MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Albert Fathi

Talk Title: Recurrence on Abelian Coverings

Date: 11 / 30 / 2018 **Time:** 3 : 30 am / pm **(circle one)**

Please summarize the lecture in 5 or fewer sentences: I honestly don't know what happened in this lecture. There are detailed slides for you to look over if you're interested.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Recurrence on abelian coverings

Albert Fathi

Berkeley, November 30, 2018

Motivations

The question I will be treating is of the following type:

Assume $f : N \to N$ is a homeomorphism of a compact manifold which is chain-recurrent (definition will be given later), and $\pi : \bar{N} \to N$ is a covering to which f lifts as $\bar{f} : \bar{N} \to \bar{N}$. Can you say something about the chain-recurrence of f?

It is motivated by John Franks approach to the Poincaré-Birkhoff theorem, and also by the problem on understanding Aubry sets for lifted Lagrangians on the torus $\mathbb{T}^n$ to $\mathbb{R}^n$.

I obtained a (partial) result that I will explain below.

However, I had to look for applications that motivated such an abstract result.

In a very contorted way, I realized that what I had obtained leads to a simpler of a result on Riemannian metrics with negative curvature.

Geodesic flows

Rather than to start by addressing the problem of lifting chain-recurrence, I will first start with the apparently totally unrelated consequence on geodesic flows.

To explain this, we fix a *c*ompact connected Riemannian manifold M.

Assume that the Riemannian metric g on M is of negative curvature.

Call $S(M)$ the unit tangent bundle.

The geodesic flow $(g_t)_{t\in\mathbb{R}}$ on $S(M)$ is Anosov and preserves the natural Riemannian measure on $S(M)$.

Therefore we know that it is ergodic, and periodic geodesics are dense.

Note also that any given non-zero homotopy class contains exactly one periodic geodesic.

Therefore we cannot have a density theorem of periodic geodesics in a given homotopy class.

Theorem

Assume the geodesic flow of the compact Riemannian manifold M with negative curvature. If $\alpha \in H_1(M, \mathbb{Z})$ is an integral homology class, then the set of closed geodesics γ with homology class $[\gamma] = \alpha$ is dense in S(M).

Although we give a low tech proof of this result, it was known in the 1990's. For surfaces of constant negative curvature, it can be found in Gottschalk-Hedlund's book. Their argument works for constant negative curvature in all dimensions.

In fact, the result is also true if we only assume that the geodesic flow is Anosov, and even for a class of general Anosov flows.

This result follows from work by many people counting closed orbits in homology classes starting from surfaces of constant curvature to general Anosov flows.

A (probably non exhaustive) list of people comprises: Katsuda, Sunada, Philipps, Sarnak, Lalley, Pollicott, Epstein, Sharp.

Although these works addressed primarily the problem of counting closed orbits in a given homology classes, some of them also obtained equidistribution of closed orbits in a homology class for a measure of full support, which implies density.

This equidistribution result is due to S. Lalley (1989) for geodesic flows on surfaces of (variable) negative curvature, to A. Katusda and T. Sunada (1990) for a class of Anosov flows that include all the geodesic flows that are Anosov. R. Sharp (1993) generalized the counting and equidistribution results to a larger class of Anosov flows

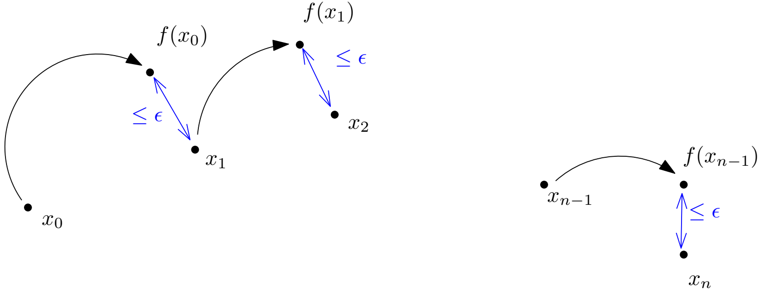
After I gave a lecture in Chapel Hill in Spring 2018, Pat Eberlein came up with an elementary "group argument" proof, which works for manifolds of (variable) negative curvature.

In the sequel we will concentrate on the case where the homology class is zero in the theorem.

