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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Maria Saprykina

Talk Title: Erratic Behavior for One-Dimensional Random Walks in a Generic Quasi-Periodic Environment

Date: 11 / 30 / 2018 **Time:** 2 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Both a random walk in a uniform environment and a random walk in a random walk have well defined limits: a diffusing Gaussian in a uniform environment and a localized distribution in a random environment. In a quasiperiodic environment, there doesn't have to be a well defined long time limit. It is possible for one random walk to have different subsequences that exhibit localization, one-sided drift, and two-sided drift.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
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R1

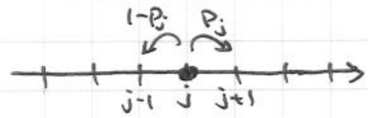
ERRATIC BEHAVIOUR FOR ONE-DIMENSIONAL RANDOM WALKS IN A GENERIC QUASI-PERIODIC ENVIRONMENT

MARIA SAPRYKINA   Notes by Jeffrey Heninger

[joint with B. Fayad, D. Dolgopyat]

Random Walk on $\mathbb{Z}$

j-space, n-time



Z_n - stochastic process

$\mathbb{P}\{Z_{n+1} = j+1 \mid Z_n = j\} = p_j$

$\mathbb{P}\{Z_{n+1} = j-1 \mid Z_n = j\} = 1 - p_j$

Random Walk on $\mathbb{T}$

$x_n = x_0 + \alpha Z_n \mod 1$

$\alpha \in \mathbb{R}/\mathbb{Q}$

Contents

① $p_j = \text{const} \ \forall j$

② p_j independent identically distributed (iid) random variables

③ $p_j = p(x_0 + j\alpha)$ $p \in C^\infty(\mathbb{T}, (0,1))$ deterministic

shift

α Diophantine-like ② ; α Liouville - many possible behaviors

① a) $p_j = \frac{1}{2} \ \forall j$ (simple random walk)

- $\mathbb{E}(Z_n) = 0$, $\mathrm{Var}(Z_n) = n$ "typical distance to 0" $\approx \sqrt{n}$
- recurrent - with probability 1, we return to 0 infinitely often
- (central limit theorem) $\mathbb{P}\left(\frac{Z_n}{\sqrt{n}} < z\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt =: \Phi(z)$
- for random walk on $\mathbb{T}$, Lebesgue measure is invariant

  $\pi(x)\, dx$ is invariant if $\pi(x) = p(x)\, \pi(x+\alpha) + (1-p(x))\, \pi(x-\alpha)$

b) $p_j = p$, $\frac{1}{2} < p < 1$ (drift random walk)

- $\mathbb{E}(Z_n) = n(p-q)$
- (central limit theorem) $\mathbb{P}\left(\frac{Z_n - n(p-q)}{\sigma(p)\sqrt{n}} < z\right) \to \Phi(z)$

② Let p_j be i.i.d. random variables
Random Walk in a Random Environment (RWRE)

[Sinai 82] Look at symmetric random walks.
Let $\mathbb{E}\left(\log \frac{p_1}{1-p_1}\right) = 0 \implies$ walk is recurrent

Then "the typical distance to 0" is $(\log n)^2$ much smaller than $\sqrt{n}$
<u>localization</u>

$\forall \delta > 0$, $\exists$ sequence $m_n(p)$ s.t. $\mathbb{P}\left(\left|\frac{Z_n}{(\log n)^2} - m_n\right| < \delta\right) \xrightarrow[n\to\infty]{} 1$ "uniformly in p"
Prob law of $m_n \to$ limit distribution
distribution of p_1

③ Random Walk in a QuasiPeriodic Environment (RWQPE)
Let $P \in C^\infty(\mathbb{T}, (0,1))$ fix x_0
random walk on $\mathbb{T}$: $p_j = P(x_0 + j\alpha)$
consider Z_n s.t. $X_n = X_0 + \alpha Z_n$
set of symmetric functions
[Sinai 99] Consider $\mathcal{P} = \{p \in C^\infty(\mathbb{T}, (0,1)) \mid \underbrace{\int_{\mathbb{T}} \log \frac{p(x)}{1-p(x)}\, dx = 0}_{\implies \text{recurrent}}\}$
for any $\alpha \in \mathcal{DC}$ (Diophantine)
- $\forall p \in \mathcal{P}$, $\exists \sigma > 0$, $\forall x \in \mathbb{T}$,
  $\mathbb{P}_x\left\{\left|\frac{Z_n}{\sigma\sqrt{n}}\right| < z\right\} \to \Phi(z)$ (central limit theorem)
- random walk on $\mathbb{T}$ has a unique invariant measure ν absolutely continuous with respect to Lebesgue
  the distribution of any point after n steps $\to \nu$

without symmetry assumptions, same conclusions, but
- in CLT, there will be a drift
- don't need Diophantine for α
  there is an absolutely continuous invariant measure $\forall$ irrational α

