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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Yasuhide Fukumoto

Talk Title: Gyroscopic Analogue of a Rotating Stratified Flow Confined in a Tilted Spheroid with a Spherical Top with the Top Axis Misaligned from the Axis of Symmetry

Date: 11 / 29 / 2018 **Time:** 3 : 30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: The Rayleigh-Taylor instability can be suppressed if the fluid is rotating rapidly enough. Using three exact steady solutions as a basis for the flow converts the fluid equations to the equations of motion for a heavy top in gravity. Analyzes various cases for the top.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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Program: Hamiltonian systems, from topology to applications through analysis
Workshop: Hamiltonian systems, from topology to applications through analysis II
November 26-30, 2018 (Nov. 29)
Mathematical Sciences Research Institute (MSRI)
Berkeley, CA

Gyroscopic analogy of a rotating stratified flow confined in a tilted spheroid with a heavy symmetrical top with the top axis misaligned from the axis of symmetry

Yasuhide Fukumoto

Institute of Mathematics for Industry,

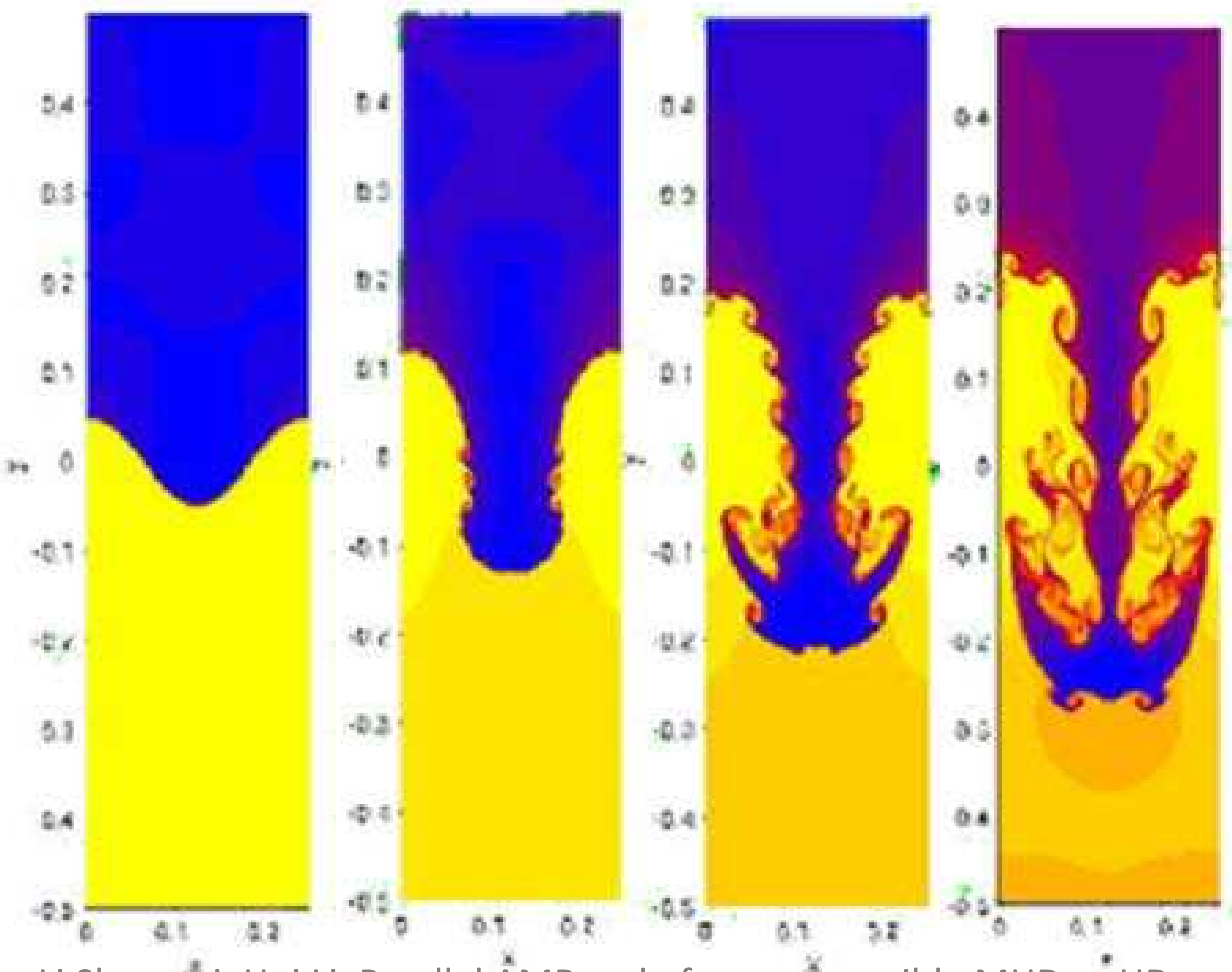
Kyushu University, Fukuoka, Japan

with

Yuki Miyachi

Toshiba Co

Can lighter fluid sustain heavier one on it?



Li Shengtai, Hui Li, Parallel AMR code for compressible MHD or HD equations. LosA. Nati. Lab.

Blue : water

Yellow : oil

Density : water > oil

Exerted by gravity

①Intermediate surface : unstable

②change position by disturbance

⇒Rayleigh-Taylor instability

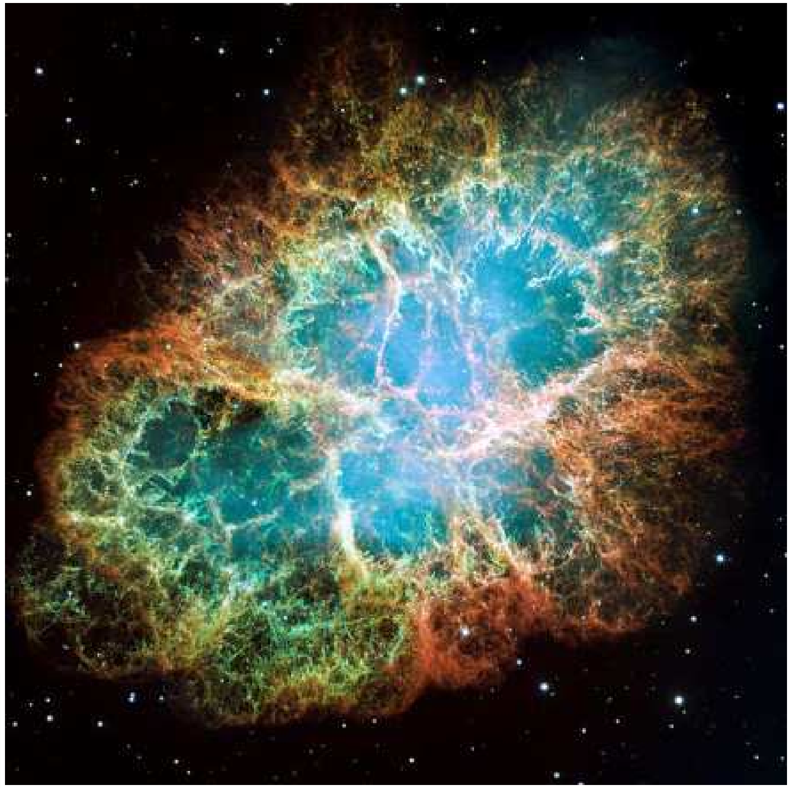
↑

Rotation?

Rayleigh-Taylor Instability (RTI)

←RT-cirrus clouds

(By D. Jewitt, University of California at Los Angeles[1])



Club Nebula→

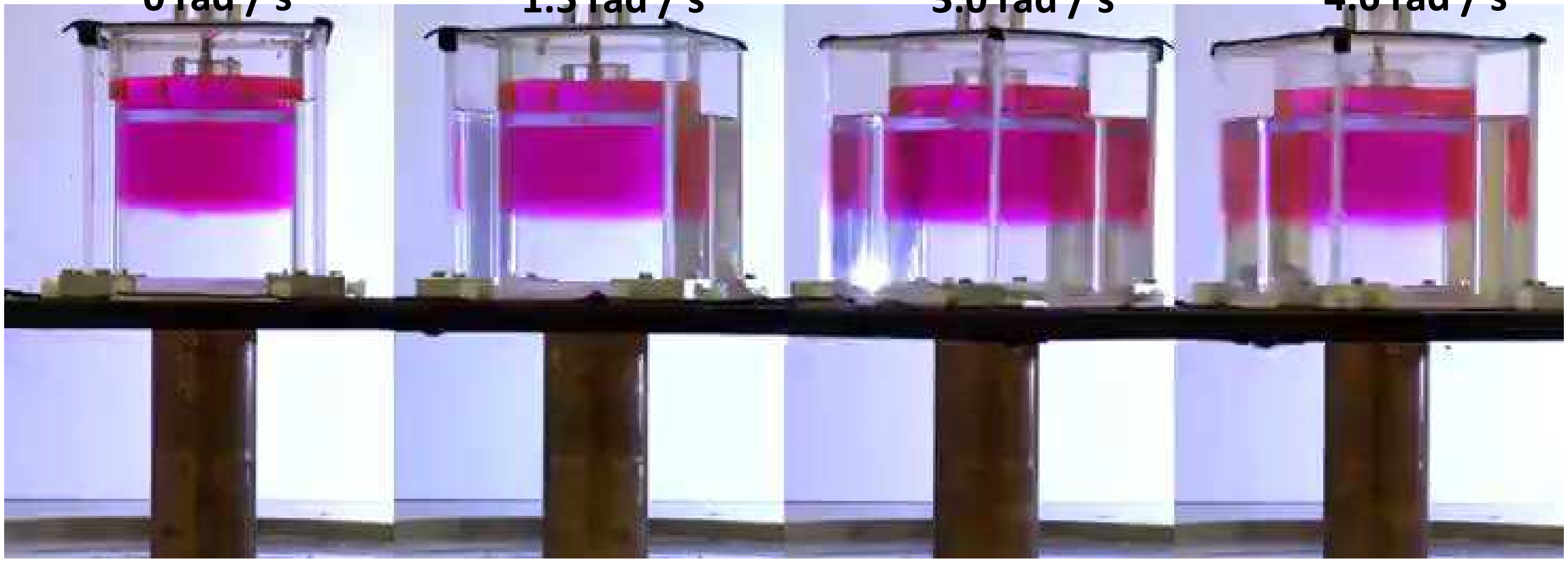
(Hubble Space Telescope , In October 1999, January 2000 and December 2000) (supernova explosions in which expanding core gas is accelerated into denser shell gas)

[1] M. J. Andrews & S. B. Dalziel, Small Atwood number Rayleigh-Taylor experiments. Phil. Trans. R. Soc. A (2010) 368, 1663-1679

Experiment : Suppression of RTI by rotation

← slow rotation

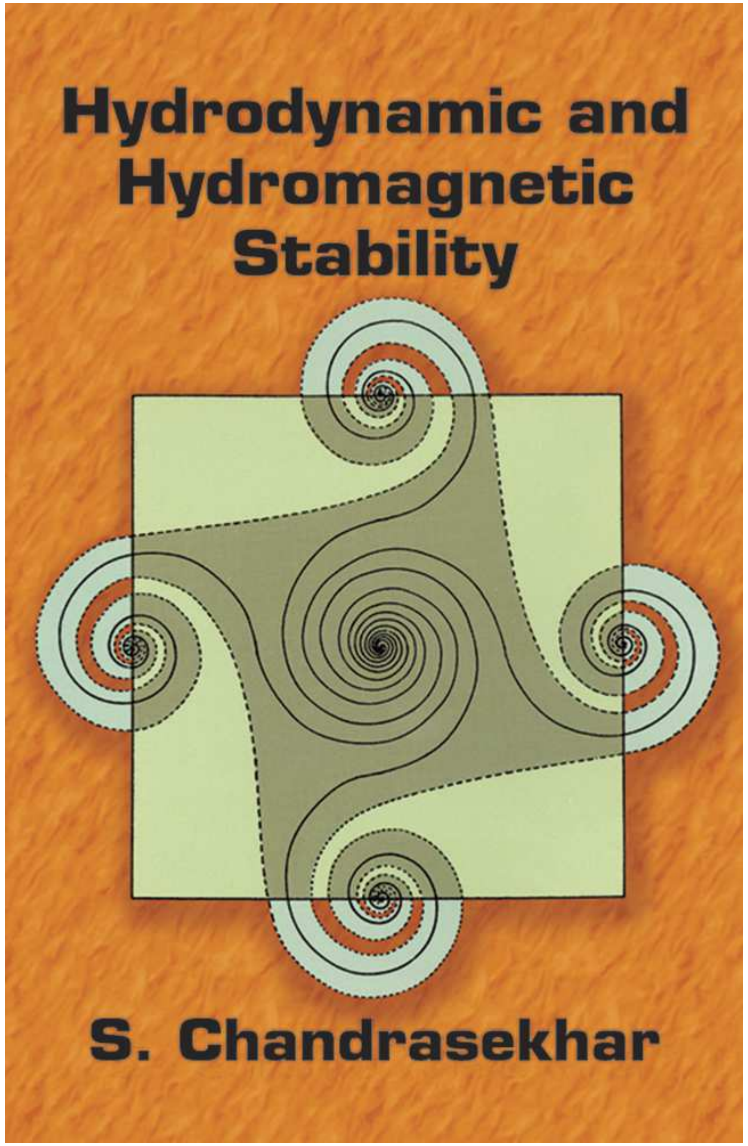
Rapid rotation →



- Purple : $MnCl_2$(aq), Transparent : NaCl (aq)
- **Effective density is controlled oppositely by magnetic field.**
- K. A. Baldwin, M. M. Scase and R. J. A. Hill, The inhabitation of the Rayleigh-Taylor instability by rotating. NATURE SCIENTIFIC REPORTS, 5:11706, DOI:10.1038/srep11706 (2015).