An important feature of Anosov flows is the so-called Anosov closing lemma (in generalized form) known also as the shadowing property.
Namely if we have an almost orbit it is close to a genuine orbit.
(In the theory of manifolds of negative curvature, this property is known as a quasi-geodesic is "close" to a geodesic.)

To explain this property, we recall a few notions from general dynamical system theory. We do it for discrete systems, adaptation to flows is well-known.
For a continuous map f of the metric space (X, d), an ϵ-pseudo orbit (or ϵ-chain), with $\epsilon > 0$, is a sequence of points $x_0, \ldots, x_n$, with $n \geq 1$, such that $d(f(x_i), x_{i+1}) < \epsilon$.

We should think of an ϵ-pseudo orbit (or ϵ-chain) as an orbit up to error ϵ.



Such an ϵ-pseudo orbit is said to be closed if $x_0 = x_n$.

A point $x \in X$ is chain-recurrent for f is for every $\epsilon > 0$, we can find an ϵ-pseudo orbit $x_0, \ldots, x_n$, for some $n \geq 1$, with $x_0 = x_n = x$, or equivalently, for every $\epsilon > 0$, there is a closed ϵ-pseudo orbit through x.

The set of chain-recurrent points for f is denoted by $\mathcal{R}(f)$.

If $X = \mathcal{R}(f)$, then we say that f is chain-recurrent.

This set $\mathcal{R}(f)$ depends on the choice of the metric d, but two uniformly equivalent metrics give rise to the same chain-recurrent set.

In particular, if X is compact, the set $\mathcal{R}(f)$ is independent of the choice of the metric d defining the topology on X.

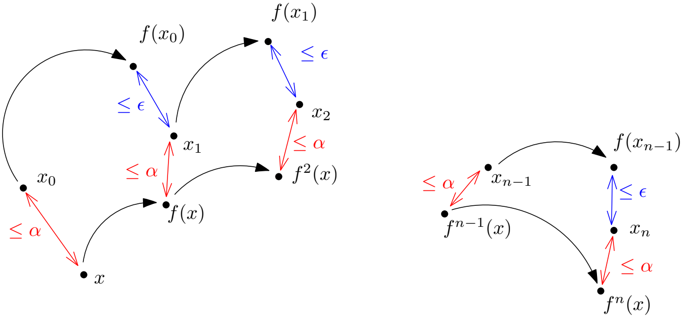
We say that $f : X \to X$ is chain-transitive if for any $x, y \in X$ and any $\epsilon > 0$, there is an ϵ-pseudo orbit $x_0, \ldots, x_n$ with $x_0 = x, x_n = y$.

If X is connected then $f : X \to X$ is chain-transitive if and only if if it is chain-recurrent.

The shadowing property for a continuous map f of the metric space (X, d) is:

Shadowing Property

For every $\alpha > 0$, there exists an $\epsilon > 0$ such that for any ϵ-chain $x_0, \dots, x_n$, with $n \geq 1$ (but arbitrary) we can find a point $x \in M$ such that $d(f^i(x), x_i) \leq \alpha$. We say that the ϵ-pseudo orbit is α-shadowed by the orbit of x.



Important fact: *ϵ does not depend on n.*

An Anosov diffeomorphism f of the compact manifold N enjoys the shadowing property. This is the Anosov closing lemma. Moreover, in this case any closed ϵ-chain of length n can be shadowed by a periodic point of period (dividing) n.
Therefore, an Anosov diffeomorphism f is chain recurrent if and only if its periodic points are dense.
Consider $\tilde{N} \to N$ a covering of the compact Riemannian manifold N. Endow $\tilde{N}$ with the distance $\tilde{d}$ coming from the lift to $\tilde{N}$ of the Riemannian metric on N.
If f is an Anosov diffeomorphism on N that lifts to a diffeomorphism $\tilde{f}$ of $\tilde{N}$, it is not difficult to see that $\tilde{f}$ has the shadowing property.

Hence, to show that the $\tilde{f}$-periodic points are dense in $\tilde{N}$, *it suffices to show that $\tilde{f}$ is chain recurrent.*

Let us try to apply this idea to reduce the following theorem

Theorem

If M is a compact Riemannian manifold whose geodesic flow is Anosov, the closed geodesics which are homologous to 0 are dense in the unit tangent bundle $S(M)$.

to a more general fact.
We denote by $\pi : \bar{M} \to M$ the maximal abelian covering of M, i.e. the regular (or Galois) covering of M whose Galois group is $H_1(M, \mathbb{Z})$.
We endow $\bar{M}$ with the Riemannian metric pulled from M. A closed geodesic in M is homologous to 0 if and only if it is the image of a closed geodesic in $\bar{M}$.
Hence, it suffices to show that the closed geodesics are dense in $\bar{M}$. Since as we saw above the geodesic flow on $S(\bar{M})$ has the shadowing property, it suffices to prove that the geodesic flow is chain recurrent on $S(\bar{M})$.

