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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Boris Khesin

Talk Title: Geometric and Hamiltonian Hydrodynamics via Madelung Transform

Date: 11 / 27 / 2018 **Time:** 11 : 30 **am** / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: The Euler equation corresponds to the geodesic equation on the space of volume preserving diffeomorphisms. For compressible fluids, we can map a metric down from the space of diffeomorphisms to densities. Volume preserving diffeomorphisms are a fiber, gradient fields are perpendicular to fibers, and the pressure force in incompressible Euler is the force required to stay on the submanifold. Considers the geometry of the hydrodynamic (Madelung) form of quantum mechanics.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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GEOMETRIC AND HAMILTONIAN HYDRODYNAMICS VIA MADELUNG TRANSFORM

BORIS KHESIN   Notes by Jeffrey Heninger

joint with G. Misiolek and K. Modin   arxiv 1711.00321
1807.07172

I. Arnold's approach to the Euler Equation

$M, (\,,\,)$ – Riemannian manifold that the fluid lives in μ – volume form

Euler Equation

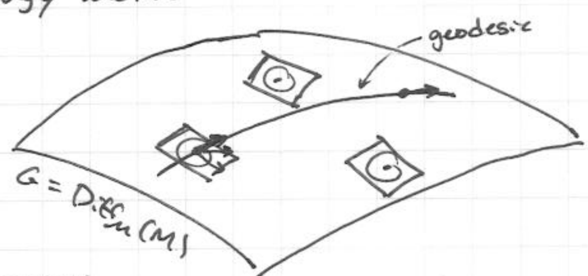
$$\begin{cases} \partial_t v + \nabla_v v = -\nabla p \\ \operatorname{div} v = 0 \quad (\text{and } v \parallel \partial M) \end{cases}$$

p determined, up to a constant, by $\operatorname{div} v = 0$. $\rho \equiv 1$

Thm [Arnold 1966]: The Euler equation $\rightleftharpoons$

the geodesic equation on $G = \mathrm{Diff}_\mu(M)$ – volume preserving diffeomorphisms with respect to the right-invariant L^2 = energy metric

$$\underset{\operatorname{div} v = 0}{E(v)} = \frac{1}{2}\int_M (v,v)\,\mu$$



Rmk Other (G, E)

$\rightsquigarrow$ Euler top, Kirchhoff, MHD, KdV, CH

Many fluid models are from this framework – not compressible

II. Compressible Fluids

Euler equations for barotropic fluids

$$\begin{cases} \partial_t v + \nabla_v v = -\frac{1}{\rho}\nabla P(\rho) \\ \partial_t \rho + \operatorname{div}(\rho v) = 0 \end{cases}$$

ρ – density

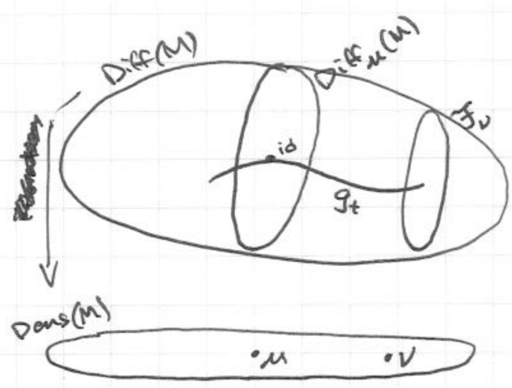
$P(\rho) = e'(\rho)\,\rho^2$ e – internal energy

Thm [Smolentsev 1979]

Euler's equations for barotropic fluids $\rightleftharpoons$ the Newton equations $\nabla_{\dot q}\dot q = -\nabla U(q)$ on $\mathrm{Diff}(M)$ with $L = \underset{= E}{K} - U$ for $U(\rho) = \int_M e(\rho)\,\rho\,\mu$

$\hookrightarrow$ doesn't have to be volume preserving

III. Geometry of Diff(M) & Optimal Transport



think of volume-preserving diffeomorphisms as a subgroup of all diffeomorphisms

take other densities $\nu > 0$, $\int_M \nu = 1$

$\mathrm{Dens}(M) = \{\text{densities on } M\}$
$\quad\llcorner$ probability measures

G2

$\mathcal{F}_\nu = \{ \varphi \in \mathrm{Diff}(M) \mid \varphi_* \mu = \nu \}$ fibration

$\exists$ L^2-metric on Diff(M) for a flat M.

$$\ell^2(\{g_t\}) = \int_0^T \| \partial_t g \|^2_{L^2(M)} \, dt$$

Thm [Otto 2000]

Map from diffeomorphisms to densities carries a metric with it.
(Diff, L^2) $\to$ (Dens, Wass) is a Riemannian submersion.