Chandrasekhar(1961) :

"... Rotation does not affect the instability or stability,
as such as a two layer stratification"



Inertia internal-gravity waves

- **Continuously stratified** rotating fluid
- The dispersion relation

$$\omega^2 = \frac{m^2}{k^2 + l^2 + m^2} f^2 + \frac{k^2 + l^2}{k^2 + l^2 + m^2} N^2$$

$$N^2 = -\frac{g}{\rho_0} \frac{d\rho}{dz}$$

$$\alpha^2 = \frac{k^2 + l^2}{m^2}$$

$w \propto \exp\ [i(kx + ly + mz - \omega t)]$

$f = 2\Omega_0$

N^2: Brunt Váisálá frequency

k, l, m: wave number

$a = \frac{|N|}{|f|}$: Rossby radius

- Short wave ($\alpha \gg 1$) : $\omega \approx \alpha N$ ←(unstable) internal gravity wave
- Long wave ($\alpha \ll 1$) : $\omega \approx f$ ←**(stable)** inertial wave

For an unstable stratified fluid

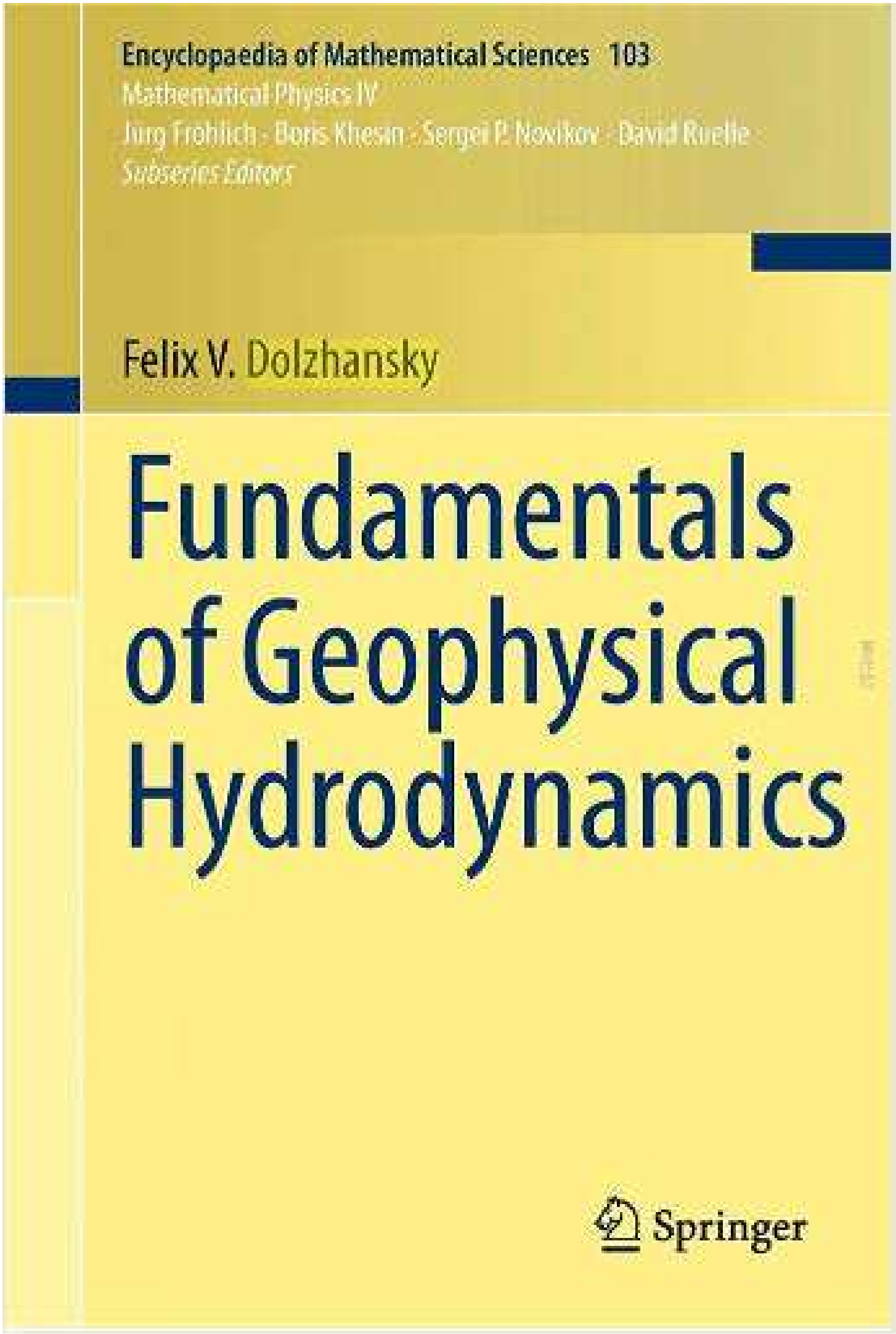
- Long waves : stable by rotation
- Short waves : unstable by gravity

Gyroscopic Analogy of Coriolis Effect on Rotating Flows Confined in a Spheroid

Why and how can an unstably stratified fluid be stabilized by ***rotation***?

*What is the **baroclinic effect**?*

Felix V. Dolzhansky
translated by **Boris A. Khesin**



Contents

Part 1 Motion of a rotating stratified flow confined in a *tilted* spheroid

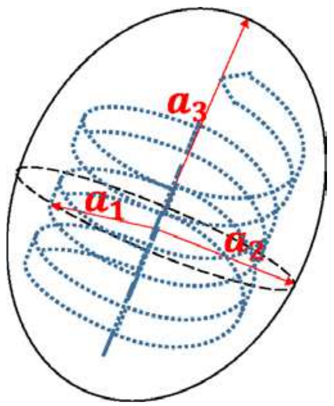
Part 2 Stability of motion of a heavy symmetric rigid body with *the top axis misaligned from the symmetric axis*

Part 3 Spatial description and energetics of motion of a heavy rigid body

Appendix Rayleigh-Taylor instability of rotating flow confined in an upright half spheroid

Part 1

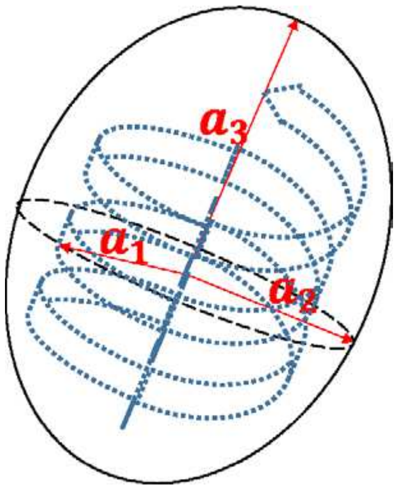
Motion of a rotating stratified flow confined in a *tilted* spheroid



Density stratification and hydrostatic balance

Spheroid:

$$S := \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} - 1 = 0$$



For Boussinesq approximation [Poincaré(1910), Dolzhansky(1977)], to set density ρ:

$$\rho = \rho_0 + \rho'(x, y, z, t),$$

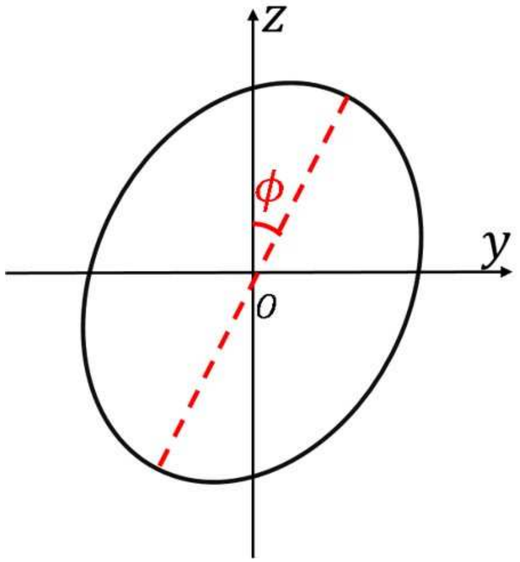
where ρ_0 is constant and assume $|\rho'| \ll \rho_0$.

A reference pressure:

$$p = p_0(z) + p'(x, y, z, t),$$

Where $|p'| \ll p_0$ and

$$\frac{\partial p_0}{\partial z} := -g\rho_0$$



Boussinesq approximation for equations of stratified fluid

Equation for buoyancy:

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho = 0$$

where the Brunt-Väisälä frequency: $\rho = \rho_0 + \rho'(x, y, z, t)$

$$N^2(z) = -\frac{g}{\rho_0}\frac{\partial \rho'}{\partial z}$$

In terms of $\boldsymbol{q} = \nabla \rho' / \rho_0$:

$$\frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\,\boldsymbol{q} = -(\boldsymbol{q} \cdot \nabla)\,\boldsymbol{v}$$

The Momentum equation:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0} g\boldsymbol{k}, \qquad \nabla \cdot \boldsymbol{v} = 0.$$

In terms of vorticity $\boldsymbol{\Omega}$ =curl$\boldsymbol{v}$:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - (\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\Omega} = -\boldsymbol{g} \times \boldsymbol{q}.$$

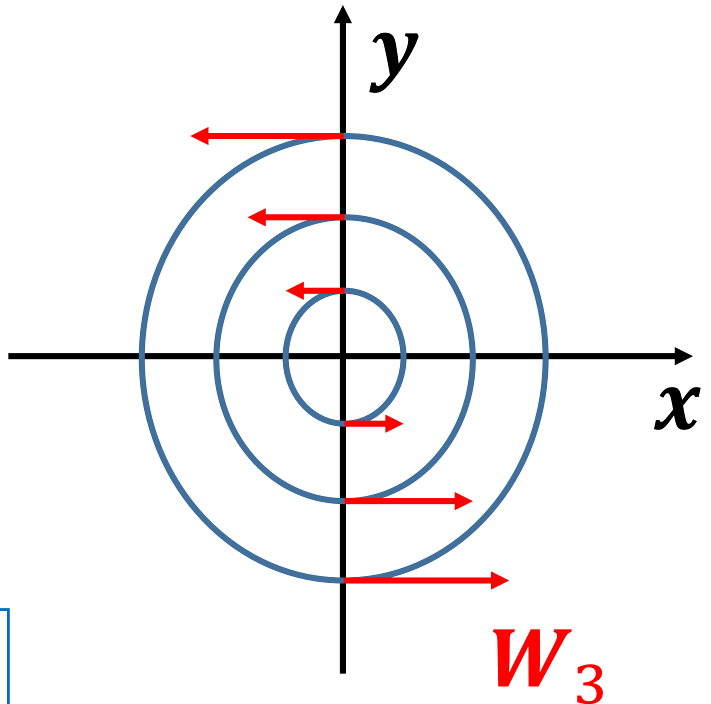
Baroclinic torque

Exact solutions of Navier-Stokes equation

Navier-Stokes equation: $\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0} g\boldsymbol{e}_z + \nu\nabla^2\boldsymbol{v}.$

The exact steady solutions [Dolzhansky'77]:

$$\boldsymbol{W}_1 = -\frac{a_2}{a_3}x_3\boldsymbol{j} + \frac{a_3}{a_2}x_2\boldsymbol{k},$$
$$\boldsymbol{W}_2 = -\frac{a_1}{a_3}x_1\boldsymbol{k} + \frac{a_1}{a_3}x_3\boldsymbol{i},$$
$$\boldsymbol{W}_3 = -\frac{a_1}{a_2}x_2\boldsymbol{i} + \frac{a_2}{a_1}x_1\boldsymbol{j},$$



Look for a general non-stationary solution:

$$\boldsymbol{v}(x,t) = \sum_{k=1}^{3}\omega_k(t)\boldsymbol{W}_k(x),$$

$$\omega_k = \frac{a_1 a_2 a_3}{a_k I_k}\Omega_k, \ (k = 1,2,3),$$

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = \begin{pmatrix} a_2^2 + a_3^2 & 0 & 0 \\ 0 & a_3^2 + a_1^2 & 0 \\ 0 & 0 & a_1^2 + a_2^2 \end{pmatrix}$$

Equations of motion for a general heavy top

Euler-Poisson equation for extended heavy top:

$$\begin{cases} \dot{\boldsymbol{m}} = \boldsymbol{\omega} \times \boldsymbol{m} + g\boldsymbol{\sigma} \times \boldsymbol{l_0} \\ \dot{\boldsymbol{\sigma}} = \boldsymbol{\omega} \times \boldsymbol{\sigma}, \quad \boldsymbol{m} = I\boldsymbol{\omega}, \end{cases}$$

Gravity torque

The vector σ are density differences

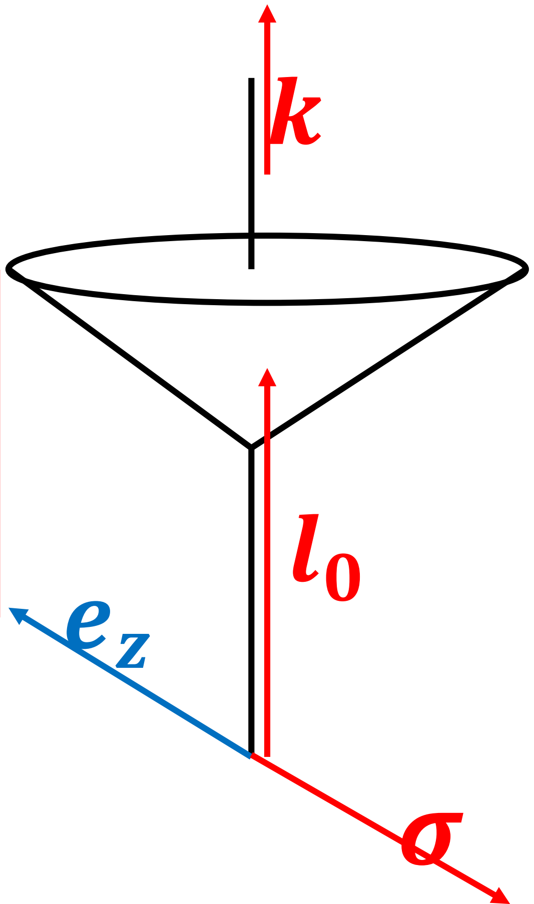
relative to the major semi-axes of ellipsoid:

$$\boldsymbol{\sigma} = \frac{1}{\rho_0}\left(a_1 \frac{\partial \rho\prime}{\partial x_1}\bigg|_0 \boldsymbol{i} + a_2 \frac{\partial \rho\prime}{\partial x_2}\bigg|_0 \boldsymbol{j} + a_3 \frac{\partial \rho\prime}{\partial x_3}\bigg|_0 \boldsymbol{k}\right).$$ constant in space