Therefore the previous theorem for the negative curvature case follows from

Theorem (Chain-recurrence for geodesic flow)

Let M be an arbitrary compact Riemannian manifold (not necessarily of negative curvature) of dimension at least 2, then the geodesic flow on $\bar{M}$, the maximal abelian cover of M, is chain-recurrent on $S(\bar{M})$.

Remark (V. Gelfreich)

Not true for $M = \mathbb{T}$: the geodesic flow on (flat) $\mathbb{R}$ is not chain recurrent. The problem is that $S(\mathbb{T})$ is not connected, and the geodesic flow on $S(\mathbb{T})$ is not chain transitive.

This Theorem is *baffling*. At first I thought it was not correct. Why? If $M = \mathbb{T}^k, k \geq 2$ with the flat metric, how can its lift to $\mathbb{R}^k$ have any recurrence property (except for fixed points on the 0 section).The geodesic flow on the unit tangent bundle is dissipative. Nothing can be more violently dissipative than that! How can it enjoy any type of recurrence?

Answer: Regular *n*-gons!

A regular *n*-gon with side equal 1 has angles $(n-2)\pi/n \to \pi$. Therefore if you follow the unit tangent vectors of the sides of such an oriented *n*-gone, it will be an ϵ-pseudo orbit for *n* large.

In fact, we will deduce, the chain-recurrence theorem above on the geodesic flow from an even more general one.
We now explain this more general setting.
Consider an abelian covering $\pi : \bar{N} \to N$ of the compact manifold N. The Galois group of $\pi : \bar{N} \to N$ is an abelian group of finite type. To simplify the setting we will make the assumption that this group has no torsion.
Hence we can assume that it is $\mathbb{Z}^k$. Therefore $N = \bar{N}/\mathbb{Z}^k$.
We denote the action of $\mathbb{Z}^k$ on $\bar{N}$ by

$$(n, \bar{x}) \mapsto \bar{x} + n, n \in \mathbb{Z}^k, \bar{x} \in \bar{N}.$$

From the general theory of covering, we can find a smooth map $\bar{\varphi} : \bar{N} \to \mathbb{R}^k$ such that

$$\bar{\varphi}(\bar{x} + n) = \bar{\varphi}(x) + n, n \in \mathbb{Z}^k, \bar{x} \in \bar{N}.$$

In fact, the map $\bar{\varphi}$ yields a map $\varphi : N \to \mathbb{T}^k$ such that the covering $\pi : \bar{N} \to N$ is the pull-back by φ of the canonical covering $\mathbb{R}^k \to \mathbb{T}^k$.
It is useful to note that by compactness of N, the smooth map φ is Lipschitz. Therefore $\bar{\varphi}$ is also Lipschitz on $\bar{N}$.

Theorem (Lifting chain-recurrence)

Assume that $f : N \to N$ is a chain-transitive homeomorphism that lifts to a homeomorphism $\bar{f} : \bar{N} \to \bar{N}$ such that

$$\bar{f}(\bar{x} + n) = \bar{f}(\bar{x}) + n, n \in \mathbb{Z}^k, \bar{x} \in \bar{N}.$$

If $\bar{f}$ is not chain-recurrent, then there exists a non-zero linear form $L : \mathbb{R}^k \to \mathbb{R}$ and a finite constant $K \geq 0$ such that

$$L\bar{\varphi}(\bar{f}^{\ell}(\bar{x})) - L\bar{\varphi}(\bar{x}) \geq -K, \textit{for every } \bar{x} \in \bar{N}, \textit{ and every } \ell \geq 0.$$

I will explain first the meaning of the theorem. Assume $M = \mathbb{T}^k$ and the cover is $\mathbb{R}^k \to \mathbb{T}^k$. In this case, we can take $\varphi = \mathrm{Id}_{\mathbb{T}^k}$.

