex. $|\det|$ is also quasiconvex, although it is not convex

Consider $\det\left((1-t)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = (1-t)t$

$|\det|$ is quasiconvex but not convex

$|\xi| \to c_0 |\xi|^2 + c_1 |\det \xi|^2$

Def The largest quasiconvex function smaller than or equal to $\bar{I}$ is called the quasiconvex envelope of $\bar{I}$ as is denoted as $Q\bar{I}$.

Thm [Morrey]

(i) I is ~~[scribbled]~~ weakly lower semicontinuous on $W^{1,2}$ iff F is quasiconvex.

Need to have some boundary condition: $F(\xi) \sim |\xi|^2$

3

<u>Corollary</u> $\inf_{u \in u_* + W^{1,2}} I[u] = \min_{u \in u_* + W^{1,2}} \int_{\mathcal{O}} QF(\nabla u)\, dx$

Quasiconvexity is the ~~appr~~ appropriate setting for these minimizers

We can turn the magnetic field problem from before into Morrey's Setting

<u>Back to Our Problem</u> (General Case)

$B(t,\cdot)$ is a k-differential form on Ω

In $\mathbb{R}^3$, vectors can be either 1-forms or 2-forms.

write $E = E_1\, dx^1 + E_2\, dx^2 + E_3\, dx^3$ — E is a 1 form

$B = B_1\, dx^2 \wedge dx^3 + B_2\, dx^3 \wedge dx^1 + B_3\, dx^1 \wedge dx^2$ — B is a 2 form

In general, $B(t,\cdot)$ is a k-differential form on $\Omega \subset \mathbb{R}^n$

$E(t,\cdot)$ is a $(k-1)$-differential form on Ω

set $\mathcal{O} = (0,1) \times \Omega$ $\quad (t,x) \quad dt, dx^i$

~~Set~~ Set $h = B - dt \wedge E$ $\quad$ unspecified d includes (t,x)

We have $dh = d_x B + dt \wedge (\partial_t B + dE)$ $\quad x_0 := t$

$\mathcal{C}(B_0, B_1)$ is the set of (E,B) such that $\begin{cases} \partial_t B + dE = 0 \\ d_x B = 0 \\ \nu_{\mathcal{O}} \wedge B = 0 \quad \nu \wedge E = 0 \\ B(0,\cdot) = B_0 \quad B(1,\cdot) = B_1 \end{cases}$

If $\nu_{\mathcal{O}} = (\nu_0, \nu_1, \dots, \nu_n) \longleftrightarrow \sum_{j=0}^{n} \nu_j\, dx^j$

Note: $dh = 0 \Rightarrow h = d\omega$ for a $(k-1)$ form ω

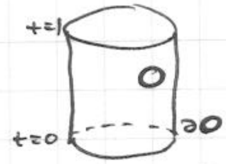
<u>Action</u> $\int_{\mathcal{O}} L(E,B) = \int_{\mathcal{O}} L(h) = \int_{\mathcal{O}} L(d\omega)$

Write $d\omega = \nabla\omega - \nabla\omega^T$ ← only antisymmetric part

Symmetric part undetermined — choose $\delta\omega = 0$ ← disappears

Gaffney Inequality $\|\nabla\omega\|_{L^s} \le C\left(\|d\omega\|_{L^s} + \|\delta\omega\|_{L^s} + \|\omega\|_{L^s}\right)$

↑ for a smooth set


not a smooth set

Compatibility Condition so the set $\mathcal{C}(B_0, B_1)$ is not empty:

$d_x B_0 = d_x B_1 = 0$ $\quad$ Set $B_0 = d_x F_0$

$(B_1 - B_0) \wedge \nu_{\mathcal{O}} = 0$ $\quad$ $B_1 = d_x F_1$

Set $\tilde{\omega}(t,x) = (1-t) F_0(x) + t F_1(x)$

$d\tilde{\omega} \wedge \nu = d\omega \wedge \nu \overset{?}{\Rightarrow} \tilde{\omega} = \omega$ on $\partial\mathcal{O}$ $\quad$ need to choose gauge so this is true

we have to assume support B_0, support $B_1 \subset \Omega$

<u>Given</u> h

$dh = 0 \Rightarrow \exists\, \omega : h = d\omega$

<u>Remark</u> Since $dd = 0$

$h = d(\omega + d\varphi)$

Def $\bar{L}$ is quasiconvex if

$$\int_{(0,1)^{n+1}} \bar{L}\left(\lambda + d_x\varphi,\ \mu - \partial_t \varphi + d_x \psi\right) \geq \bar{L}(\lambda, \mu)$$

$\lambda \in \Lambda^k$, $\mu \in \Lambda^{k-1}$, $\varphi \in W_0^{1,\infty}(\Omega, \Lambda^{k-1})$, $\psi \in W_0^{1,\infty}(\Omega, \Lambda^{k-2})$

Thm $\inf A_{\bar{L}} = \inf A_{Q\bar{L}}$