The vector $\boldsymbol{l_0}$ defined by ellipsoid's orientation in space:

$$\boldsymbol{l_0} = a_1 \cos \alpha_1 \boldsymbol{i} + a_2 \cos \alpha_2 \boldsymbol{j} + a_3 \cos \alpha_3 \boldsymbol{k}\,.$$

α_i $(i = 1,2,3)$: angle of gravity vector with principal ellipsoid axis

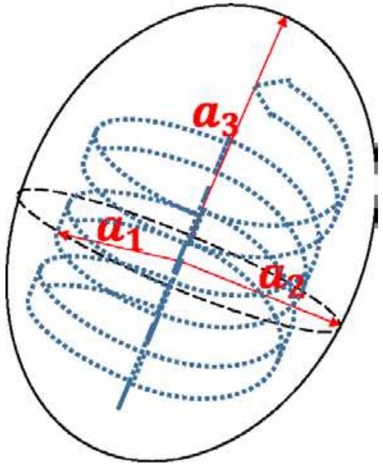


$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - (\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\Omega} = -\boldsymbol{g} \times \boldsymbol{q}.$$

Baroclinic torque

Motion of a heavy rigid body

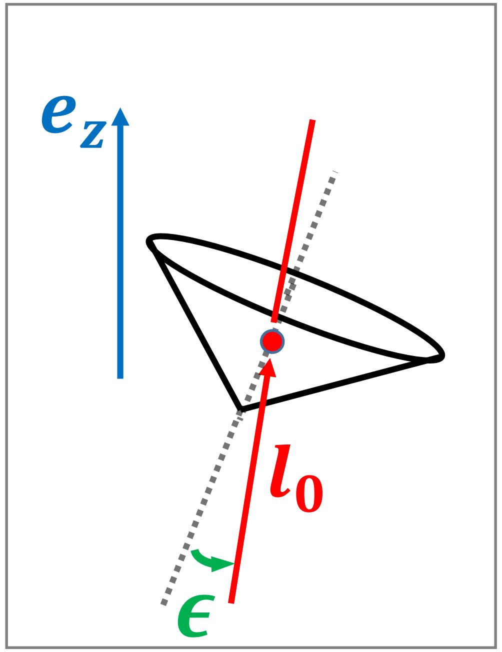
$$\begin{cases} \dot{\mathbf{m}} = \omega \times \mathbf{m} + g\sigma \times \mathbf{l}_0, \quad \mathbf{m} = I\omega \\ \dot{\sigma} = \omega \times \sigma \end{cases}$$



Change of sign

$$\omega \to -\omega, \quad \mathbf{m} \to -\mathbf{m}, \quad \sigma \to -\sigma$$

$$\begin{cases} \dot{\mathbf{m}} = \mathbf{m} \times \omega + Mgl\gamma \times \mathbf{l}_0, \quad \mathbf{m} = I\omega \\ \dot{\gamma} = \gamma \times \omega, \quad \gamma := \sigma/|\sigma| \end{cases}$$



First Integrals

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0} g\boldsymbol{k}, \quad \frac{\partial b'}{\partial t} + (\boldsymbol{v}\cdot\nabla)b' = 0.$$

Full energy and potential vorticity of the fluid:

$$E = \frac{1}{2}\rho_0 \int_D u^2 dxdydz - \int_D \rho' \boldsymbol{g}\cdot\boldsymbol{x} dxdydz,$$

$$\Pi = \boldsymbol{\Omega}\cdot\nabla\rho'$$

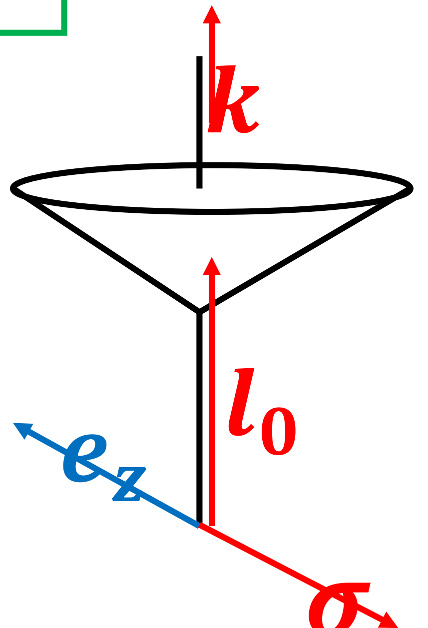
$$\dot{\boldsymbol{m}} = \boldsymbol{\omega}\times\boldsymbol{m} + g\boldsymbol{\sigma}\times\boldsymbol{l}_0, \qquad \dot{\boldsymbol{\sigma}} = \boldsymbol{\omega}\times\boldsymbol{\sigma}, \qquad \boldsymbol{m} = I\boldsymbol{\omega}$$

have **three** first integrals of motion

$$E_m = \frac{1}{2}\boldsymbol{m}\cdot\boldsymbol{\omega} - g\boldsymbol{l_0}\cdot\boldsymbol{\sigma}, \qquad |\boldsymbol{\sigma}|^2,$$

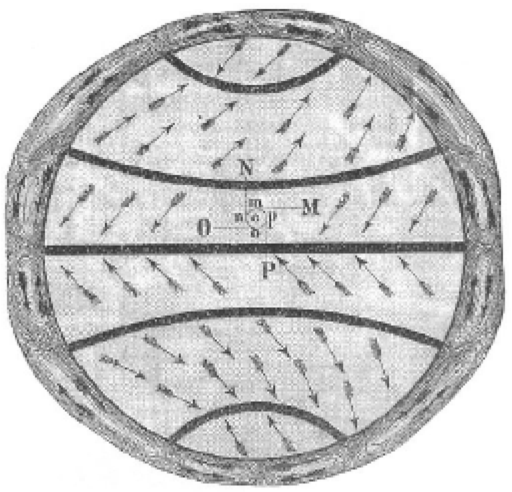
$$\Pi_z = \mathbf{m}\cdot\boldsymbol{\sigma}$$

Angular momentum about the z-axis

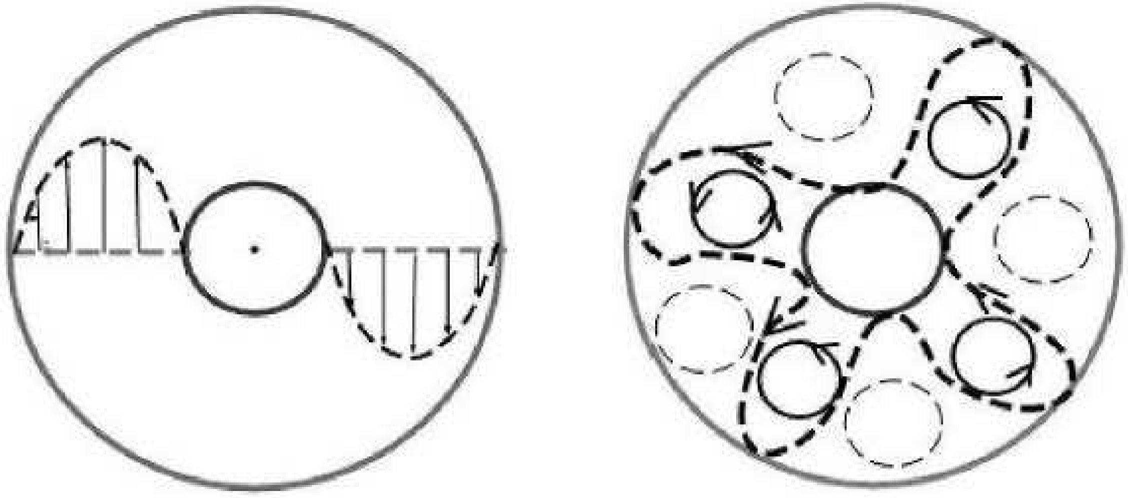


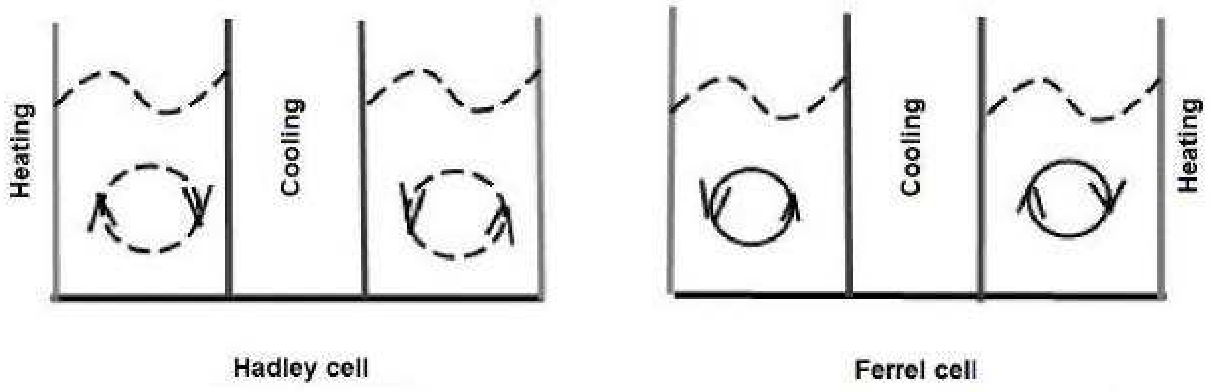
Application: Toy circulation of Hadley and Rossby

Dolzhansky *Fundamentals of Geophysical Hydrodynamics* (2006)



General atomospheric circulation according to Ferrel (1859)





system rotation, friction (viscosity)

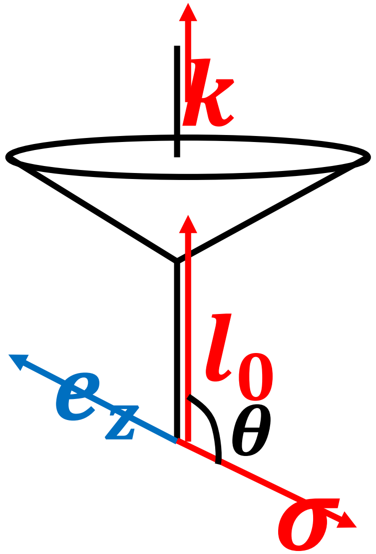
$$\dot{\mathrm{m}} = \omega \times (\mathrm{m} + 2\mathrm{m}_0) + g\mathrm{l}_0 \times \sigma - \lambda \mathrm{m}$$

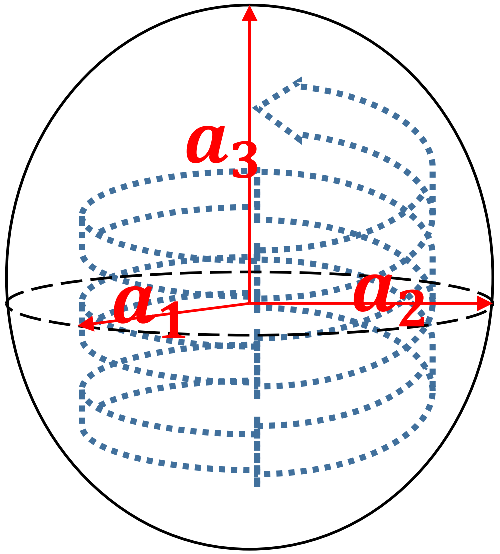
$$\dot{\sigma} = \omega \times \sigma + \mu(\sigma_B - \sigma)$$

thermal conductivity

Flow field corresponding to stationary top : upright spheroid

$$\boldsymbol{\sigma} = \frac{1}{\rho_0}\left(a_1 \frac{\partial \rho'}{\partial x}\bigg|_0 \boldsymbol{i} + a_2 \frac{\partial \rho'}{\partial y}\bigg|_0 \boldsymbol{j} + a_3 \frac{\partial \rho'}{\partial z}\bigg|_0 \boldsymbol{k} \right), \qquad \boldsymbol{l_0} = (0,0,-a_3)$$





$\sigma_3 = -\cos\theta$

$\sigma \parallel l_0$ ⇔sleeping top or hanging down top

○Sleeping top:

$\sigma_3 = +1(\theta = \pi) \Leftrightarrow \frac{\partial \rho'}{\partial z} > 0 \Leftrightarrow$ unstable stratification ($N^2(z) = -\frac{g}{\rho_0}\frac{\partial \rho'}{\partial z} < 0$)

○Hanging down top:

$\sigma_3 = -1(\theta = 0) \Leftrightarrow \frac{\partial \rho'}{\partial z} < 0 \Leftrightarrow$ stable stratification ($N^2(z) = -\frac{g}{\rho_0}\frac{\partial \rho'}{\partial z} > 0$)

Linear stability of sleeping top

The steady state corresponding to the sleeping top ($a_1 = a_2$):

$$\omega_1 = \omega_2 = 0, \qquad \sigma_1 = \sigma_2 = 0, \qquad \omega_3 = \omega_{30}, \qquad \sigma_3 = \sigma_{30}$$

$$\omega_k = \frac{a_1 a_2 a_3}{a_k I_k} \Omega_k, \quad \sigma_{30} \propto \frac{\partial \rho'}{\partial z}$$

This steady state is perturbed by $(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)$.