The theorem for the $\mathbb{T}^k$ case is then:

Theorem (Lifting chain-recurrence for $\mathbb{T}^k$)

Assume that $f : \mathbb{T}^k \to \mathbb{T}^k$ is a chain-transitive homeomorphism that lifts to a homeomorphism $\bar{f} : \mathbb{R}^k \to \mathbb{R}^k$ such that $\bar{f}(\bar{x} + n) = \bar{f}(\bar{x}) + n$, for all $n \in \mathbb{Z}^k$ and $\bar{x} \in \mathbb{R}^k$. If $\bar{f}$ is not chain-recurrent, then there exists a non-zero linear form $L : \mathbb{R}^k \to \mathbb{R}$ and a finite constant $K \geq 0$ such that

$$L(\bar{f}^\ell(\bar{x})) - L(\bar{x}) = L(\bar{f}^\ell(\bar{x}) - \bar{x}) \geq -K,$$

for every $\bar{x} \in \mathbb{R}^k$, and every $\ell \geq 0$.

That "basically" means that the orbits of $\bar{f}$ are contained in a unique affine half space of $\mathbb{R}^k$.
This of course not exactly true since orbits can start outside any affine half space, so you have to subtract the origin $\bar{x}$ of the orbit.
Therefore, the theorem says that if $\bar{f}$ is not chain-recurrent, then the set

$$\mathcal{SDI}(\bar{f}) = \{\bar{f}^\ell(\bar{x}) - \bar{x} \mid \bar{x} \in \mathbb{R}^k, \ell \geq 0\}$$

is contained in an affine half space of $\mathbb{R}^k$.

We can also reformulate the general case as:

Theorem (Lifting chain-recurrence)

Assume that $f : N \to N$ is a chain-transitive homeomorphism that lifts to a homeomorphism $\bar{f} : \bar{N} \to \bar{N}$ such that

$$\bar{f}(\bar{x} + n) = \bar{f}(\bar{x}) + n, n \in \mathbb{Z}^k, \bar{x} \in \bar{N}.$$

If $\bar{f}$ is not chain-recurrent, then set

$$\mathcal{SDI}_{\bar{\varphi}}(\bar{f}) = \{\bar{\varphi}(\bar{f}^{\ell}(\bar{x})) - \bar{\varphi}(\bar{x}) \mid \bar{x} \in \mathbb{R}^k, \ell \geq 0\}$$

is contained in an affine half space.

Corollary

In the situation above, assume that, for every $n \in \mathbb{Z}^k \setminus \{0\}$, we can find a point $\bar{x}_n$ such that $\bar{f}^{\ell_n}(\bar{x}_n) = \bar{x}_n + n$, for some $\ell_n \geq 1$. Then $\bar{f}$ is chain recurrent.

To prove the corollary, note that $\bar{\varphi}$ commutes with the translation by the vectors $n \in \mathbb{Z}^k$.
Therefore from $\bar{f}^{\ell_n}(\bar{x}_n) = \bar{x}_n + n$, we get

$$\bar{\varphi}(\bar{f}^{\ell_n}(\bar{x}_n)) = \bar{\varphi}(\bar{x}_n + n) = \bar{\varphi}(\bar{x}_n) + n.$$

Hence

$$\bar{\varphi}(\bar{f}^{\ell_n}(\bar{x}_n)) - \bar{\varphi}(\bar{x}_n) = n.$$

This forces the set

$$\mathcal{SDI}_{\bar{\varphi}}(\bar{f}) = \{\bar{\varphi}(\bar{f}^{\ell}(\bar{x})) - \bar{\varphi}(\bar{x}) \mid \bar{x} \in \mathbb{R}^k, \ell \geq 0\}$$

to contain $\mathbb{Z}^k \setminus \{0\}$.
Therefore $\mathcal{SDI}_{\bar{\varphi}}(\bar{f})$ cannot be contained in an affine half space.
The lifting theorem implies that $\bar{f}$ ischain-recurrent.

The corollary can be applied to the case of the geodesic flow since any non-trivial homotopy (hence homology) class of curves contains a closed geodesic. Note that $S(M)$ is connected if the dimension of M is ≥ 2. Therefore, in this case, the geodesic flow is *chain-transitive*

This provides a proof of

Theorem

Let M be an arbitrary compact connected Riemannian manifold (not necessarily of negative curvature) of dimension ≥ 2, then the geodesic flow on $\bar{M}$, the maximal abelian cover of M, is chain-recurrent on $S(\bar{M})$.