This is an ∞-dim version of Richard Montgomery's talk yesterday.

Here, Wass metric is related to the distance function:

$$\mathrm{dist}^2(\mu, \nu) = \inf_{\substack{\varphi \in \mathrm{Diff} \\ \varphi_*\mu = \nu}} \int_M \mathrm{dist}^2(x, \varphi(x))\, \mu$$

This gives us a way to describe geodesics downstairs.

Rmk Geodesics in Diff(M) $\leftrightharpoons$ solutions of the Burger's equation $\partial_t v + \nabla_v v = 0$

tangent space of $\mathrm{Diff}_\mu(M)$ are divergence free vector fields
perpendicular to these are gradient fields - orthogonal to fibers

Incompressible Euler $\leftrightharpoons$ restriction of L^2 to $\mathrm{Diff}_\mu(M)$
the $-\nabla P$ force is the force that keeps the solution on the submanifold

IV Madelung Transformation

Thm [Madelung 1927 / von Renesse 2012]

$\exists$ hydrodynamic form of quantum mechanics: $i\partial_t \psi = -\Delta\psi + f(|\psi|^2)\psi + V(x)\psi$

$\psi = \sqrt{\rho}\, e^{i\theta}$ Call this transformation φ. $\psi : M \to \mathbb{C} \setminus \{0\}$

$$\rightsquigarrow \begin{cases} \partial_t v + \nabla_v v + 2\nabla\left(V + f(\rho) - \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \\ \partial_t \rho + \mathrm{div}(\rho v) = 0 \text{ where } v = \nabla\theta \end{cases}$$

What is the geometry behind this transformation?

This is compressionable fluid mechanics with a "quantum pressure" - $P = P(\rho, \nabla\rho, \nabla^2\rho)$
not just $P = P(\rho)$

Q. Geometry behind Madelung

(ρ, θ) $\rho > 0$, $\int \rho = 1$ consider $(\rho, [\theta])$

$[\theta] = \{ \theta + c \mid c \in \mathbb{R} \}$ coset of θ

Claim: $(\rho, [\theta]) \in T^*\mathrm{Dens}$ in cotangent bundle of density space

Also consider $[\Psi] = \{e^{i\alpha}\Psi \mid \alpha \in \mathbb{R}\}$ $\qquad [\Psi] \in \mathbb{P}C^\infty(M, \mathbb{C}\setminus\{0\})$

We also want $\int_M |\Psi|^2 = 1$

Thm 1 $\quad \Phi : (T^*\text{Dens}, \omega_{can}) \to (\mathbb{P}C^\infty, \omega_{FS})$ is a symplectomorphism

Cor Φ maps Hamiltonians to Hamiltonians

If we also include the appropriate metrics, this is an isometry.
metric on Diff(M) from H^1 norm

V $\quad$ H^1 Metrics on Diff(M)
related to (CS, HS)

$\forall$ compact manifold M
introduce $\dot{H}^1$ metric $\qquad (v,v)_{\dot{H}^1} = \frac{1}{4}\|\operatorname{div} v\|_{L^2}^2$
degenerate along fibers

Thm [K, Lenells, Misiolek, Preston]

Diff $\to$ Dens
$\dot{H}^1 \to$ FR – metric in information geometry $\qquad$ FR = Fisher–Rao
~~[illegible]~~ spherical metric

Thm 2 Φ is an isometry between the FR metric on T^*Dens
(really the lift of FR to cotangent bundle) $\quad$ and FS metric on $\mathbb{P}C^\infty$

Cor Φ is Kahler