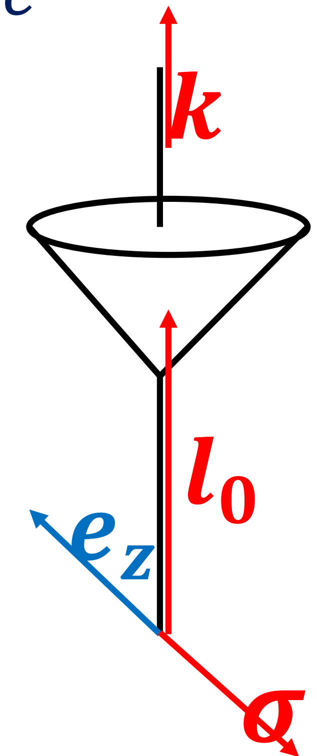
The forth-order system admits solutions with $(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3) \propto e^{-i\omega t}$

For stability, we get:

$$\omega_{30}{}^2 > \frac{4\sigma_{30} g a_3 I_1}{I_3^2}$$

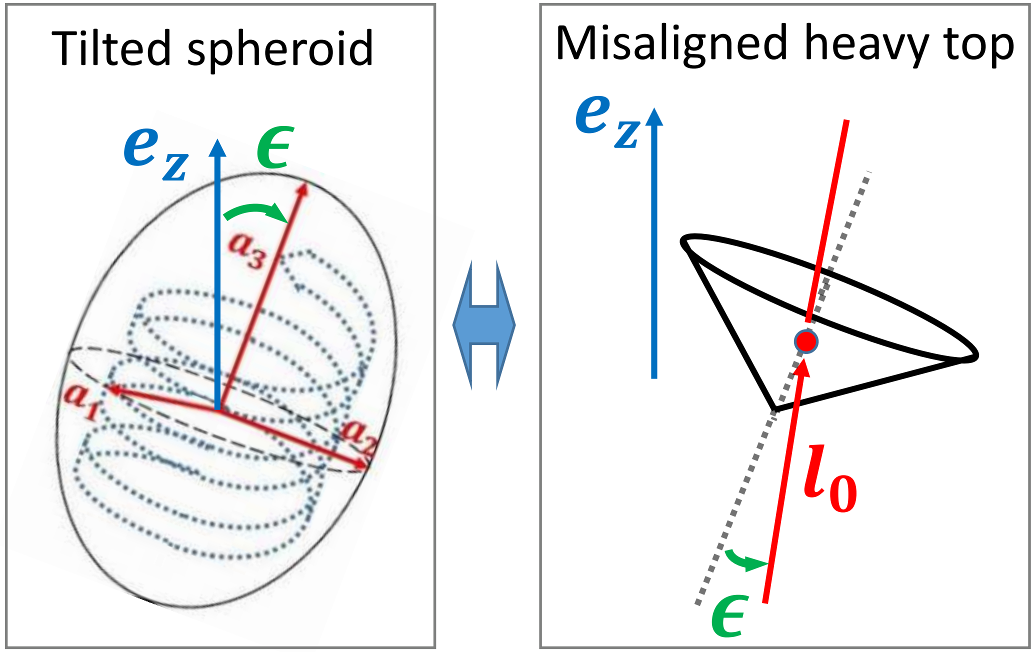
Thus, for $\sigma_{30} > 0$, we see

lighter fluid can keep lifting up heavier one on it when ω_{30} is large enough to satisfy the above inequality.

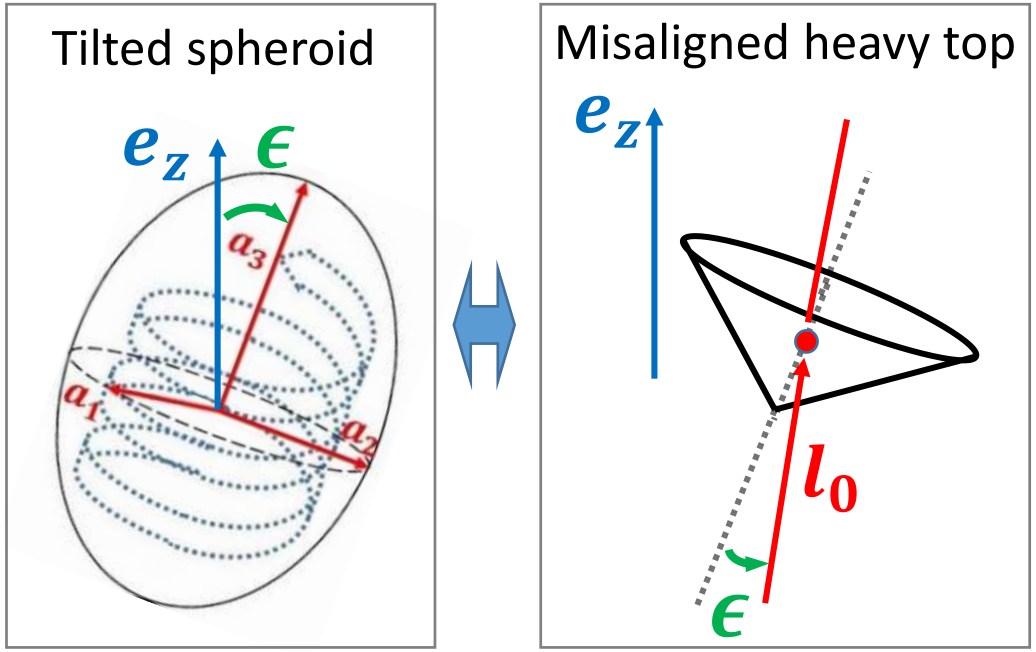


Part 2

Stability of motion of a heavy symmetric rigid body with *the top axis misaligned from the symmetric axis*



Tilted spheroid & Misaligned symmetrical top



Euler-Poisson equation

$$\dot{\mathbf{m}}_b = \mathbf{m}_b \times \boldsymbol{\omega}_b + Mgl\boldsymbol{\sigma}_b \times \mathbf{l}_0$$

$$\dot{\boldsymbol{\sigma}}_b = \boldsymbol{\sigma}_b \times \boldsymbol{\omega}_b$$

with $\mathbf{l}_0 = (0, \sin\varepsilon, \cos\varepsilon)$

Heavy symmetrical top with rotating axis tilted from symmetric axis

- By solving $\dot{\boldsymbol{m}}_b = 0$ & $\dot{\boldsymbol{\sigma}}_b = 0$, steady solutions are derived

$$\omega_1 = 0, \quad \omega_2 = \frac{\sigma_{30} Mgl\omega_{30} \sin\varepsilon}{(I_1 - I_3)\omega_{30}^2 + Mgl\sigma_{30}\cos\varepsilon}, \quad \omega_3 = \omega_{30},$$

$$\sigma_1 = 0, \quad \sigma_2 = \frac{\sigma_{30}^2 Mgl \sin\varepsilon}{(I_1 - I_3)\omega_{30}^2 + Mgl\sigma_{30}\cos\varepsilon}, \quad \sigma_3 = \sigma_{30},$$

- $\sigma_{30} > 0 \Rightarrow$ upright top; $\sigma_{30} < 0 \Rightarrow$ hanging down top

Reconstruction problem in terms of Euler angles

Reconstruction problem: $(\omega_1, \omega_2, \omega_3, \sigma_1, \sigma_2, \sigma_3) \to (\theta, \phi, \psi)$

$$C_2 = \frac{Mgl\sigma_{30}}{(I_1 - I_3)\omega_{30}^2 + Mgl\sigma_{30}}$$

For small $\epsilon \ll 1$

$\omega_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi = 0,$ → ② → $\dot\theta = 0$

$\omega_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi = \omega_{30} C_2 \varepsilon,$ → ③ → $\dot\phi = \omega_{30}$

$\omega_3 = \dot\phi \cos\theta + \dot\psi = \omega_{30},$ → ③ → $\dot\phi = \omega_{30}$

$\sigma_1 = \sin\theta \sin\psi = 0,$ → ① → $\psi = 0$

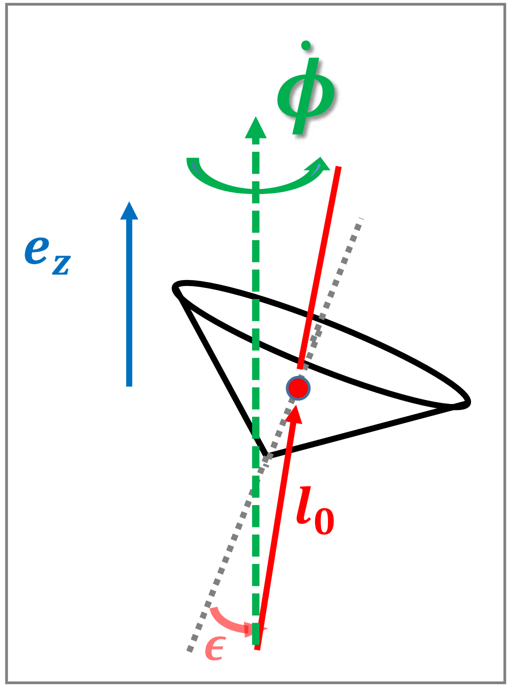
$\sigma_2 = \sin\theta \cos\psi = C_2 \varepsilon,$ → ② → $\theta = C_2 \epsilon$

$\sigma_3 = \cos\theta = \sigma_{30},$ → ③ → $\theta = O(\epsilon)$

In summary,

$$\theta = \frac{Mgl}{(I_1 - I_3)\omega_{30}^2 + Mgl}\varepsilon, \quad \phi = \omega_{30} t + \phi_0, \quad \psi = 0$$

→ The misaligned top behaves as precession with velocity $\dot\phi$, though it has not ψ



Linear stability analysis & characteristic equation

$$\mathbf{m}_b = I\boldsymbol{\omega}_b$$

- Linear stability analysis

$$(\boldsymbol{\omega}_b, \boldsymbol{\sigma}_b) = (\boldsymbol{\omega}_0, \boldsymbol{\sigma}_0) + (\tilde{\boldsymbol{\omega}}_b, \tilde{\boldsymbol{\sigma}}_b) \propto \exp(i\omega t)$$

$$\dot{\mathbf{m}}_b = \mathbf{m}_b \times \boldsymbol{\omega}_b + Mgl\boldsymbol{\sigma}_b \times \mathbf{l}_0$$

$$\dot{\boldsymbol{\sigma}}_b = \boldsymbol{\sigma}_b \times \boldsymbol{\omega}_b$$

with $\mathbf{l}_0 = (0, \sin\varepsilon, \cos\varepsilon)$

Characteristic equation: $D = \omega^2 \times (\text{quartic eq. in } \omega) = 0$

$$\omega_2 = \frac{\sigma_{30} Mgl\omega_{30}\sin\varepsilon}{(I_1 - I_3)\omega_{30}^2 + Mgl\sigma_{30}\cos\varepsilon}$$

$$\sigma_2 = \frac{\sigma_{30}^2 Mgl\sin\varepsilon}{(I_1 - I_3)\omega_{30}^2 + Mgl\sigma_{30}\cos\varepsilon}$$

$$\omega_1 = 0,\ \omega_3 = \omega_{30},\ \sigma_1 = 0,\ \sigma_3 = \sigma_{30},$$

- Non-dimensional form (quartic part)

$$\hat{D} = \hat{\omega}^4 + \hat{P}_1\hat{\omega}^2 + \hat{P}_2 = 0$$

$$\hat{P}_1 = -\alpha^2 + (1+C)\sigma_{30}l_2\hat{g}\alpha - (2 - 2C + C^2 - 2\sigma_{30}l_3\hat{g}C)$$

$$\begin{aligned}\hat{P}_2 = &-\sigma_{30}\hat{g}l_2(1-C)\alpha^3 \\ &+\sigma_{30}\hat{g}C\left\{\sigma_{30}l_2^2\hat{g} + (1-C)\left[\frac{1-C}{C\sigma_{30}\hat{g}} + l_3\right]\right\}\alpha^2 \\ &+\sigma_{30}\hat{g}l_2\left\{\sigma_{30}l_3\hat{g}C(1+C) - 3(1-C)^2\right\}\alpha \\ &+(1 - C + \sigma_{30}l_3\hat{g}C)^2\end{aligned}$$

$$\hat{\omega} = \frac{\omega}{\omega_{30}} \quad \hat{g} = \frac{Mgl}{I_3\omega_{30}^2} \quad C = \frac{I_3}{I_1}$$

$$\alpha := \frac{\sigma_{20}}{\sigma_{30}} = \frac{\omega_{20}}{\omega_{30}} = \frac{\hat{g}C\sigma_{30}l_2}{1 - C + \hat{g}C\sigma_{30}l_3}$$

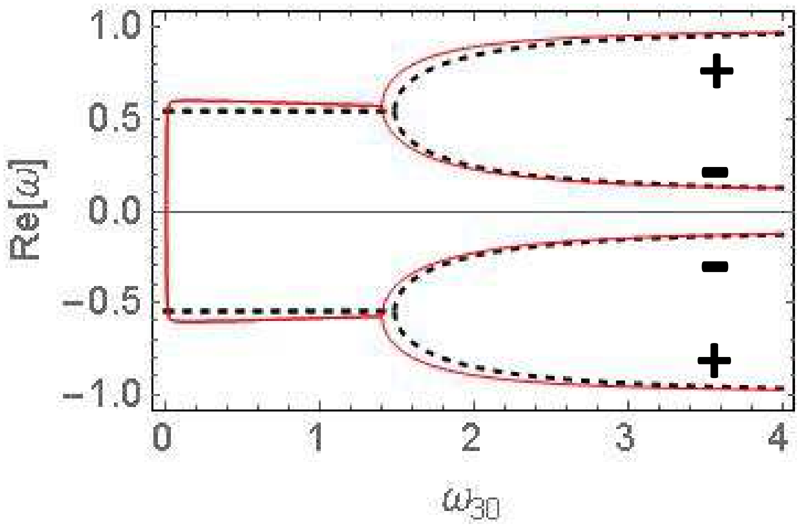
Eigenvalues=Spectra: prolate 'sleeping' top

- Prolate **sleeping** misaligned top ($a_1 = 1, a_3 = 1.1$)