We now sketch the proof of the Lifting chain-recurrence theorem.

Theorem (Lifting chain-recurrence)

Assume that $f : N \to N$ is a chain-transitive homeomorphism that lifts to $\bar{f} : \bar{N} \to \bar{N}$ with $\bar{f}(\bar{x} + n) = \bar{f}(\bar{x}) + n, n \in \mathbb{Z}^k, \bar{x} \in \bar{N}$. If $\bar{f}$ is not chain-recurrent, there exists a non-zero linear form $L : \mathbb{R}^k \to \mathbb{R}$ and a finite constant $K \geq 0$ such that $L\bar{\varphi}(\bar{f}^\ell(\bar{x})) - L\bar{\varphi}(\bar{x}) \geq -K$.

We choose a Riemannian metric on N, and we lift it to $\bar{N}$ via the covering projection $\pi : \bar{N} \to N$.
We use on N and $\bar{N}$ the distances d and $\bar{d}$ coming from the Riemannian metrics.

Note that we have:

(a) $\bar{d}(\bar{x} + n, \bar{y} + n) = \bar{d}(\bar{x}, \bar{y})$, for all $n \in \mathbb{Z}^k$.

(b) $d(\pi(\bar{x}), \pi(\bar{y})) = \min_{n \in \mathbb{Z}^k} \bar{d}(\bar{x}, \bar{y} + n)$.

(c) Moreover, the covering map $\pi : \bar{N} \to N$ is a local isometry.

We now introduce the Pageault barriers $\mathcal{P}$ on N and $\bar{\mathcal{P}}$ on $\bar{N}$.
For $x_0, \ldots, x_\ell, \ell \geq 1$ in N, we set

$$p(x_0, \ldots, x_\ell) = \max_{i=0,\ell-1} d(x_{i+1}, f(x_i)).$$

(resp. For $\bar{x}_0, \ldots, \bar{x}_\ell, \ell \geq 1$ in $\bar{N}$, we set

$$\bar{p}(\bar{x}_0, \ldots, \bar{x}_\ell) = \max_{i=0,\ell-1} \bar{d}(\bar{x}_{i+1}, \bar{f}(\bar{x}_i)).)$$

Then for $x, y \in N$, we set

$$\mathcal{P}(x, y) = \inf\{p(x_0, \ldots, x_\ell) \mid \ell \geq 1, x_0 = x, x_\ell = y\}.$$

(resp. for $\bar{x}, \bar{y} \in \bar{N}$, we set

$$\bar{\mathcal{P}}(\bar{x}, \bar{y}) = \inf\{\bar{p}(\bar{x}_0, \ldots, \bar{x}_\ell) \mid \ell \geq 1, \bar{x}_0 = \bar{x}, \bar{x}_\ell = \bar{y}\}.)$$

The properties that we are going to use are:
(1) $\mathcal{P}(x,x) = 0$ (resp. $\bar{\mathcal{P}}(\bar{x},\bar{x}) = 0$) if and only if x is chain-recurrent for f (resp. $\bar{x}$ is chain-recurrent for $\bar{f}$).
(2) $\mathcal{P}(x,f(x)) = 0$, since $p(x,f(x)) = d(f(x),f(x)) = 0$ (resp. $\bar{\mathcal{P}}(\bar{x},\bar{f}(\bar{x})) = 0$).
(3) [Ultrametric property] $\mathcal{P}(x,z) \leq \max[\mathcal{P}(x,y),\mathcal{P}(y,z)]$ by concatenation of chains (resp. $\bar{\mathcal{P}}(\bar{x},\bar{z}) \leq \max[\bar{\mathcal{P}}(\bar{x},\bar{y}),\bar{\mathcal{P}}(\bar{y},\bar{z})]$).
(4) $\bar{\mathcal{P}}(\bar{x}+n,\bar{y}+n) = \bar{\mathcal{P}}(\bar{x},\bar{y})$, for all $n \in \mathbb{Z}^k$.
(5) $\mathcal{P}(\pi(\bar{x}),\pi(\bar{y})) = \inf_{n\in\mathbb{Z}^k} \bar{\mathcal{P}}(\bar{x},\bar{y}+n)$.