<u>Liouville Case</u>
<u>Thm A</u> $\forall \alpha \in$ Liouv. $\exists$ dense G_δ set $S \subset \mathcal{P}$ such that
$\forall p \in \mathcal{P}$, for almost every $x \in \mathbb{T}$,
$\exists$ sequences $\{r_k\}, \{s_k\}, \{t_k\}$

- For $T = r_k$ (for this sequence of times), you will have localization
  $\mathbb{P}_x\left(|Z_T| > (\log T)^2\right) < T^{-1/2}$

- For $T = s_k$, you will have one-sided drift
  $\exists \mu_k \simeq s_k$, $\sigma_k \simeq \sqrt{s_k}$ such that $\mathbb{P}_x\left(\left|\frac{Z_T - \mu_k}{\sigma_k}\right| < z\right) \to \Phi(z)$
  expectations, variances

B3

we can also produce drift in the other direction using another sequence

- For $T = t_k$, you will have "two-sided drift"

$\exists (b_k)$, $\varepsilon_k \to 0$ $(k > 0)$ such that

$$\mathbb{P}\left(\left|\frac{Z_T - b_k}{b_k}\right| < \varepsilon_k\right) > 0.1$$

$$\mathbb{P}\left(\left|\frac{Z_T + b_k}{b_k}\right| < \varepsilon_k\right) > 0.1$$

walk concentrated around two ~~located~~ locations

<u>Cor</u> $\alpha \in$ Liouville $\Rightarrow$ random walk on $\mathbb{T}$ for generic $p \in \mathcal{P}$ has no absolutely continuous invariant measure.

Different subsequences converge to uniform measure & to atomic measure

<u>Thm</u> $\exists$ dense G_δ set $\mathcal{D} \subset \mathbb{R}$ such that $\forall \alpha \in \mathcal{D}$, $\forall p \in \mathcal{P}$ $\exists$ sequences (T_n), (σ_n) such that for almost every $x \in \mathbb{T}$,

$$\underbrace{\mathbb{P}(Z_t < \sigma_n \sqrt{t}\, z_n) \xrightarrow[n\to\infty]{} \Phi(z)}_{|\mathbb{P}(Z_t < \sigma_n \sqrt{t}\, z_n) - \Phi(z)| < \frac{1}{n}} \quad \text{for all } T_n < t < e^{T_n}$$

Intuitive Idea

$$\Sigma(n) = \begin{cases} \sum_{j=1}^{n} \log\left(\frac{1-p_j}{p_j}\right) & n \geq 1 \\ 0 & n = 0 \\ -\sum_{j=1}^{n} \log\left(\frac{1-p_j}{p_j}\right) & n \leq 1 \end{cases} \qquad p_j = P(x_0 + j\alpha)$$

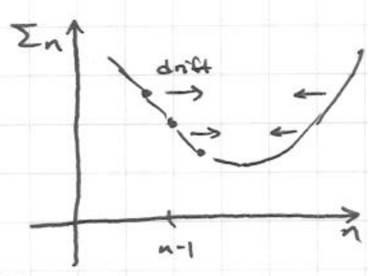
look at $n \geq 1$.

If $p_n > \frac{1}{2}$, then $p_n > 1 - p_n \iff \Sigma(n-1) > \Sigma(n)$ since last term is negative

<u>Lemma</u> If $\exists T$ s.t. $\Sigma(T) > \sqrt{T}$ and $\Sigma(-T) > \sqrt{T}$ (increasing in both directions)

then localization at time T

$$\mathbb{P}\left(|Z_T| > (\log T)^2 < T^{-1/2}\right)$$



Given $\alpha \in$ Liouville, construct a dense G_δ set of p in $\mathcal{P}$ with the above property

<u>Notice</u>: If p_j are i.i.d., then $\left(\log \frac{p_j}{1-p_j}\right)$ are also iid.

$$\mathbb{P}\left(\left|\frac{\sum_{j=1}^{n} \log \frac{p_j}{1-p_j}}{\sigma\sqrt{n}}\right| < z\right) \to \Phi(z)$$

So $\Sigma(n) \approx \sqrt{n}$ with some large probability $\Rightarrow$ localization

Notice : If $\alpha \in$ Diophantine

$$\log \frac{p(x)}{1-p(x)} = g(x+\alpha) - g(x) \text{ has a smooth solution}$$

So $\displaystyle\sum_{j=1}^{n} \log \frac{p(x+j\alpha)}{1-p(x+j\alpha)}$ is bounded independent of n

For Liouville α, we can make this sum grow the way we want