$\boldsymbol{\sigma_{30} > 0}$

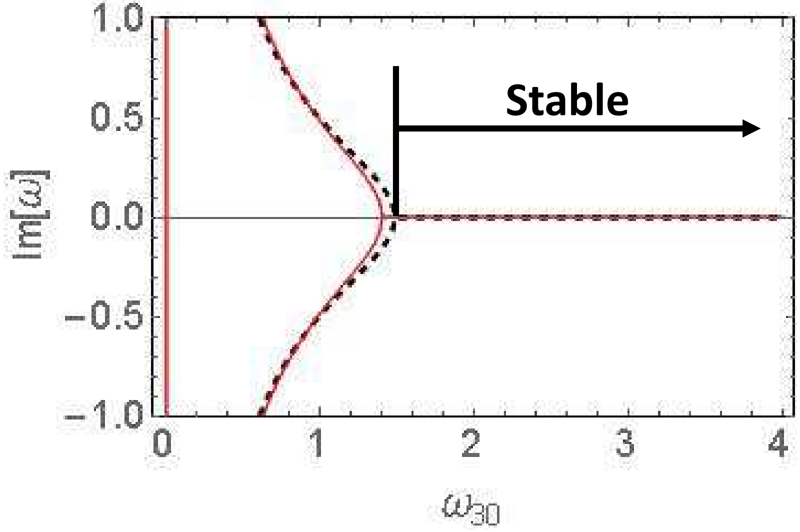
$$C = \frac{I_3}{I_1} < 1$$

$$(\tilde{\omega}_b, \tilde{\sigma}_b) \propto \exp(i\omega t)$$



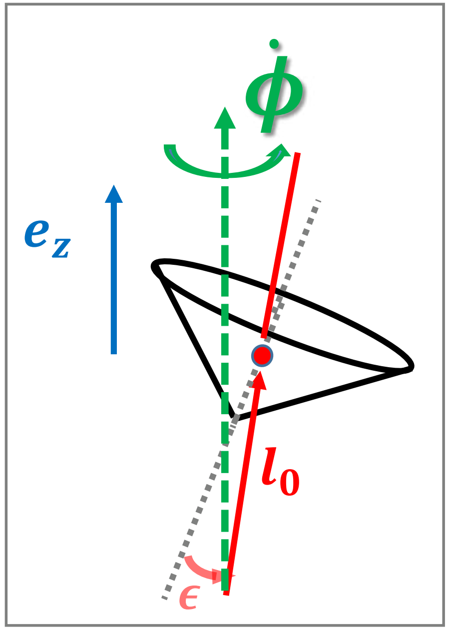
+: PEM

-: NEM



Black dashed line: $\epsilon = 0$

Red solid line: $\epsilon = 0.5$



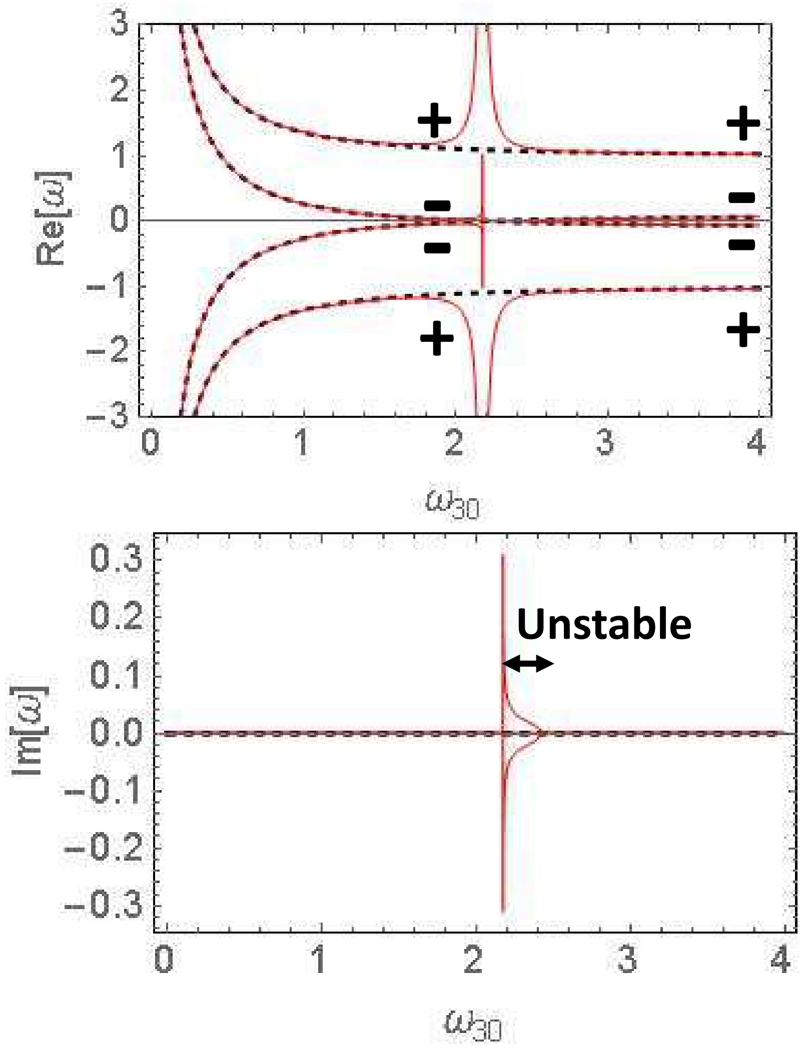
Spectra: prolate hanging-down top

$\sigma_{30} < 0$

- Prolate **hanging down** misaligned top ($a_1 = 1, a_3 = 1.1$)

$$C = \frac{I_3}{I_1} < 1$$

$$(\tilde{\omega}_b, \tilde{\sigma}_b) \propto \exp(i\omega t)$$

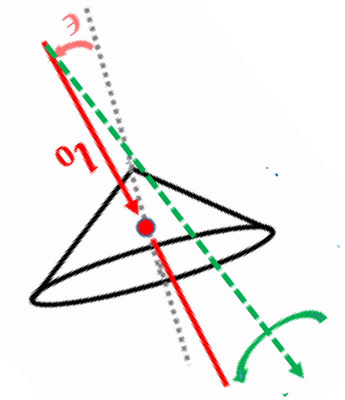


+: PEM

−: NEM

Black dashed line: $\epsilon = 0$

Red solid line: $\epsilon = 0.1$



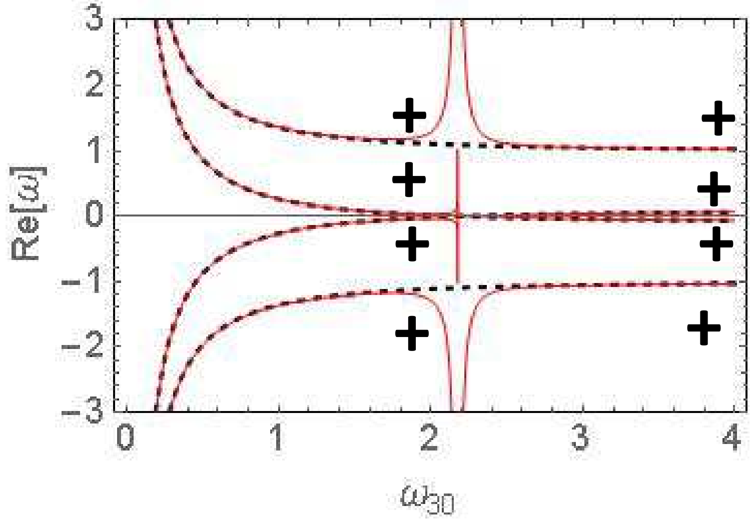
Spectra: prolate hanging-down top

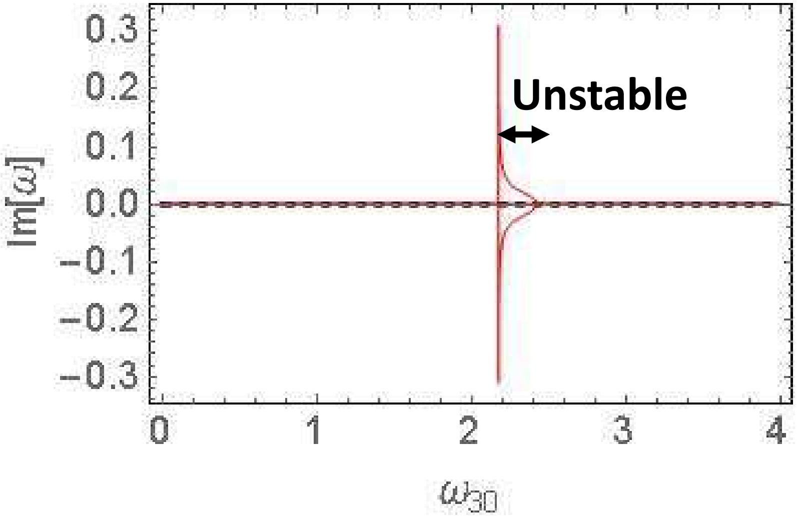
- Prolate **hanging down** misaligned top $(a_1 = 1, a_3 = 1.1)$

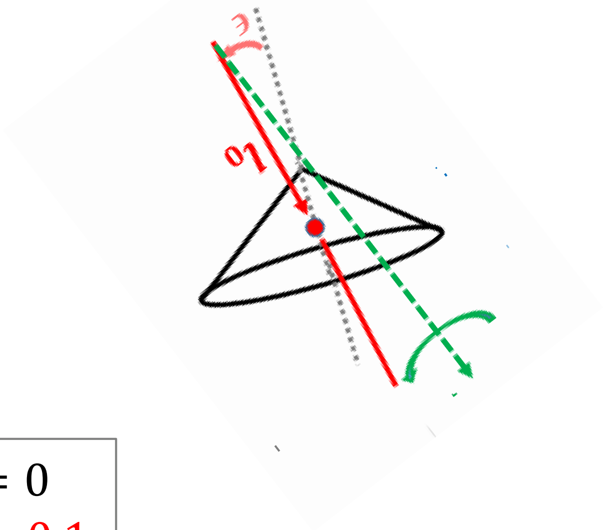
$\sigma_{30} < 0$

$$C = \frac{I_3}{I_1} < 1$$

$$(\tilde{\omega}_b, \tilde{\sigma}_b) \propto \exp(i\omega t)$$







+: PEM

−: NEM

Black dashed line: $\epsilon = 0$

Red solid line: $\epsilon = 0.1$

$$\omega_2 = \frac{\sigma_{30} Mgl\omega_{30} \sin\varepsilon}{(I_1 - I_3)\omega_{30}^2 + Mgl\sigma_{30}\cos\varepsilon}$$

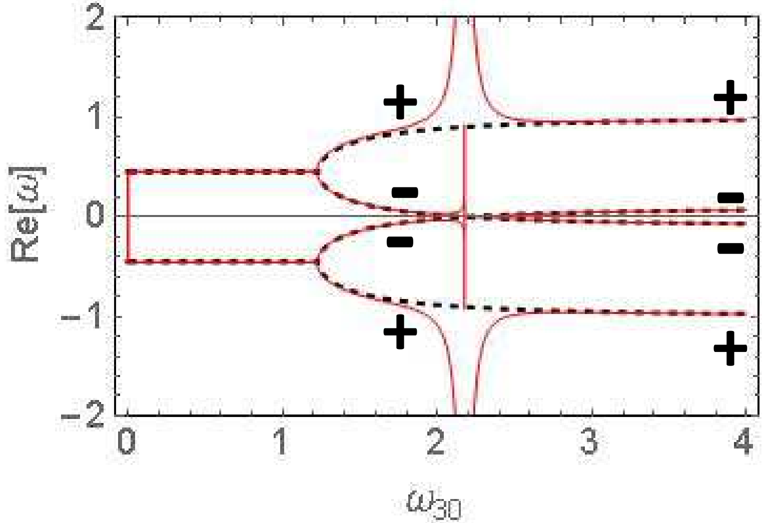
$$\sigma_2 = \frac{\sigma_{30}^2 Mgl \sin\varepsilon}{(I_1 - I_3)\omega_{30}^2 + Mgl\sigma_{30}\cos\varepsilon}$$

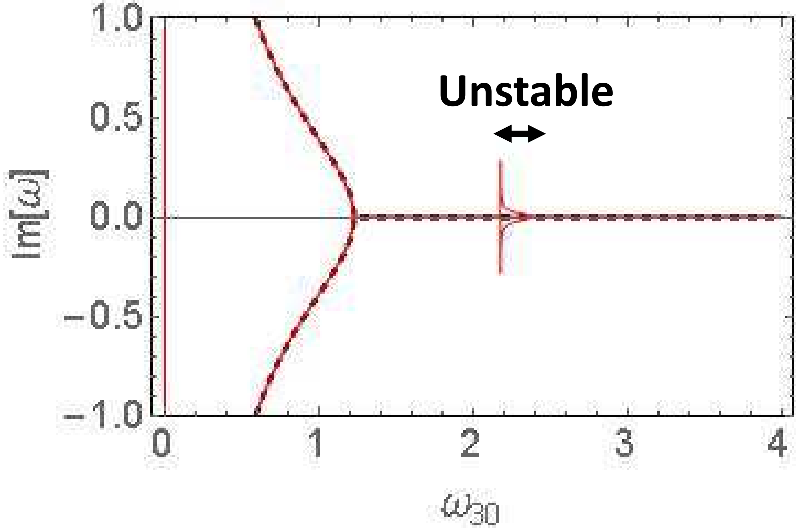
$$\omega_1 = 0,\ \omega_3 = \omega_{30},\ \sigma_1 = 0,\ \sigma_3 = \sigma_{30},$$

Spectra: oblate 'sleeping' top

- **Oblate sleeping misaligned top** ($a_1 = 1.1, a_3 = 1$) $\boldsymbol{\sigma_{30} > 0}$

$C = \frac{I_3}{I_1} > 1$





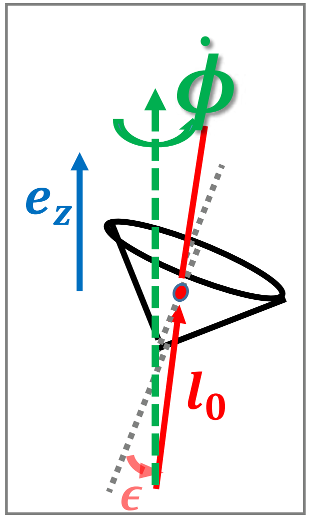
$(\tilde{\omega}_b, \tilde{\sigma}_b) \propto \exp(i\omega t)$