Since we are assuming that f is chain-recurrent, from (1) we obtain that $\mathcal{P}(x,x) = 0$, for all $x \in N$.
Therefore from (5) we get

$$\inf_{n\in\mathbb{Z}^k} \bar{\mathcal{P}}(\bar{x},\bar{x}+n) = 0, \text{ for every } \bar{x} \in \bar{N}.$$

If we assume that $\bar{f}$ is not chain-recurrent, we can find $\bar{x}_0 \in \bar{N}$ such that $\bar{\mathcal{P}}(\bar{x}_0, \bar{x}_0) > 0$. If ϵ is fixed, with $\bar{\mathcal{P}}(\bar{x}_0, \bar{x}_0) \geq \epsilon > 0$, we define the subset $\Gamma \subset \mathbb{Z}^k$ by

$$\Gamma = \{n \in \mathbb{Z}^k \mid \bar{\mathcal{P}}(\bar{x}_0, \bar{x}_0 + n) < \epsilon\}.$$

By choice of ϵ, this subset Γ does not contain $\{0\}$.
From $\inf_{n\in\mathbb{Z}^k} \bar{\mathcal{P}}(\bar{x}_0, \bar{x}_0 + n) = 0$, it is not empty.
Moreover Γ *is closed under addition.*
In fact, by the ultrametric property (3)
$\bar{\mathcal{P}}(\bar{x}, \bar{z}) \leq \max[\bar{\mathcal{P}}(\bar{x}, \bar{y}), \bar{\mathcal{P}}(\bar{y}, \bar{z})]$, we obtain
$\bar{\mathcal{P}}(\bar{x}, \bar{x} + n + m) \leq \max[\bar{\mathcal{P}}(\bar{x}, \bar{x} + n), \bar{\mathcal{P}}(\bar{x} + n, \bar{x} + n + m)]$. But by property (4) $\bar{\mathcal{P}}(\bar{x} + n, \bar{y} + n) = \bar{\mathcal{P}}(\bar{x}, \bar{y})$. Hence
$\bar{\mathcal{P}}(\bar{x} + n, \bar{x} + n + m) = \bar{\mathcal{P}}(\bar{x}, \bar{x} + m)$. Therefore

$$\bar{\mathcal{P}}(\bar{x}, \bar{x} + n + m) \leq \max[\bar{\mathcal{P}}(\bar{x}, \bar{x} + n), \bar{\mathcal{P}}(\bar{x}, \bar{x} + m)],$$

which clearly implies that Γ is stable under addition.

Since $\Gamma \subset \mathbb{Z}^k$ *is stable under addition and does not contain* 0, we can conclude that the convex envelop of Γ does not contain 0.
By Hahn-Banach theorem, we can find a non-zero linear form $L : \mathbb{R}^k \to \mathbb{R}$ such that $L \geq 0$ on Γ.
Using that f is *chain-transitive*, we now explain why there exists a finite constant $K \geq 0$, such for every $\bar{x}, \bar{y} \in \bar{N}$, with $\bar{\mathcal{P}}(\bar{x}, \bar{y}) < \epsilon$, we have

$$L\bar{\varphi}(\bar{y}) - L\bar{\varphi}(\bar{x}) \geq -K.$$

Since $\bar{\mathcal{P}}(\bar{x}, \bar{f}^\ell(\bar{x})) = 0$, this will prove

$$L\bar{\varphi}(\bar{f}^\ell(\bar{x})) - L\bar{\varphi}(\bar{x}) \geq -K.$$

To find K, we need:

Lemma

Assuming f chain-transitive on N, there exists a constant $K_\epsilon < +\infty$ (depending on ϵ) such that, for every $x, y \in N$, we can find $\bar{x}, \bar{y} \in \bar{N}$, with $x = \pi(\bar{x}), y = \pi(\bar{y})$, satisfying

$$\bar{\mathcal{P}}(\bar{x}, \bar{y}) < \epsilon \text{ and } \bar{d}(\bar{x}, \bar{y}) \leq K_\epsilon.$$

Assume $\bar{\mathcal{P}}(\bar{x}, \bar{y}) < \epsilon$. We apply the Lemma to $\pi(\bar{x}_0), \pi(\bar{x})$ to find $m_1, m_2 \in \mathbb{Z}^k$ such that

$$\bar{\mathcal{P}}(\bar{x}_0 + m_1, \bar{x} + m_2) < \epsilon \text{ and } \bar{d}(\bar{x}_0 + m_1, \bar{x} + m_2) \leq K_\epsilon.$$