+: PEM

−: NEM

Black dashed line: $\epsilon = 0$

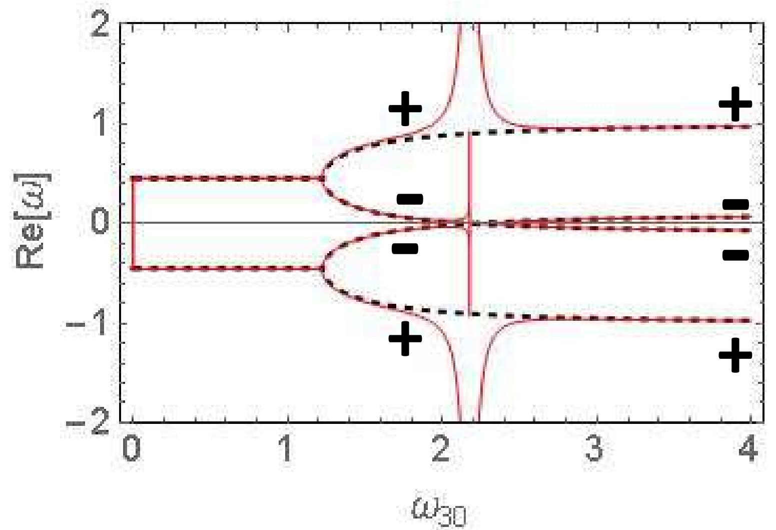
Red solid line: $\epsilon = 0.1$



Spectra: oblate 'sleeping' top

- Oblate sleeping misaligned top ($a_1 = 1.1, a_3 = 1$) $\sigma_{30} > 0$

$$C = \frac{I_3}{I_1} > 1$$



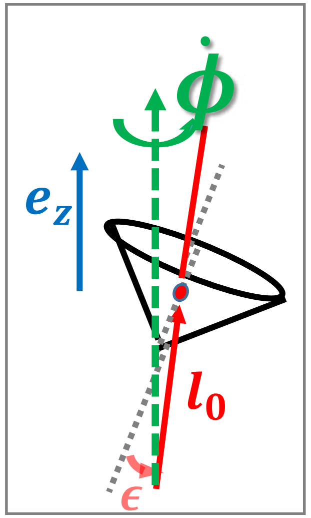
$$(\tilde{\omega}_b, \tilde{\sigma}_b) \propto \exp(i\omega t)$$

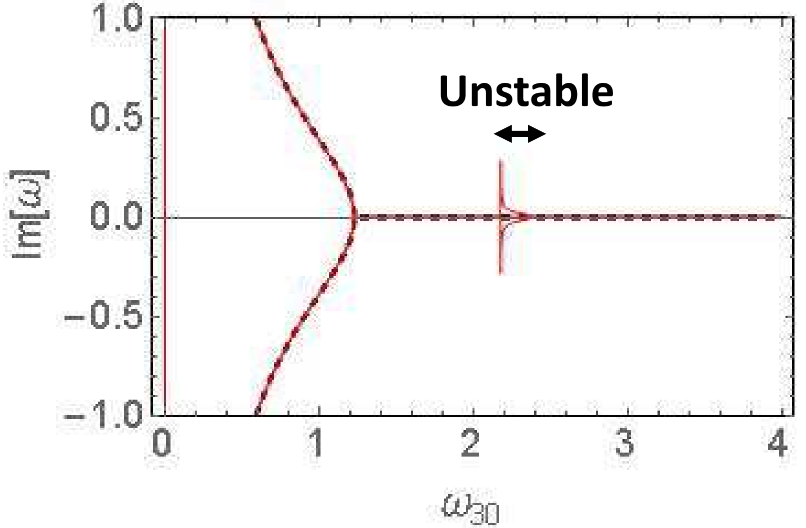
+: PEM

-: NEM

Black dashed line: $\epsilon = 0$

Red solid line: $\epsilon = 0.1$





$$\omega_2 = \frac{\sigma_{30} M g l \omega_{30} \sin\varepsilon}{(I_1 - I_3)\omega_{30}^2 + M g l \sigma_{30} \cos\varepsilon}$$

$$\sigma_2 = \frac{\sigma_{30}^2 M g l \sin\varepsilon}{(I_1 - I_3)\omega_{30}^2 + M g l \sigma_{30} \cos\varepsilon}$$

$$\omega_1 = 0,\ \omega_3 = \omega_{30},\ \sigma_1 = 0,\ \sigma_3 = \sigma_{30},$$

Instability of sleeping top due to misalignment (oblate top) (i)

- Oblate misaligned top ($a_1 = 1.1, a_3 = 1, C = I_3/I_1 > 1$)
- Instability arise at collision point ($\hat{g} = \hat{g}_c$)

$$\alpha := \frac{\hat{g}C\sigma_{30}l_2}{1-C+\hat{g}C\sigma_{30}l_3}$$

$$\hat{D} = \hat{\omega}^4 + \hat{P}_1\hat{\omega}^2 + \hat{P}_2 = 0$$

$$\hat{P}_1 = -\alpha^2 + (1+C)\sigma_{30}l_2\hat{g}\alpha - (2-2C+C^2-2\sigma_{30}l_3\hat{g}C)$$

$$\hat{P}_2 = -\sigma_{30}\hat{g}l_2(1-C)\alpha^3$$

$$+\sigma_{30}\hat{g}C\left\{\sigma_{30}l_2^2\hat{g} + (1-C)\left[\frac{1-C}{C\sigma_{30}\hat{g}} + l_3\right]\right\}\alpha^2$$

$$+\sigma_{30}\hat{g}l_2\left\{\sigma_{30}l_3\hat{g}C(1+C) - 3(1-C)^2\right\}\alpha$$

$$+(1-C+\sigma_{30}l_3\hat{g}C)^2$$

In summary, $\hat{\omega}^2 \propto O(\epsilon^2), \hat{P}_1 \propto O(1), \hat{P}_2 \propto O(\epsilon^2)$,

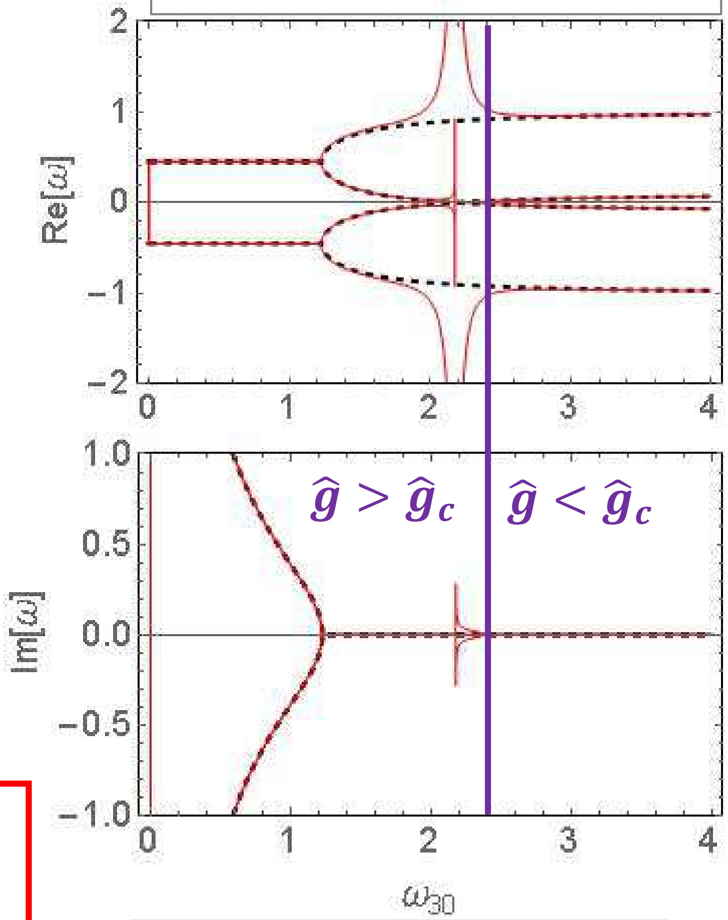
$$\Rightarrow \quad \hat{P}_1\hat{\omega}^2 + \hat{P}_2 = 0$$

$$\hat{\omega}^2 = \frac{(1-C+\sigma_{30}\hat{g}l_3C)^2 + \sigma_{30}\hat{g}l_2F(\hat{g})}{2(1-C+\sigma_{30}l_3\hat{g}C)+C^2}$$

with $F(\hat{g}) := \left\{\sigma_{30}l_3C(1+C)\hat{g} - 3(1-C)^2\right\}\alpha$

Black dashed line: $\epsilon = 0$

Red solid line: $\epsilon = 0.1$



$$\mathbf{l}_0 = (0, \sin\varepsilon, \cos\varepsilon)$$

$$\hat{g} = \frac{Mgl}{I_3\omega_{30}^2} \qquad C = \frac{I_3}{I_1}$$

Eigenvalues at divergence point (oblate top)

- Oblate misaligned top ($a_1 = 1.1, a_3 = 1, C = I_3/I_1 > 1$)
- $\alpha \to \infty$ at divergence point($\hat{g} = \hat{g}_d$)

$$\alpha := \frac{\hat{g}C\sigma_{30}l_2}{1-C+\hat{g}C\sigma_{30}l_3}$$

$$\hat{D} = \hat{\omega}^4 + \hat{P}_1\hat{\omega}^2 + \hat{P}_2 = 0$$

$$\hat{P}_1 = \boxed{-\alpha^2 + (1+C)\sigma_{30}l_2\hat{g}\alpha} \; \cancel{-(2-2C+C^2-2\sigma_{30}l_3\hat{g}C)}$$

$$\hat{P}_2 = \boxed{-\sigma_{30}\hat{g}l_2(1-C)\alpha^3}$$

$$\cancel{+\sigma_{30}\hat{g}C\left\{\sigma_{30}l_2^2\hat{g} + (1-C)\left[\frac{1-C}{C\sigma_{30}\hat{g}} + l_3\right]\right\}\alpha^2}$$

$$\cancel{+\sigma_{30}\hat{g}l_2\left\{\sigma_{30}l_3\hat{g}C(1+C) - 3(1-C)^2\right\}\alpha}$$

$$\cancel{+(1-C+\sigma_{30}l_3\hat{g}C)^2}$$

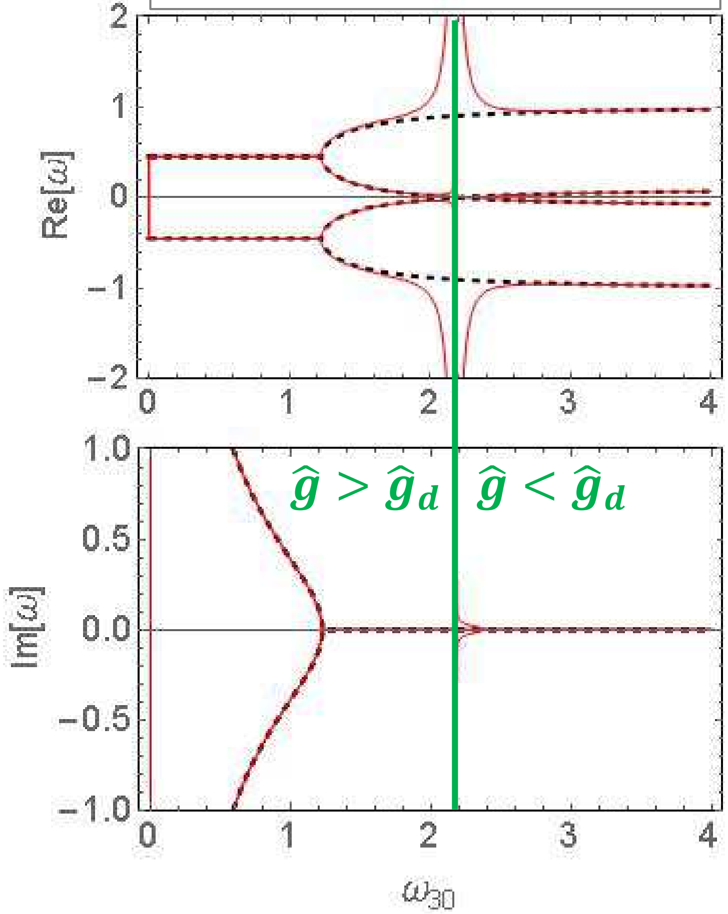
In summary, $\hat{\omega}^2 \propto O(\alpha), \hat{P}_1 \propto O(\alpha^2), \hat{P}_2 \propto O(\alpha^3)$,

$$\Rightarrow \quad \hat{P}_1\hat{\omega}^2 + \hat{P}_2 = 0$$

$$\hat{\omega} = \pm|\sigma_{30}|\hat{g}C\sin\varepsilon\sqrt{\frac{1}{C}\cdot\frac{C-1}{\hat{g}C\sigma_{30}\cos\varepsilon-(C-1)}}$$

→ $\hat{\omega} \in \mathbb{R}$ (stable) if $\hat{g} > \hat{g}_d$; $\hat{\omega} \in \mathbb{C}$ (unstable) if $\hat{g} < \hat{g}_d$

Black dashed line: $\epsilon = 0$
Red solid line: $\epsilon = 0.1$

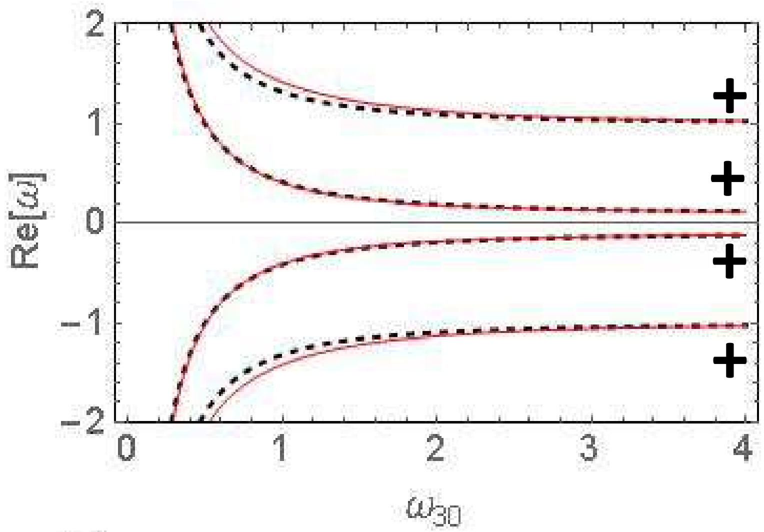


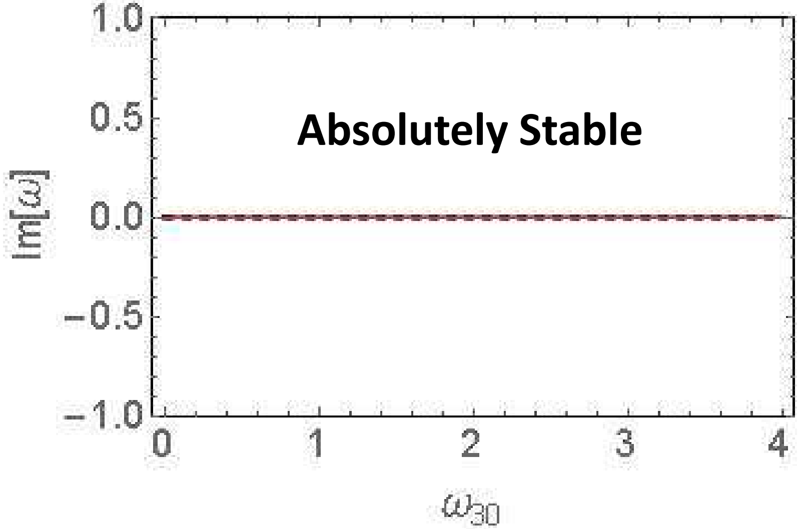
$$\hat{g} = \frac{Mgl}{I_3\omega_{30}^2} \qquad C = \frac{I_3}{I_1}$$

Spectra: oblate hanging-down top

$\sigma_{30} < 0$

- Oblate hanging down misaligned top ($a_1 = 1.1, a_3 = 1$)





$(\tilde{\omega}_b, \tilde{\sigma}_b) \propto \exp(i\omega t)$

+: PEM

−: NEM

Black dashed line: $\epsilon = 0$

Red solid line: $\epsilon = 0.5$

Cf. Sufficient conditions for (nonlinear) stability
Linear stability analysis

Nonlinearity 5 (1992) 1–48. Printed in the UK

They did not consider misalignment when $I_1 = I_2$

The heavy top: a geometric treatment

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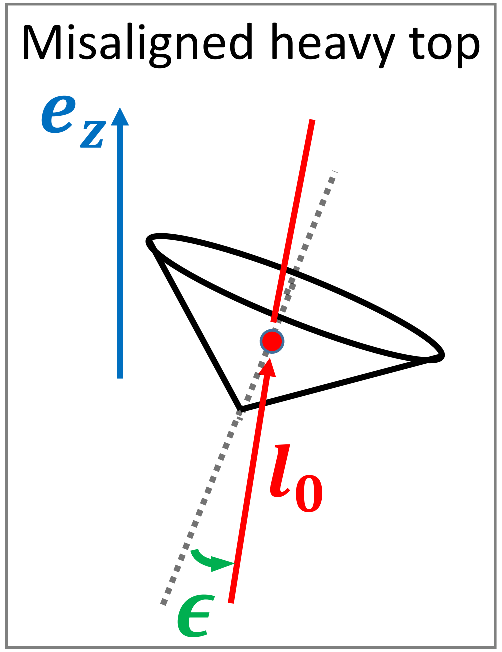
Received 6 June 1991
Accepted by J D Gibbon

Part 3

Spatial description and energetics of motion of a heavy rigid body



Spatial description of a heavy symmetrical top

Fukumoto '97
Paerhati & Fukumoto '13

$\mathbf{t}(t)$ unit vector along axis of the symmetry

Basis of the body coordinates $\{\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{t}(t)\}$

$$\dot{\mathbf{t}} = \boldsymbol{\omega} \times \mathbf{t}$$

$\boldsymbol{\omega}(t)$: angular velocity of the body

Taking $\mathbf{t}\times$

$$\boldsymbol{\omega} = \mathbf{t} \times \dot{\mathbf{t}} + \omega_3 \mathbf{t}$$

$\mathbf{m}_\mathrm{S}$: angular momentum relative to the stationary point O,
viewed from the inertial frame

$$\mathbf{m}_\mathrm{S} = A\mathbf{t} \times \dot{\mathbf{t}} + C\omega_3 \mathbf{t}$$

C , A: Moments of inertia about O w.r.t axial direction and perpendicular to it

Equation of motion of a heavy symmetric top
Lagrange's top

$$\dot{\mathbf{m}}_S = l\mathbf{t} \times (-Mg\mathbf{e}_z)$$

$$\mathbf{m}_S = A\mathbf{t} \times \dot{\mathbf{t}} + C\omega_3 \mathbf{t}$$

M : mass of the body,

l: length of line segment connecting O to the centre of mass.

$-g\mathbf{e}_z$: the gravity acceleration with $\mathbf{e}_z$ unit vector in z-direction.

$$\dot{\mathbf{m}}_S = A\mathbf{t} \times \ddot{\mathbf{t}} + C\omega_3 \dot{\mathbf{t}} + C\dot{\omega}_3 \mathbf{t}$$

$$(\dot{\omega}_3 = 0 \text{ for Lagrange's top})$$

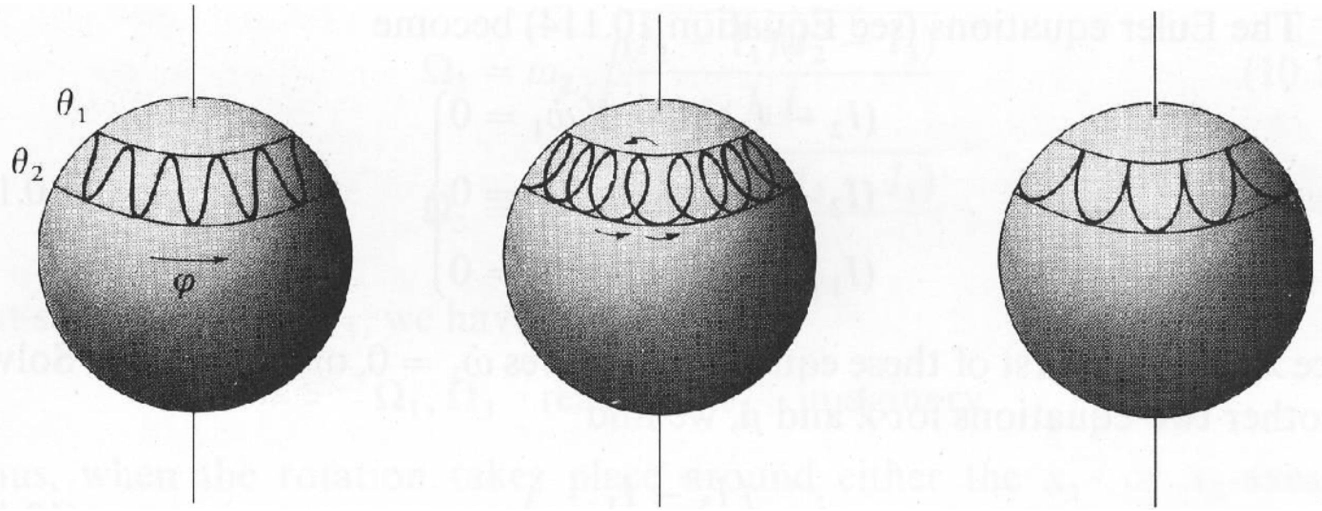
Taking $\mathbf{t}\times$

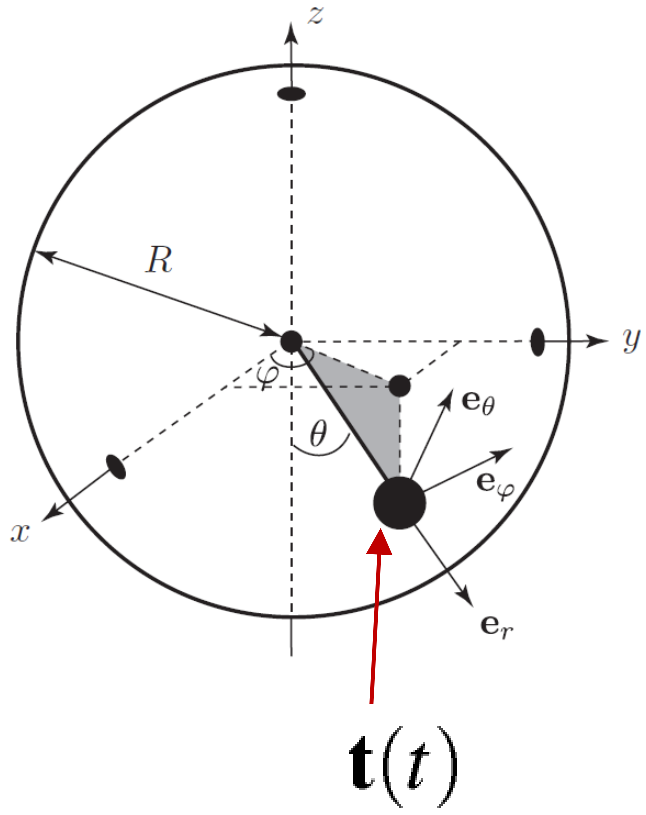
$$\dot{\mathbf{m}}_S = l\mathbf{t} \times (-Mg\mathbf{e}_z)$$

Motion of a charged spherical pendulum

$$A\ddot{\mathbf{t}} = -Mgl[\mathbf{e}_z - (\mathbf{t}\cdot\mathbf{e}_z)\mathbf{t}] - A(\dot{\mathbf{t}}\cdot\dot{\mathbf{t}})\mathbf{t} - C\omega_3\dot{\mathbf{t}}\times\mathbf{t}$$

Motion of a charged spherical pendulum with gravity force in negative z-axis, in the field of ***magnetic monopole*** located at O.





https://hepweb.ucsd.edu/ph110b/110b_notes/node36.html

Componentwise form of motion of Lagrange's top

$$A\ddot{\mathbf{t}} = -Mgl[\mathbf{e}_z - (\mathbf{t}\cdot\mathbf{e}_z)\mathbf{t}] - A(\dot{\mathbf{t}}\cdot\dot{\mathbf{t}})\mathbf{t} - C\omega_3\dot{\mathbf{t}}\times\mathbf{t}$$

Position of a pendulum

$$\mathbf{t}(t) = (X(t), Y(t), Z(t)) \ (\in S^2)$$

$$A\ddot{X} = MglZX - A\dot{\mathbf{X}}^2X - C\omega_3(\dot{Y}Z - \dot{Z}Y),$$

$$A\ddot{Y} = MglZY - A\dot{\mathbf{X}}^2Y - C\omega_3(\dot{Z}X - \dot{X}Z),$$

$$A\ddot{Z} = Mgl(Z^2-1) - A\dot{\mathbf{X}}^2Z - C\omega_3(\dot{X}Y - \dot{Y}X)$$

Equilibria $(X_0, Y_0, Z_0) = (0, 0, -1)$ *Hanging-down top*

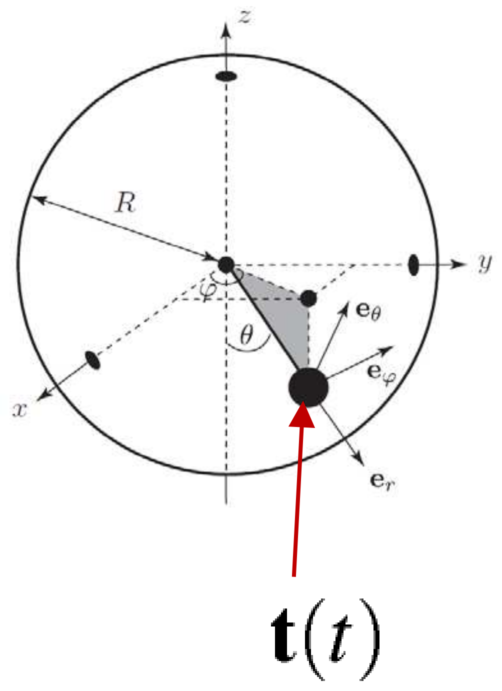
$(X_0, Y_0, Z_0) = (0, 0, 1)$ *'Sleeping top'*

Perturbation around equlibria

$$\mathbf{t}(t) = (X_0, Y_0, Z_0) + (x(t), y(t), z(t))$$

$$\begin{cases} A\ddot{x} + C\omega_3 Z_0\dot{y} - MglZ_0x = 0, \\ A\ddot{y} - C\omega_3 Z_0\dot{x} - MglZ_0y = 0 \end{cases}$$

$$[A\ddot{Z} - 2MglZ = -A\dot{x}^2Z_0 - C\omega_3(\dot{x}y - \dot{y}x) \approx 0]$$



Perturbation about equilibria of Lagrange's top

Hanging-down top $Z_0 = -1$

$$\begin{cases} A\ddot{x} - C\omega_3\dot{y} + Mglx = 0, \\ A\ddot{y} + C\omega_3\dot{x} + Mgly = 0 \end{cases}$$

'Sleeping top' $Z_0 = 1$

$$\begin{cases} A\ddot{x} + C\omega_3\dot{y} - Mglx = 0, \\ A\ddot{y} - C\omega_3\dot{x} - Mgly = 0 \end{cases}$$

Energy of disturbances

$$E_2 = \frac{1}{2}A\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}Mgl\left(x^2 + y^2\right) \ (> 0)$$

$$E_2 = \frac{1}{2}A\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}Mgl\left(x^2 + y^2\right)$$