By the invariance properties of both $\bar{\mathcal{P}}$ and $\bar{d}$, setting $m = m_1 - m_2$, we get

$$\bar{\mathcal{P}}(\bar{x}_0 + m, \bar{x}) < \epsilon \text{ and } \bar{d}(\bar{x}_0 + m, \bar{x}) \leq K_\epsilon.$$

In the same way, we can find $m' \in \mathbb{Z}$ such that

$$\bar{\mathcal{P}}(\bar{y}, \bar{x}_0 + m') < \epsilon \text{ and } \bar{d}(\bar{y}, \bar{x}_0 + m') \leq K_\epsilon.$$

In particular, by the ultrametric property of $\bar{\mathcal{P}}$, we get

$$\bar{\mathcal{P}}(\bar{x}_0 + m, \bar{x}_0 + m') < \epsilon.$$

Since $\bar{\mathcal{P}}(\bar{x}_0 + m, \bar{x}_0 + m') = \bar{\mathcal{P}}(\bar{x}_0, \bar{x}_0 + m' - m)$, this implies $m' - m \in \Gamma$. Therefore

$$L(m' - m) \geq 0.$$

Since $\bar{\varphi}$ is Lipschitz, from $\bar{d}(\bar{x}_0 + m, \bar{x}) \leq K_\epsilon$ and $\bar{d}(\bar{y}, \bar{x}_0 + m') \leq K_\epsilon$, we obtain

$$|L\bar{\varphi}(\bar{x}_0 + m') - L\bar{\varphi}(\bar{y})| \leq \|L\| \operatorname{Lip}(\bar{\varphi}) K_\epsilon$$
$$|L\bar{\varphi}(\bar{x}) - L\bar{\varphi}(\bar{x}_0 + m)| \leq \|L\| \operatorname{Lip}(\bar{\varphi}) K_\epsilon.$$

But $L\bar{\varphi}(\bar{x}_0 + z) = L[\bar{\varphi}(\bar{x}_0) + z] = L\bar{\varphi}(\bar{x}_0) + L(z)$, for every $z \in \mathbb{Z}^k$. Hence, with $K' = 2\|L\| \operatorname{Lip}(\bar{\varphi}) K_\epsilon$, we obtain from the inequalities above

$$|L(m' - m) - [L\bar{\varphi}(\bar{y}) - L\bar{\varphi}(\bar{x})]| \leq K'.$$

Using $L(m' - m) \geq 0$, this yields the desired inequality

$$L\bar{\varphi}(\bar{y}) - L\bar{\varphi}(\bar{x}) \geq -K'.$$

It remains to prove the Lemma:

Lemma

Assuming f chain-transitive on N, there exists a constant $K_\epsilon < +\infty$ (depending on ϵ) such that, for every $x, y \in N$, we can find $\bar{x}, \bar{y} \in \bar{N}$, with $x = \pi(\bar{x}), y = \pi(\bar{y})$, satisfying

$$\bar{\mathcal{P}}(\bar{x}, \bar{y}) < \epsilon \text{ and } \bar{d}(\bar{x}, \bar{y}) \leq K_\epsilon.$$

Since h is *chain-transitive*, we have

$$0 = \mathcal{P}(x, y) = \inf\{\bar{\mathcal{P}}(\bar{x}, \bar{y}) \mid \pi(\bar{x}) = x, \pi(\bar{y}) = y\}.$$

Hence

$$N \times N = \pi \times \pi\{(\bar{x}, \bar{y}) \mid \bar{\mathcal{P}}(\bar{x}, \bar{y}) < \epsilon\}.$$

The open set $O = \{(\bar{x}, \bar{y}) \mid \bar{\mathcal{P}}(\bar{x}, \bar{y}) < \epsilon\}$ is the increasing union of the open sets

$$O_n = \{(\bar{x}, \bar{y}) \mid \bar{\mathcal{P}}(\bar{x}, \bar{y}) < \epsilon, \bar{d}(\bar{x}, \bar{y}) < n\}.$$