1. Gyroscopic systems

One might think that systems with NEMs are artifacts or unphysical, purely mathematical, oddities; this, however, is not the case. They occur in fluid and plasma systems[38] for a reason that will become clear below. Generally, they occur in mechanical systems with gyroscopic forces, like the Coriolis force, and they occur in the dynamics of particles in magnetic fields. An example that exhibits both of these is described by a Lagrangian of the form

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + G(\dot{y}x - \dot{x}y) + \tfrac{1}{2}k(x^2 + y^2), \qquad (372)$$

Morrison: Rev. Mod. Phys. **70** (1998) 467

Linear stability of sleeping Lagrange's top

$$x(t) \propto e^{\alpha t}. \quad y(t) \propto e^{\alpha t}$$

$$\begin{vmatrix} A\alpha^2 - Mgl & C\omega_3\alpha \\ -C\omega_3\alpha & A\alpha^2 - Mgl \end{vmatrix} = 0$$

$$\omega_{\mathrm{G}} := \frac{C\omega_3}{2A}, \quad \omega_K^2 := \frac{Mgl}{A}$$

$$\alpha^2 = \omega_K^2 - 2\omega_{\mathrm{G}}^2 \pm 2\omega_{\mathrm{G}}\sqrt{\omega_{\mathrm{G}}^2 - \omega_K^2} \; (= i\omega \text{ for stability})$$

A sufficient condition for (neutral) stabiliy

$$\alpha^2 < 0 \quad \Longleftrightarrow \quad \omega_{\mathrm{G}}^2 - \omega_K^2 > 0 \;\&\; 2\omega_{\mathrm{G}}^2 - \omega_K^2 > 2\omega_{\mathrm{G}}\sqrt{\omega_{\mathrm{G}}^2 - \omega_K^2}$$

$$\Longleftrightarrow \quad C^2\omega_3^2 > 4Mgl$$

Energy of perturbations

$$E_2 = \frac{1}{2}A\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}Mgl\left(x^2 + y^2\right)$$

$$\overline{E_2^{\pm}} = \frac{1}{2}A\left(\alpha^2 - \frac{Mgl}{A}\right)\overline{(x^2 + y^2)} = A\sqrt{\omega_{\mathrm{G}}^2 - \omega_K^2}\left(\sqrt{\omega_{\mathrm{G}}^2 - \omega_K^2} \pm \omega_{\mathrm{G}}\right)\overline{(x^2 + y^2)}$$

Morrison (1998)

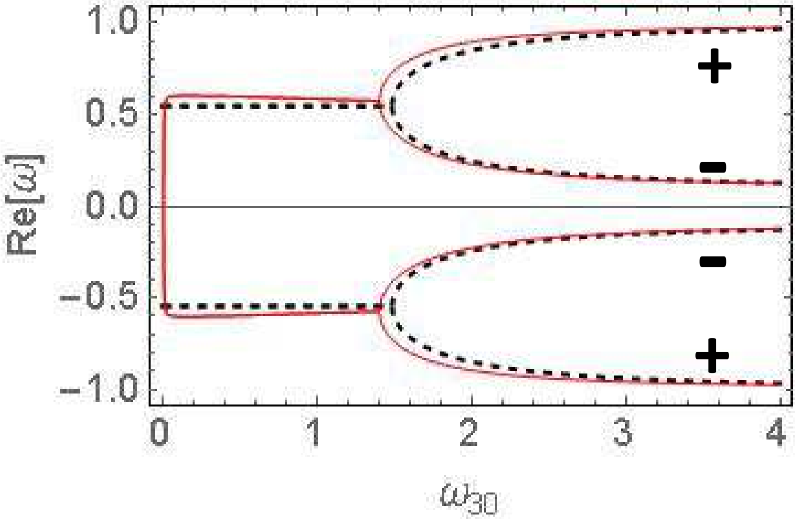
$$\overline{E_2^{+}} > 0, \quad \overline{E_2^{-}} < 0$$

Eigenvalues=Spectra: prolate sleeping top

- Prolate **sleeping** misaligned top ($a_1 = 1, a_3 = 1.1$) $\boldsymbol{\sigma_{30} > 0}$

$$C = \frac{I_3}{I_1} < 1$$

$$(\tilde{\omega}_b, \tilde{\sigma}_b) \propto \exp(i\omega t)$$

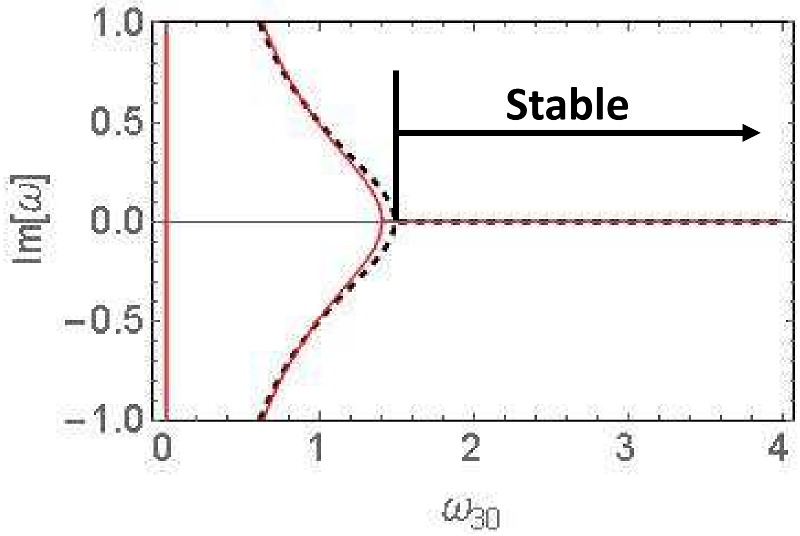


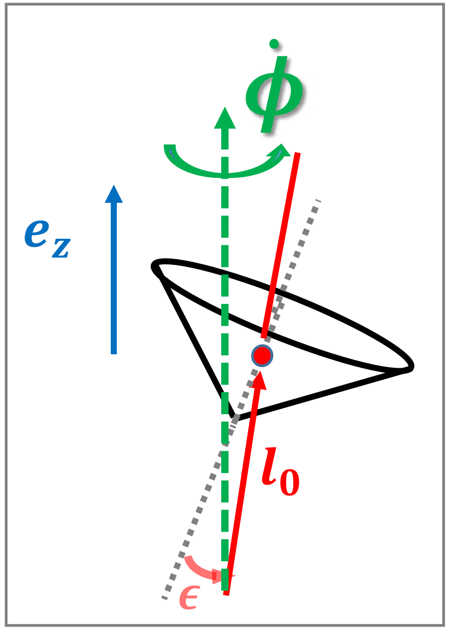
+: PEM

−: NEM

Black dashed line: $\epsilon = 0$

Red solid line: $\epsilon = 0.5$

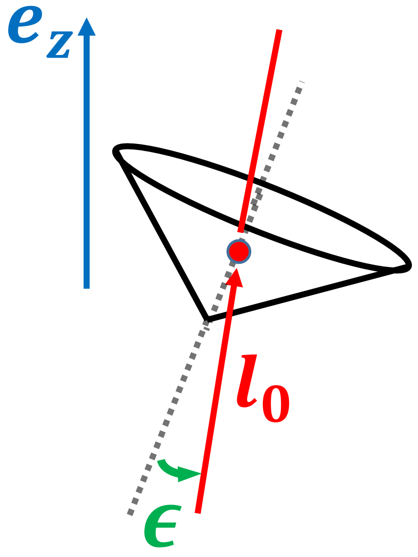




Spatial description of a heavy symmetrical top with axis misaligned from symmetric axis

$$\dot{\mathbf{m}}_S = l\gamma \times (-Mg\mathbf{e}_z)$$

$$\gamma = \cos\varepsilon \mathbf{t} + \sin\varepsilon \mathbf{e}_2 (\neq \mathbf{t})$$



$$\dot{\mathbf{m}}_S = A\mathbf{t} \times \ddot{\mathbf{t}} + C\omega_3 \dot{\mathbf{t}} + C\dot{\omega}_3 \mathbf{t}$$

$$(\dot{\omega}_3 \neq 0 \text{ for misaligned top})$$

$\mathbf{t} \times$

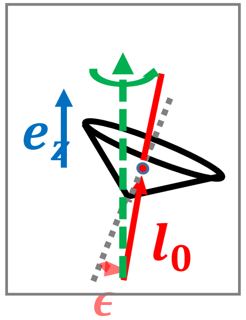
$$A\ddot{\mathbf{t}} = -Mgl\cos\varepsilon[\mathbf{e}_z - (\mathbf{t}\cdot\mathbf{e}_z)\mathbf{t}] - A(\dot{\mathbf{t}}\cdot\dot{\mathbf{t}})\mathbf{t} - C\omega_3 \dot{\mathbf{t}} \times \mathbf{t} - \sin\varepsilon(\mathbf{t}\cdot\mathbf{e}_z)\mathbf{e}_2$$

Summary

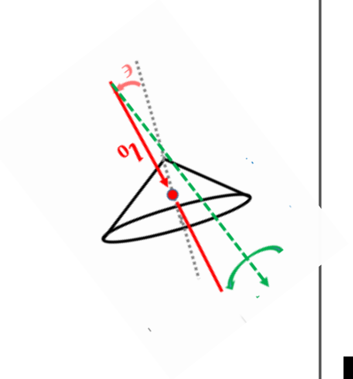
- **Obliquely tilted spheroid** corresponds to the heavy symmetrical top with the top axis tilted from the symmetric axis (**misaligned heavy top**)

- **Steady solutions** are understood via **Euler angles**

→ Only precession velocity $\dot{\phi}$ survives with small angle $\theta \approx O(\epsilon)$; $(\psi = 0)$



- Spectrum for oblate **upright** misaligned top has singularity (divergence point) ($\hat{g} = \hat{g}_d$) and a new unstable region ($\hat{g} = \hat{g}_c$) due to the asymmetry, which does not appear for the usual symmetrical top



- Although the ususal **hanging down** top is always absolutely stable, spectrum for prolate **hanging down** misaligned top has singularity (divergence point) ($\hat{g} = \hat{g}_d$) and unstable region ($\hat{g} = \hat{g}_c$) due to the asymmetry

- For large ω_{30}, Instability occurs for oblate sleeping top whereas two NEMs collide at zero growth rate and the energy become zero

- The **asymptotic eigenvalues** are derived and represent these curious spectrum well

Gyroscopic Analogy of a Rotating Stratified Flow Confined in a Tilted Spheroid with a Heavy Symmetrical Top with the Top Axis Misaligned from the Axis of Symmetry

Yasuhide Fukumoto Notes by Jeffrey Heninger

Rayleigh-Taylor Instability

heavy fluid sitting on top of a light fluid occurs in cirrus clouds, nebula

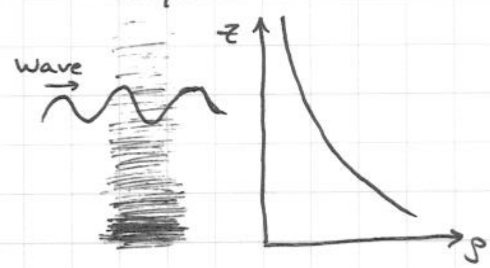
experiment - suppression of Rayleigh-Taylor Instability by rotation - delayed

Inertial Internal Gravity Waves

in a continuously stratified rotating fluid

$\omega^2 \propto f^2 + N^2$ (f^2 from rotation, N^2 from density difference)

even if $N^2 < 0$ (unstable), ω^2 could still be positive



Motion of a Rotating Stratified Flow in a Tilted Spheroid / Ellipsoid

Boussinesq approx - only small changes in density $\rho = \rho_0 + \rho'$

Brunt-Väisälä $N^2 = -\frac{g}{\rho_0} \frac{\partial \rho'}{\partial z}$

write fluid equations in terms of vorticity Ω baroclinic torque

3 exact steady solutions [Dolzhansky 77] - use these as a basis - coefficients ω_k

this converts the equations for the coefficients into the equations for a heavy top in gravity.

First Integrals

full energy, angular momentum about z-axis appear both for fluid & top

Application: toy circulation model for earth's atmosphere from hot equator to cold poles

Hadley & Rossby

Sleeping (right side up) top - like heavier fluid on top vs. Hanging Down Top (Under table)

if rotation is fast enough, the motion is stable

Top Axis Misaligned with Symmetry Axis

top described by Euler-Poisson equation - find a steady solution

use Euler angles

precesses with velocity $\dot{\phi}$

look at linear stability

spectrum shows stability for large ω, instability for small ω if $\varepsilon = 0$, upright (ε: deviation from vertical)

ε small, nonzero → also small region of stability near $\omega = 0$.

hanging down case can also be unstable for finite ε, small ω

Oblate case: ~~disk~~ flat disk

another weird zone of instability – when denominator diverges

always stable if hanging down

Spatial Description & Energetics of Motion of a Heavy Rigid Body

how to understand intuitively

arrange one of the (body) axes along the axis of symmetry

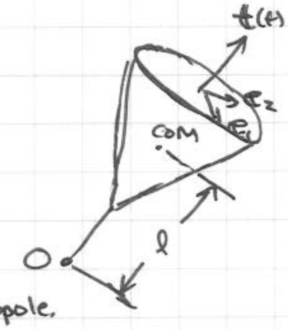
body: $\mathbb{t}(t)$, $\mathbb{e}_1(t)$, $\mathbb{e}_2(t)$

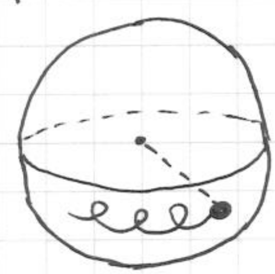
Lagrange's top

~~equ~~ equivalent to motion of spherical pendulum with extra force.

charged spherical pendulum in the field of a point magnetic monopole.

pendulum likes to rotate locally





equilibria – straight up & straight down

perturb around here to check stability

hanging top – potential energy min, sleeping – potential energy max

if axes not aligned, addition term – like internal dynamics