Since $\pi \times \pi$ is an open map, the compactness of $N \times N$ implies that

$$N \times N = \pi \times \pi(O_n), \text{ for } n \text{ large}.$$

Recurrence on Abelian Coverings

Albert Fathi Notes by Jeffrey Heninger

Motivations

f homomorphism

John-Franks approach to Poincaré-Birkhoff

understanding Aubry-Mather set

Geodesic Flows

compact connected Riemannian manifold – negative curvature

geodesic flow is Anosov, preserves natural Riemannian metric

ergodic, periodic orbits are dense

Thm closed geodesics of an integer homology class are dense

Proof – focus on case where homology class is zero

Anosov closing lemma i.e. shadowing property

almost orbit is close to a genuine orbit – error of ε

ε pseudo-orbit or ε chain

$x \in X$ is chain-recurrent for f if $\forall \varepsilon > 0$, we can find an ε pseudoorbit $x_0 \cdots x_n$ for some $n \geq 1$ with $x_0 = x_n = x$

set of chain-recurrent points for f is $\mathcal{R}(f)$

if chain recurrent every, say f is chain-recurrent

chain transitive – can make an ε pseudoorbit between any two points in X

Shadowing Property

ε pseudo orbit is α close to a real orbit

↑ does not depend on n.

Ansov diffeo f of compact manifold N has a shadowing property

closed ε chain can be approximated by a periodic orbit of the same period

Anosov diffeo f chain recurrent ⟺ periodic points are dense

Covering of the Riemannian manifold $\tilde{N} \to N$

lifted $\tilde{f}$ also has shadowing property (even though $\tilde{N}$ not compact)

we just have to show $\tilde{f}$ is chain recurrent

Thm M – compact Riemannian manifold, geodesic flow is Anosov

$\Uparrow$ closed geodesic homologous to 0 are dense in the ~~[struck-out word]~~ unit tangent bundle

Thm <u>Chain-Recurrence for Geodesic Flow</u>

M – arbitrary compact Riemannian manifold   dim > 2 (not true for circle)

$\overline{M}$ – maximal abelian cover of M

$\Uparrow$ then the geodesic flow is chain-recurrent of $S(\overline{M})$ (unit tangent bundle)

Thm

Setting: $\pi : \overline{N} \to N$ abelian covering of a compact manifold

assume no torsion – assume it is $\mathbb{Z}^k$ so $N = \overline{N} / \mathbb{Z}^k$

smooth map $\overline{\varphi}$ – pull back of φ

<u>Lifting Chain Recurrence</u>

$f : N \to N$ chain-transitive homeomorphism, lifts to homeomorphism $\overline{f}$

$\overline{f}$ commutes with action of $\mathbb{Z}^k$

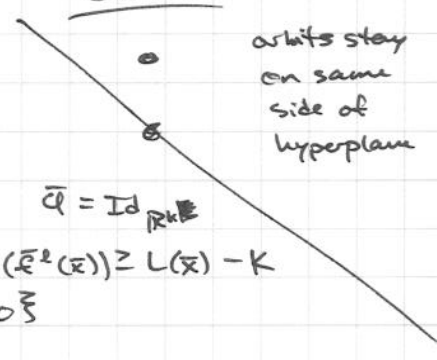
if $\overline{f}$ not chain-recurrent, $\exists$ non-zero linear form $L : \mathbb{R}^k \to \mathbb{R}$

K – finite constant

such that $L\overline{\varphi}(\overline{f}^{\ell}(\overline{x})) - L\overline{\varphi}(\overline{x}) \geq -K$

for every $\overline{x} \in \overline{N}$, every $\ell \geq 0$

On Torus



Reformulate:

$\overline{f}(\overline{x} + n) = \overline{f}(\overline{x}) + n$ $\overline{\varphi} = Id_{\mathbb{R}^k}$

if $\overline{f}$ is not chain-recurrent, $L(\overline{f}^{\ell}(\overline{x})) \geq L(\overline{x}) - K$

then the set $SDI_{\overline{\varphi}}(\overline{f}) = \{\overline{\varphi}(\overline{f}^{\ell}(\overline{x})) - \overline{\varphi}(\overline{x}) \mid \overline{x} \in \mathbb{R}^k, \ell \geq 0\}$

is contained in an affine half space

<u>Cor</u> if $\forall n \in \mathbb{Z}^k \setminus \{0\}$, we can find $\overline{x}_n$ such that $\overline{f}^{\ell_n}(\overline{x}_n) = \overline{x}_n + n$ for some $\ell_n \geq 1$

then $\overline{f}$ is chain-recurrent

Apply Cor to geodesic ~~flow~~ flow since any homology class of curves contains a closed geodesic.

See the ~~[struck-out word]~~ slides for details about how the proofs work.

uses Hahn-Banach thm